Miura transformations from Novikov algebras

Ian Strachan

5th September 2019
Consider a symplectic 2-form $\omega$. 

$d\omega = 0$. 

There exists Darboux coordinates such that $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$. 

Darboux coordinates - coordinates in which some tensor associated with dynamics, takes its simplest form, i.e. constant components.
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Darboux coordinates - coordinates in which some tensor associated with dynamics, takes its simplest form, i.e. constant components.
Hamiltonian structures

\[ \{ F, G \} = \int \frac{\delta f}{\delta u^i} \mathcal{H}^{ij} \frac{\delta g}{\delta u^j} dX \]

where

\[ F = \int f(u^i, u^i_X, \ldots) dX \]

Properties:

- skew;
- Jacobi identity.
The operator

\[ \mathcal{H}^{ij} = g^{ij}(u) \frac{d}{dx} + \Gamma_k^{ij}(u) u^k_x \]

defines a Hamiltonian operator if and only if \((g, \Gamma)\) is flat.
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Simplest case (Darboux-form):

$$g^{ij} = \text{constant}$$

Any flat metric can be put into constant coefficient form. So the construction of Darboux coordinates is just the problem of finding flat coordinates for a flat metric.
Two Hamiltonian structures:

\[ \mathcal{H}_1 = \frac{d}{dX}, \]
\[ \mathcal{H}_2 = \frac{d^3}{dX^3} + u \frac{d}{dX} + \frac{1}{2} u X \]

such that \( \mathcal{H}_1 + \lambda \mathcal{H}_2 \) is also Hamiltonian.
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such that \( \mathcal{H}_1 + \lambda \mathcal{H}_2 \) is also Hamiltonian.

Any local Hamiltonian system may be transformed, via suitable change of variable, to a constant, or Darboux, form.
The Miura transformation

Miura map

\[ u = -v_x + \frac{1}{4}v^2 \]

\[ \mathcal{H}_1 \mapsto \mathcal{H}_2 \]
Miura map

\[ u = -v_X + \frac{1}{4}v^2 \]

\[ \mathcal{H}_1 \mapsto \mathcal{H}_2 \]

For the rest of the talk I will turn off the dispersive terms:

\[ u = \frac{1}{4}v^2 \]

\[ \mathcal{H}_1 = \frac{d}{dX} \mapsto \mathcal{H}_2 = u \frac{d}{dX} + \frac{1}{2}u_X \]
The Gauss-Manin equations

To find flat coordinates for a metric $g$ one has to solve the system:

$$g \nabla dv = 0.$$
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Solutions have monodromy:

$$ \Sigma = \{ u | \det (g^{ij}(u)) = 0 \} $$
The Gauss-Manin equations

To find flat coordinates for a metric $g$ one has to solve the system:

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This is over-determined, but integrability conditions are precisely the flatness of the connection.
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$$\Sigma = \{ u | \det (g^{ij}(u)) = 0 \}$$

The continuation of a solution under a closed path $\gamma$ on $M \setminus \Sigma$ yields a transformation

$$\tilde{\nu}^a(u) = A^a_b(\gamma) \nu^b(u) + B^a(\gamma)$$

with $A$ orthogonal with respect to the metric $g(u)$, and these generate a subgroup of $O(N)$, the Monodromy group

$$W(M) = \mu(\pi_1(M \setminus \Sigma))$$
\[ \dot{g'} = \omega \otimes \omega \]

\[ \dot{g'} = u \omega_u \otimes \omega_u \]
\[ v = 2u^{1/2} \]

\[ u = \phi \]

\[ \tilde{g}^i = u \partial_u \otimes \partial_u \]

**Gauss-Manin:**

\[ u \frac{d^2 v}{du^2} + \frac{1}{2} \frac{dv}{du} = 0 \]
$v = 2u^{1/2}$

$\bar{g}^i = \partial_v \otimes \partial_u$

$\bar{g} = u \partial_u \otimes \partial_u$

$u = \frac{1}{4} \frac{v^2}{u}$

Gauss-Manin:

$u \frac{d^2 v}{d u^2} + \frac{1}{2} \frac{d v}{d u} = 0$
\[ v = 2u^{1/2} \]

\[ u = \frac{1}{4} \sqrt{v^2} \in \mathbb{C}^{\mathbb{Z}_2}[v] \]

\[ \tilde{g} = \partial_v \otimes \partial_v \]

\[ \tilde{g}' = u \partial_u \otimes \partial_u \]

Gauss-Manin: \[ u \frac{d^2v}{du^2} + \frac{1}{2} \frac{dv}{du} = 0 \]
Linear case:

\[ \mathcal{H}_{ij} = g^{ij}(u) \frac{d}{dx} + b^{ij}_k u^k_x, \]

with \( g^{ij}(u) = c^{ij}_k u^k \) symmetric and \( b^{ij}_k, c^{ij}_k \) are constants.
Linear case:

$$\mathcal{H}^{ij} = g^{ij}(u) \frac{d}{dx} + b_k^{ij} u_x^k,$$

with $g^{ij}(u) = c_k^{ij} u^k$ symmetric and $b_k^{ij}, c_k^{ij}$ are constants.

**Gelfand-Dorfmann/Balinskii-Novikov**

The operator defines a Hamiltonian operator if and only if

- $c_k^{ij} = b_k^{ij} + b_k^{ji}$;
- $b_k^{ij}$ are the structure constants of an algebra $A$ (now called a Novikov Algebra), that is $e_i \circ e_j = b_k^{ij} e_k$ where $e_1, \ldots, e_n$ are basis vectors, such that

$$(a \circ b) \circ c = (a \circ c) \circ b,$$

$$(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c).$$
Properties

Rewrite in terms of left and right multiplications

$L_{a \cdot b} = R_{b \cdot a} = a \circ b$:

$[R_{a}, R_{b}] = 0$, 
$[L_{a}, L_{b}] = L_{[a, b]}$.

where $[a, b] = a \circ b - b \circ a$.

- $[a, b] = a \circ b - b \circ a$ defines a Lie algebra $g(A)$;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

Example

Let $\cdot$ be a commutative, associative multiplication with derivation $\delta$, then $a \circ b = a \cdot \delta b + c \cdot a \cdot b$ defines a Novikov algebra.

Classification: Bei and Meng (for $\text{dim} \leq 3$).
Rewrite in terms of left and right multiplications
$L_a b = R_b a = a \circ b$ :

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To every solution of the Gauss-Manin equation

$$\nabla dv = 0$$
Finding flat coordinates: Solving the Gauss-Manin Equations

To every solution of the Gauss-Manin equation

$$\nabla dv = 0$$

$$\frac{\partial^2 v}{\partial u^i \partial u^j} - \Gamma^a_{ij}(u) \frac{\partial v}{\partial u^a} = 0$$

we have an associated monodromy group $\mathcal{W}(A)$. 
Finding flat coordinates: Solving the Gauss-Manin Equations

To every solution of the Gauss-Manin equation

\[ \nabla d\nu = 0 \]

\[ \frac{\partial^2 \nu}{\partial u^i \partial u^j} - \Gamma^a_{ij}(u) \frac{\partial \nu}{\partial u^a} = 0 \]

we have an associated monodromy group \( \mathcal{W}(A) \).

Matrix Form:

\[ L^{(k)}\xi + \Lambda^k\xi = 0 \]

where

\[ L^{(k)} = g^{kj} \frac{\partial}{\partial u^j}, \]

\[ \left( \Lambda^k \right)_{rc} = b_r^{kc} \]
Easy to see:

\[ [L^{(i)}, L^{(j)}] = (b^{ij}_k - b^{ji}_k)L^{(k)}, \quad [\Lambda^i, \Lambda^j] = (b^{ij}_k - b^{ji}_k)\Lambda^k. \]

Also

\[ [\Xi^i, \Xi^j] = (b^{ij}_k - b^{ji}_k)\Xi^k. \]

where

\[ (\Xi^i)_{rc} = -(b^i_r + b^r_i). \]
Easy to see:

\[ [L^{(i)}, L^{(j)}] = (b^{ij}_k - b^{ji}_k)L^{(k)} , \quad [\Lambda^i, \Lambda^j] = (b^{ij}_k - b^{ji}_k)\Lambda^k . \]

Also

\[ [\Xi^i, \Xi^j] = (b^{ij}_k - b^{ji}_k)\Xi^k . \]

where

\[ (\Xi^i)_{rc} = - (b^{ir}_c + b^{ri}_c) . \]

Three representations of the same Lie algebra!
Many simplifications occur when the Lie algebra is Abelian.
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**Associative/Commutative case**

**Frobenius’ Theorem:**

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\[ L^{(i)} = \frac{\partial}{\partial w^i} \]

Easy to solve:

\[ \xi = \prod e^{-\Lambda^{(k)} w^k} \xi_0 \]

Solution in wrong coordinates - straightforward to find \( w = w(u) \).
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Solution in wrong coordinates - straightforward to find $w = w(u)$.

**Novikov-Balinskii**

$$u^i = \frac{1}{2} b^i_{ab} v^a v^b$$
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\[ g(A) \text{ is solvable.} \]
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**Burde/Demkimpe**

\( g(A) \) is solvable.

Recall: Let \( g^{(0)} = g \) and \( g^{(k+1)} = [g^{(k)}, g^{(k)}] \).

If \( g^{(N)} = 0 \) for some \( N \) the Lie algebra is solvable.
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If \( g^{(N)} = 0 \) for some \( N \) the Lie algebra is solvable.

Key to solving the Gauss-Manin equations in the general case:

\[ g = g_0 \supset g_1 \supset \ldots \supset g_r = 0 \]

with \( g_{i+1} \) an ideal in \( g_i \) and \( g_i/g_{i+1} \) Abelian.
A Frobenius Theorem for Solvable Vector Fields

A Partial Straightening out of Vector Fields.

\[ L_1 = 2u_1 \frac{\partial}{\partial u_1} + 3u_2 \frac{\partial}{\partial u_2}, \]

\[ L_2 = 3u_2 \frac{\partial}{\partial u_1}, \]

so

\[ [L_1, L_2] = L_2. \]

Take the vector field at the end of the sequence of ideals:

\[ 3u_2 \frac{\partial}{\partial u_1} = \frac{\partial}{\partial w_2}. \]

Introduce a vector field

\[ v = L_1 - \alpha(w_1, w_2) L_2. \]

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Miura transformations from Novikov algebras
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Take the vector field at the end of the sequence of ideals:

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3u^2 \frac{\partial}{\partial u^1} = \frac{\partial}{\partial w^2}.
\]

Introduce a vector field

\[
v = L^{(1)} - \alpha(w^1, w^2)L^{(2)}
\]
Require:

\[ [v, L^{(2)}] = [v, \partial_{w^2}] = 0. \]
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\[
[v, L^{(2)}] = [v, \partial_{w^2}] = 0.
\]

This gives \( \alpha = -w_2 \). Hence from these commuting vector fields one may introduce coordinates so

\[
L^{(1)} = \frac{\partial}{\partial w^1} - w^2 \frac{\partial}{\partial w^2},
\]

\[
L^{(2)} = \frac{\partial}{\partial w^2}
\]

and a simple calculation gives \( u^1 = w^2 e^{3w^1}, u^2 = e^{3w^1} \).
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Miura transformations from Novikov algebras
How to proceed:

Proof by extension:

\[ g_i - 1 = C \Xi \]

work up from the end of the sequence.

Technical part: definition of the \( \alpha \)'s.

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Miura transformations from Novikov algebras
Proof by extension:

\[ g_{i-1} = \mathbb{C} \Xi^i \oplus g^i \]

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Technical part: definition of the \( \alpha \)'s.
Theorem

Let $\mathcal{A}$ be a Novikov algebra with a right-identity and satisfying certain non-degeneracy conditions. The transformation $u = u(v)$ is found by eliminating the $w$-variables from the equations

$$v_i(w) = \left( \prod e^{+\Lambda(r) w^r} \right)_{i1},$$

$$u^i(w) = \left( \prod e^{-\Xi(r) w^r} \right)_{jn}.$$

where $v_i = \eta_{ij} v^j$. 
Example

Take $\mathcal{A} = \mathbb{C}[z]/\langle z^n \rangle$ with basis

$$e^i = z^{i-1}, \quad i = 1, \ldots, n$$

This a commutative, associate algebra $e^i \cdot e^j = e^{i+j-1}$. 
Example

Take $\mathcal{A} = \mathbb{C}[z]/\langle z^n \rangle$ with basis

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This a commutative, associate algebra $e^i \cdot e^j = e^{i+j-1}$. Take the derivation $\partial e^i = (i - 1)e^i$ and form the Novikov algebra

$$e^i \circ e^j = e^i \cdot e^j + e^i \cdot \partial e^j,$$

$$= je^{i+j-1}$$

Solvable Lie algebra $[e^i, e^j] = (j - i)e^{i+j-1}$
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$$= je^{i+j-1}$$

Solvable Lie algebra $[e^i, e^j] = (j - i)e^{i+j-1}$

Flat metric

$$g^{-1} = \sum_{i,j} (i + j)u^{i+j-1} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}.$$
Example

The monodromy group associated to the Gauss-Manin equations is

\[ \mathcal{W}(\mathcal{A}) = \mathbb{Z}_{1+n} [1, 2, \ldots, n] \]

which acts on the \( v^i \) coordinates by the action

\[ v^i \mapsto \varepsilon^{n+1-i} v^i \]

where \( \varepsilon^{n+1} = 1 \).
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The functions \( u^i \) are invariant polynomials of degrees 2, \ldots, \( n+1 \):

\[ u^i(v) \in \mathbb{C}^{\mathcal{W}(A)}[v^1, \ldots, v^n] \]
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N.B.

$$k[V]^{\mathcal{W}(\mathcal{A})} = k[u^1, \ldots, u^N]/J$$
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Cyclic Quotient singularity.
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Miura transformations from Novikov algebras
Example

There exists a dispersive Miura transformation between the third-order Hamiltonian operator

\[ \mathcal{H}_{2}^{ij} = \left\{ \eta_{2} \frac{d^{3}}{dX^{3}} \right\} + \left\{ (\Gamma_{r}^{ij} + \Gamma_{r}^{ji}) u^{r} \frac{d}{dX} + \Gamma_{r}^{ij} u_{X}^{r} \right\} + \left\{ \eta_{ij} \frac{d}{dX} \right\} \].

and the constant operator

\[ \mathcal{H}_{1}^{ij} = \eta_{ij} \frac{d}{dX} \].

Question:

What is it?
There exists a dispersive Miura transformation between the third-order Hamiltonian operator

\[ \mathcal{H}^{ij}_2 = \left\{ \eta^{ij} \frac{d^3}{dX^3} \right\} + \left\{ (\Gamma^{ij}_r + \Gamma^{ji}_r) u^r \frac{d}{dX} + \Gamma^{ij}_r u_X^r \right\} + \left\{ \eta^{ij} \frac{d}{dX} \right\}. \]

and the constant operator

\[ \mathcal{H}^{ij}_1 = \eta^{ij} \frac{d}{dX}. \]

Question:
Dispersive Miura transformations

Example

There exists a dispersive Miura transformation between the third-order Hamiltonian operator

$$\mathcal{H}^{ij}_2 = \left\{ \eta^{ij}_2 \frac{d^3}{dX^3} \right\} + \left\{ (\Gamma^{ij}_r + \Gamma^{ji}_r) u^r \frac{d}{dX} + \Gamma^{ij}_r u^r_X \right\} + \left\{ \eta^{ij} \frac{d}{dX} \right\}.$$

and the constant operator

$$\mathcal{H}^{ij}_1 = \eta^{ij} \frac{d}{dX}.$$ 

Question: What is it?
Apply Miura map to third KdV Hamiltonian structure to get second Hamiltonian structure of mKdV:

\[ \mathcal{H}^{mKdV}_2 = D^3 - DvD^{-1}vD. \]
mKdV and (modified)-Camassa-Holm bi-Hamiltonian structures

Apply Miura map to third KdV Hamiltonian structure to get second Hamiltonian structure of mKdV:

$$\mathcal{H}_2^{mKdV} = D^3 - DvD^{-1}vD.$$  

**Proposition**

In the associative case:

$$\left(\mathcal{H}_1^{mKdV}\right)^{ij} = \eta^{ij}D,$$

$$\left(\mathcal{H}_2^{mKdV}\right)^{ij} = \eta^{ij}D^3 - c_p^{ij}c_m^p Dv^m D^{-1}v^n D.$$  

A-valued KdV and mKdV equations can now easily be constructed.
Here we construct $\mathcal{A}$-valued modified Camassa-Holm equations.

**Example**

One may apply the standard tri-Hamiltonian ‘tricks’ to obtain the $\mathcal{A}$-valued bi-Hamiltonian pair:

$$
\begin{align*}
C_1^{ij} &= \eta^{ij}(D^3 + D), \\
C_2^{ij} &= c^{ij}_p c^{p}_{mn} D^{m} D^{-1} v^n D.
\end{align*}
$$

Multi-component modified Camassa-Holm equation

$$
\begin{align*}
v_T + v_{XXX} &= 1 \frac{1}{2} v_{XXX} \circ v \circ v + v_{XX} \circ v_{XX} \circ v_x \\
&+ 1 \frac{1}{2} v_{XXX} \circ v \circ v + 2 v_{XX} \circ v_x \circ v + \frac{1}{2} v_X \circ v_X \circ v_X \\
&+ \frac{3}{2} v_X \circ v \circ v.
\end{align*}
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$$C_1^{ij} = \eta^{ij} (D^3 + D),$$
$$C_2^{ij} = c_{p}^{ij} c_{mn}^{p} D v^m D^{-1} v^n D .$$

**Multi-component modified Camassa-Holm equation**

$$v_T + v_{XXT} = \frac{1}{2} v_{XXX} \circ v_X \circ v_X + v_{XX} \circ v_{XX} \circ v_X$$
$$+ \frac{1}{2} v_{XXX} \circ v \circ v + 2 v_{XX} \circ v_X \circ v + \frac{1}{2} v_X \circ v_X \circ v_X$$
$$+ \frac{3}{2} v_X \circ v \circ v .$$

**Question:**

Novikov-algebra valued Camassa-Holm equation?
Here we construct $\mathcal{A}$-valued modified Camassa-Holm equations.

**Example**

One may apply the standard tri-Hamiltonian ‘tricks’ to obtain the $\mathcal{A}$-valued bi-Hamiltonian pair:

\[
C_1^{ij} = \eta^{ij}(D^3 + D),
\]

\[
C_2^{ij} = c_{p}^{ij} c_{mn}^{p} D v^m D^{-1} v^n D.
\]

Multi-component modified Camassa-Holm equation

\[
v_T + v_{XXT} = \frac{1}{2} v_{XXX} \circ v_x + v_{XX} \circ v_{XX} \circ v_x + \frac{1}{2} v_{XX} \circ v \circ v + 2 v_{XX} \circ v_x \circ v + \frac{1}{2} v_x \circ v_x \circ v_x + \frac{3}{2} v_x \circ v \circ v.
\]

Question: Novikov-algebra valued Camassa-Holm equation?