Miura transformations from Novikov algebras

Ian Strachan

5^{th} September 2019

Darboux Coordinates: the prototype

Consider a sympletic 2-form ω .

Darboux Coordinates: the prototype

Consider a sympletic 2-form ω .
So $d\omega = 0$.

Darboux Coordinates: the prototype

Consider a sympletic 2-form ω.
So $d\omega = 0$.
There exists Darboux coordinates such that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \,.$$

Darboux Coordinates: the prototype

Consider a sympletic 2-form ω .
So $d\omega = 0$.
There exists Darboux coordinates such that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \, .$$

Darboux coordinates - coordinates in which some tensor associated with dynamics, takes its simplest form, i.e. constant components.

Hamiltonian structures

$$\{F, G\} = \int \frac{\delta f}{\delta u^i} \mathcal{H}^{ij} \frac{\delta g}{\delta u^j} dX$$

where

$$F = \int f(u^i, u^i_X, \ldots) dX$$

Properties:

- skew;
- Jacobi identity.

First-order Hamiltonian Structures

Dubrovin/Novikov

The operator

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + \Gamma^{ij}_{k}(u)u^{k}_{x}$$

defines a Hamiltonian operator if and only if (g, Γ) is flat.

First-order Hamiltonian Structures

Dubrovin/Novikov

The operator

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + \Gamma^{ij}_k(u)u^k_x$$

defines a Hamiltonian operator if and only if (g, Γ) is flat.

Simplest case (Darboux-form):

$$g^{ij} = \text{constant}$$

First-order Hamiltonian Structures

Dubrovin/Novikov

The operator

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + \Gamma^{ij}_k(u)u^k_x$$

defines a Hamiltonian operator if and only if (g, Γ) is flat.

Simplest case (Darboux-form):

$$g^{ij} = \text{constant}$$

Any flat metric can be put into constant coefficient form.

First-order Hamiltonian Structures

Dubrovin/Novikov

The operator

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + \Gamma^{ij}_k(u)u^k_x$$

defines a Hamiltonian operator if and only if (g, Γ) is flat.

Simplest case (Darboux-form):

$$g^{ij} = \text{constant}$$

Any flat metric can be put into constant coefficient form.
So the construction of Darboux coordinates is just the problem of finding flat coordinates for a flat metric.

Bi-Hamiltonian systems

Two Hamiltonian structures:

$$\begin{aligned} \mathcal{H}_1 &= \frac{d}{dX}, \\ \mathcal{H}_2 &= \frac{d^3}{dX^3} + u\frac{d}{dX} + \frac{1}{2}u_X \end{aligned}$$

such that $\mathcal{H}_1 + \lambda\mathcal{H}_2$ is also Hamiltonian.

Bi-Hamiltonian systems

Two Hamiltonian structures:

$$\mathcal{H}_1 = \frac{d}{dX},$$
$$\mathcal{H}_2 = \frac{d^3}{dX^3} + u\frac{d}{dX} + \frac{1}{2}u_X$$

such that $\mathcal{H}_1 + \lambda\mathcal{H}_2$ is also Hamiltonian.

Any local Hamiltonian system may be transformed, via suitable change of variable, to a constant, or Darboux, form.

The Miura transformation

Miura map

$$u = -v_X + \frac{1}{4}v^2$$

$$\mathcal{H}_1 \mapsto \mathcal{H}_2$$

The Miura transformation

Miura map

$$u = -v_X + \frac{1}{4}v^2$$

$$\mathcal{H}_1 \mapsto \mathcal{H}_2$$

For the rest of the talk I will turn off the dispersive terms:

$$u = \frac{1}{4}v^2$$

$$\mathcal{H}_1 = \frac{d}{dX} \mapsto \mathcal{H}_2 = u\frac{d}{dX} + \frac{1}{2}u_X$$

The Gauss-Manin equations

To find flat coordinates for a metric g one has to solve the system:

$$ {}^{g}\nabla d\mathbf{v} = 0 \,. $$

The Gauss-Manin equations

To find flat coordinates for a metric g one has to solve the system:

$$ {}^{g}\nabla d\mathbf{v} = 0 \,. $$

This is over-determined, but integrability conditions are precisely the flatness of the connection.

The Gauss-Manin equations

To find flat coordinates for a metric g one has to solve the system:

$$ {}^{g}\nabla d\mathbf{v} = 0\,. $$

This is over-determined, but integrability conditions are precisely the flatness of the connection.
Solutions have monodromy:

$$ \Sigma = \{\mathbf{u}\,|\, \det\left(g^{ij}(\mathbf{u})\right) = 0\} $$

The Gauss-Manin equations

To find flat coordinates for a metric g one has to solve the system:

$$^{g}\nabla d\mathbf{v} = 0\,.$$

This is over-determined, but integrability conditions are precisely the flatness of the connection.
Solutions have monodromy:

$$\Sigma = \{\mathbf{u}\,|\,\det\left(g^{ij}(\mathbf{u})\right) = 0\}$$

The continuation of a solution under a closed path γ on $M\backslash\Sigma$ yields a transformation

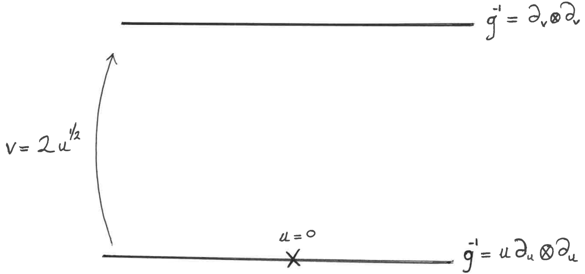
$$\tilde{v}^a(\mathbf{u}) = A^a_b(\gamma)v^b(\mathbf{u}) + B^a(\gamma)$$

with A orthogonal with respect to the metric $g(\mathbf{u})$, and these generate a subgroup of $O(N)$, the Monodromy group

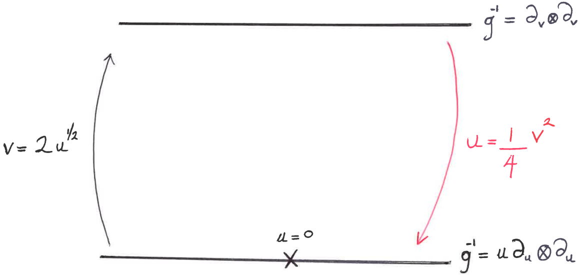
$$W(M) = \mu\left(\pi_1(M\backslash\Sigma)\right)$$

$$g^{-1} = \partial_v \otimes \partial_v$$

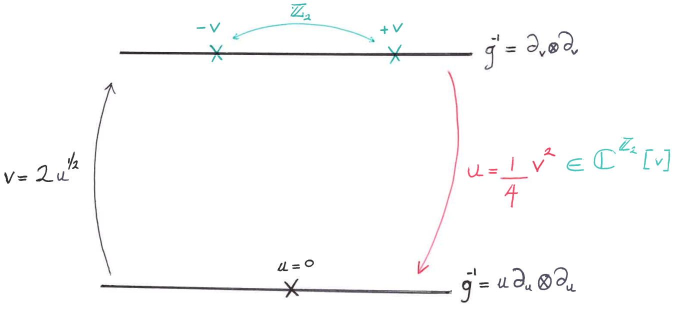
$$g^{-1} = u\,\partial_u \otimes \partial_u$$



Gauss–Manin: $u \dfrac{d^2 v}{du^2} + \dfrac{1}{2} \dfrac{dv}{du} = 0$



Gauss–Manin: $u \dfrac{d^2 v}{du^2} + \dfrac{1}{2} \dfrac{dv}{du} = 0$



Gauss-Manin: $u \dfrac{d^2 v}{du^2} + \dfrac{1}{2} \dfrac{dv}{du} = 0$

Linear case:

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k u^k_x \,,$$

with $g^{ij}(u) = c^{ij}_k u^k$ symmetric and b^{ij}_k, c^{ij}_k are constants.

Linear case:

$$\mathcal{H}^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k u^k_x\,,$$

with $g^{ij}(u) = c^{ij}_k u^k$ symmetric and b^{ij}_k, c^{ij}_k are constants.

Gelfand-Dorfmann/Balinskii-Novikov

The operator defines a Hamiltonian operator if and only if

- $c^{ij}_k = b^{ij}_k + b^{ji}_k$;
- b^{ij}_k are the structure constants of an algebra A (now called a Novikov Algebra), that is $e^i \circ e^j = b^{ij}_k\, e^k$ where $e^1, \ldots, e^n$ are basis vectors, such that

$$\begin{aligned}(a\circ b)\circ c &= (a\circ c)\circ b\,,\\ (a\circ b)\circ c - a\circ(b\circ c) &= (b\circ a)\circ c - b\circ(a\circ c)\,.\end{aligned}$$

Properties

Properties

Rewrite in terms of left and right multiplications
$L_a b = R_b a = a \circ b$:

$$[R_a, R_b] = 0\,, \qquad [L_a, L_b] = L_{[a,b]}\,.$$

where $[a, b] = a \circ b - b \circ a$.

- $[a, b] = a \circ b - b \circ a$ defines a Lie algebra $\mathfrak{g}(A)$;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

Properties

Rewrite in terms of left and right multiplications
$L_a b = R_b a = a \circ b$:

$$[R_a, R_b] = 0\,, \qquad [L_a, L_b] = L_{[a,b]}\,.$$

where $[a, b] = a \circ b - b \circ a$.

- $[a, b] = a \circ b - b \circ a$ defines a Lie algebra $\mathfrak{g}(A)$;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

Example

Let $\cdot$ be a commutative, associative multiplication with derivation δ , then

$$a \circ b = a \cdot \delta b$$

Properties

Rewrite in terms of left and right multiplications
$L_a b = R_b a = a \circ b$:

$$[R_a, R_b] = 0\,, \qquad [L_a, L_b] = L_{[a,b]}\,.$$

where $[a, b] = a \circ b - b \circ a$.

- $[a, b] = a \circ b - b \circ a$ defines a Lie algebra $\mathfrak{g}(A)$;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

Example

Let $\cdot$ be a commutative, associative multiplication with derivation δ , then

$$a \circ b = a \cdot \delta b + c \cdot a \cdot b$$

defines a Novikov algebra.

Properties

Rewrite in terms of left and right multiplications
$L_a b = R_b a = a \circ b$:

$$[R_a, R_b] = 0\,, \qquad [L_a, L_b] = L_{[a,b]}\,.$$

where $[a, b] = a \circ b - b \circ a$.

- $[a, b] = a \circ b - b \circ a$ defines a Lie algebra $\mathfrak{g}(A)$;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

Example

Let $\cdot$ be a commutative, associative multiplication with derivation δ , then

$$a \circ b = a \cdot \delta b + c \cdot a \cdot b$$

defines a Novikov algebra.

Classification: Bei and Meng (for $dim \leq 3$).

Finding flat coordinates: Solving the Gauss-Manin Equations

To every solution of the Gauss-Manin equation

$$\nabla dv = 0$$

Finding flat coordinates: Solving the Gauss-Manin Equations

To every solution of the Gauss-Manin equation

$$\nabla dv = 0$$

$$\frac{\partial^2 v}{\partial u^i \partial u^j} - \Gamma^a_{ij}(\mathbf{u}) \frac{\partial v}{\partial u^a} = 0$$

we have an associated monodromy group $\mathcal{W}(\mathcal{A})$.

Finding flat coordinates: Solving the Gauss-Manin Equations

To every solution of the Gauss-Manin equation

$$\nabla dv = 0$$

$$\frac{\partial^2 v}{\partial u^i \partial u^j} - \Gamma^a_{ij}(\mathbf{u}) \frac{\partial v}{\partial u^a} = 0$$

we have an associated monodromy group $\mathcal{W}(\mathcal{A})$.

Matrix Form:

$$L^{(k)}\xi + \Lambda^k \xi = 0$$

where

$$\begin{aligned} L^{(k)} &= g^{kj} \frac{\partial}{\partial u^j}, \\ \left(\Lambda^k\right)_{rc} &= b^{kc}_r \end{aligned}$$

Easy to see:

$$[L^{(i)}, L^{(j)}] = (b^{ij}_k - b^{ji}_k) L^{(k)}\,, \qquad [\Lambda^i, \Lambda^j] = (b^{ij}_k - b^{ji}_k)\Lambda^k\,.$$

Also

$$[\Xi^i, \Xi^j] = (b^{ij}_k - b^{ji}_k)\Xi^k\,.$$

where

$$\left(\Xi^i\right)_{rc} = -(b^{ir}_c + b^{ri}_c)\,.$$

Easy to see:

$$[L^{(i)}, L^{(j)}] = (b^{ij}_k - b^{ji}_k) L^{(k)}\,, \qquad [\Lambda^i, \Lambda^j] = (b^{ij}_k - b^{ji}_k)\Lambda^k\,.$$

Also

$$[\Xi^i, \Xi^j] = (b^{ij}_k - b^{ji}_k)\Xi^k\,.$$

where

$$\left(\Xi^i\right)_{rc} = -(b^{ir}_c + b^{ri}_c)\,.$$

Three representations of the same Lie algebra!

The Abelian Case

Many simplifications occur when the Lie algebra is Abelian.

The Abelian Case

Many simplifications occur when the Lie algebra is Abelian.

Associative/Commutative case

Frobenius' Theorem:

$$L^{(i)} = \frac{\partial}{\partial w^i}$$

The Abelian Case

Many simplifications occur when the Lie algebra is Abelian.

Associative/Commutative case

Frobenius' Theorem:

$$L^{(i)} = \frac{\partial}{\partial w^i}$$

Easy to solve:

$$\xi = \prod e^{-\Lambda^{(k)} w^k} \, \xi_0$$

Solution in wrong coordinates - straightforward to find $\mathbf{w} = \mathbf{w}(\mathbf{u})$.

The Abelian Case

Many simplifications occur when the Lie algebra is Abelian.

Associative/Commutative case

Frobenius' Theorem:

$$L^{(i)} = \frac{\partial}{\partial w^i}$$

Easy to solve:

$$\xi = \prod e^{-\Lambda^{(k)} w^k} \xi_0$$

Solution in wrong coordinates - straightforward to find $\mathbf{w} = \mathbf{w}(\mathbf{u})$.

Novikov-Balinskii

$$u^i = \frac{1}{2} b^i_{ab} v^a v^b$$

General Case

By a result of Zelmanov:

Burde/Demkimpe

$$\mathfrak{g}(A) \text{ is solvable.}$$

General Case

By a result of Zelmanov:

Burde/Demkimpe

$\mathfrak{g}(A)$ is solvable.

Recall: Let $g^{(0)} = g$ and $g^{(k+1)} = \left[g^{(k)}, g^{(k)}\right]$
If $g^{(N)} = 0$ for some N the Lie algebra is solvable.

General Case

By a result of Zelmanov:

Burde/Demkimpe

$$\mathfrak{g}(A) \text{ is solvable.}$$

Recall: Let $g^{(0)} = g$ and $g^{(k+1)} = \left[g^{(k)}, g^{(k)}\right]$
If $g^{(N)} = 0$ for some N the Lie algebra is solvable.
Key to solving the Gauss-Manin equations in the general case:

$$g = g_0 \supset g_1 \supset \ldots \supset g_r = 0$$

with g_{i+1} an ideal in g_i and g_i / g_{i+1} Abelian.

A Frobenius Theorem for Solvable Vector Fields

A Partial Straightening out of Vector Fields.

A Partial Straightening out of Vector Fields.

$$
\begin{aligned}
L^{(1)} &= 2u^1 \frac{\partial}{\partial u^1} + 3u^2 \frac{\partial}{\partial u^2}\,, \\
L^{(2)} &= 3u^2 \frac{\partial}{\partial u^1}
\end{aligned}
$$

so $\left[L^{(1)}, L^{(2)}\right] = L^{(2)}\,.$

A Partial Straightening out of Vector Fields.

$$
\begin{aligned}
L^{(1)} &= 2u^1 \frac{\partial}{\partial u^1} + 3u^2 \frac{\partial}{\partial u^2}\,, \\
L^{(2)} &= 3u^2 \frac{\partial}{\partial u^1}
\end{aligned}
$$

so $\left[L^{(1)}, L^{(2)}\right] = L^{(2)}$.
Take the vector field at the *end* of the sequence of ideals:

$$
3u^2 \frac{\partial}{\partial u^1}
$$

A Partial Straightening out of Vector Fields.

$$\begin{aligned} L^{(1)} &= 2u^1 \frac{\partial}{\partial u^1} + 3u^2 \frac{\partial}{\partial u^2}\,, \\ L^{(2)} &= 3u^2 \frac{\partial}{\partial u^1} \end{aligned}$$

so $\left[L^{(1)}, L^{(2)}\right] = L^{(2)}$.
Take the vector field at the *end* of the sequence of ideals:

$$3u^2 \frac{\partial}{\partial u^1} = \frac{\partial}{\partial w^2}\,.$$

A Partial Straightening out of Vector Fields.

$$
\begin{aligned}
L^{(1)} &= 2u^1 \frac{\partial}{\partial u^1} + 3u^2 \frac{\partial}{\partial u^2}, \\
L^{(2)} &= 3u^2 \frac{\partial}{\partial u^1}
\end{aligned}
$$

so $\left[L^{(1)}, L^{(2)}\right] = L^{(2)}$.
Take the vector field at the *end* of the sequence of ideals:

$$
3u^2 \frac{\partial}{\partial u^1} = \frac{\partial}{\partial w^2}.
$$

Introduce a vector field

$$
\mathbf{v} = L^{(1)} - \alpha(w^1, w^2) L^{(2)}
$$

Require:

$$\left[\mathbf{v}, L^{(2)}\right] = [\mathbf{v}, \partial_{w^2}] = 0\,.$$

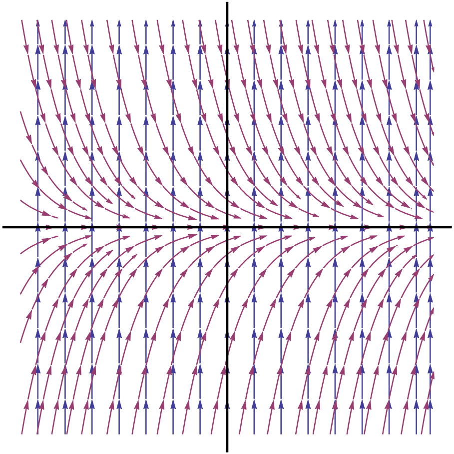
Require:

$$\left[\mathbf{v}, L^{(2)}\right] = [\mathbf{v}, \partial_{w^2}] = 0\,.$$

This gives $\alpha = -w_2$. Hence from these commuting vector fields one may introduce coordinates so

$$\begin{aligned} L^{(1)} &= \frac{\partial}{\partial w^1} - w^2 \frac{\partial}{\partial w^2}\,, \\ L^{(2)} &= \frac{\partial}{\partial w^2} \end{aligned}$$

and a simple calculation gives $u^1 = w^2 e^{3w^1}$, $u^2 = e^{3w^1}$.



How to proceed:

How to proceed:

Proof by extension:

$$g_{i-1} = \mathbb{C}\Xi^{i} \oplus g^{i}$$

work up from the end of the sequence.

How to proceed:

Proof by extension:

$$g_{i-1} = \mathbb{C}\Xi^i \oplus g^i$$

work up from the end of the sequence.

Technical part: definition of the α's.

Main Theorem

Theorem

Let $\mathcal{A}$ be a Novikov algebra with a right-identity and satisfying certain non-degeneracy conditions. The transformation $\mathbf{u} = \mathbf{u}(\mathbf{v})$ is found by eliminating the **w**-variables from the equations

$$\begin{aligned} v_i(\mathbf{w}) &= \left(\overrightarrow{\prod} e^{+\Lambda^{(r)} w^r} \right)_{i1} , \\ u^i(\mathbf{w}) &= \left(\overleftarrow{\prod} e^{-\Xi^{(r)} w^r} \right)_{jn} . \end{aligned}$$

where $v_i = \eta_{ij} v^j$.

Example

Take $\mathcal{A} = \mathbb{C}[z]/\langle z^n \rangle$ with basis

$$e^i = z^{i-1}\,, \quad i = 1\,, \ldots\,, n$$

This a commutative, associate algebra $e^i \cdot e^j = e^{i+j-1}\,.$

Example

Take $\mathcal{A} = \mathbb{C}[z]/\langle z^n \rangle$ with basis

$$e^i = z^{i-1}\,, \quad i = 1\,, \ldots\,, n$$

This a commutative, associate algebra $e^i \cdot e^j = e^{i+j-1}$ .Take the derivation $\partial e^i = (i-1)e^i$ and form the Novikov algebra

$$\begin{aligned} e^i \circ e^j &= e^i \cdot e^j + e^i \cdot \partial e^j\,, \\ &= j e^{i+j-1} \end{aligned}$$

Solvable Lie algebra $[e^i, e^j] = (j-i)e^{i+j-1}$

Monodromy and Orbit Spaces

Example

Take $\mathcal{A} = \mathbb{C}[z]/\langle z^n \rangle$ with basis

$$e^i = z^{i-1}\,, \quad i = 1\,, \ldots\,, n$$

This a commutative, associate algebra $e^i \cdot e^j = e^{i+j-1}$ .Take the derivation $\partial e^i = (i-1)e^i$ and form the Novikov algebra

$$\begin{aligned} e^i \circ e^j &= e^i \cdot e^j + e^i \cdot \partial e^j\,, \\ &= j e^{i+j-1} \end{aligned}$$

Solvable Lie algebra $[e^i, e^j] = (j-i)e^{i+j-1}$
Flat metric

$$g^{-1} = \sum_{i,j} (i+j) u^{i+j-1} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}\,.$$

Example

The monodromy group associated to the Gauss-Manin equations is

$$\mathcal{W}(\mathcal{A}) = \mathbb{Z}_{1+n}[1, 2, \ldots, n]$$

which acts on the v^i coordinates by the action

$$v^i \mapsto \varepsilon^{n+1-i} v^i$$

where $\varepsilon^{n+1} = 1$.

Example

The monodromy group associated to the Gauss-Manin equations is

$$\mathcal{W}(\mathcal{A}) = \mathbb{Z}_{1+n}[1, 2, \ldots, n]$$

which acts on the v^i coordinates by the action

$$v^i \mapsto \varepsilon^{n+1-i} v^i$$

where $\varepsilon^{n+1} = 1$.
The functions u^i are invariant polynomials of degrees $2, \ldots, n+1$:

$$u^i(\mathbf{v}) \in \mathbb{C}^{\mathcal{W}(\mathcal{A})}[v^1, \ldots, v^n]$$

Example

The monodromy group associated to the Gauss-Manin equations is

$$\mathcal{W}(\mathcal{A}) = \mathbb{Z}_{1+n}[1, 2, \ldots, n]$$

which acts on the v^i coordinates by the action

$$v^i \mapsto \varepsilon^{n+1-i} v^i$$

where $\varepsilon^{n+1} = 1$.
The functions u^i are invariant polynomials of degrees $2, \ldots, n+1$:

$$u^i(\mathbf{v}) \in \mathbb{C}^{\mathcal{W}(\mathcal{A})}[v^1, \ldots, v^n]$$

N.B.

$$k[V]^{\mathcal{W}(\mathcal{A})} = k[u^1, \ldots, u^N]/J$$

Example

The monodromy group associated to the Gauss-Manin equations is

$$\mathcal{W}(\mathcal{A}) = \mathbb{Z}_{1+n}[1, 2, \ldots, n]$$

which acts on the v^i coordinates by the action

$$v^i \mapsto \varepsilon^{n+1-i} v^i$$

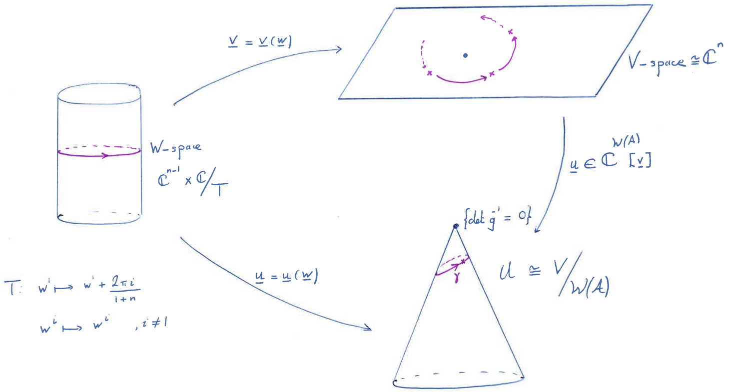
where $\varepsilon^{n+1} = 1$.
The functions u^i are invariant polynomials of degrees $2, \ldots, n+1$:

$$u^i(\mathbf{v}) \in \mathbb{C}^{\mathcal{W}(\mathcal{A})}[v^1, \ldots, v^n]$$

N.B.

$$k[V]^{\mathcal{W}(\mathcal{A})} = k[u^1, \ldots, u^N]/J$$

Cyclic Quotient singularity.


V = V(W)
V-space ≅ ℂ^n
W-space
ℂ^{n-1} × ℂ/T
u ∈ ℂ^{W(A)}[v]
{det g' = 0}
u = u(W)
U ≅ V/W(A)
T: w^1 ↦ w^1 + 2πi/(1+n)
w^i ↦ w^i , i ≠ 1

Dispersive Miura transformations

Example

There exists a dispersive Miura transformation between the third-order Hamiltonian operator

$$\mathcal{H}_2^{ij} = \left\{ \eta_2^{ij} \frac{d^3}{dX^3} \right\} + \left\{ (\Gamma_r^{ij} + \Gamma_r^{ji}) u^r \frac{d}{dX} + \Gamma_r^{ij} u_X^r \right\} + \left\{ \eta^{ij} \frac{d}{dX} \right\} .$$

and the constant operator

$$\mathcal{H}_1^{ij} = \eta^{ij} \frac{d}{dX} .$$

Dispersive Miura transformations

Example

There exists a dispersive Miura transformation between the third-order Hamiltonian operator

$$\mathcal{H}_2^{ij} = \left\{ \eta_2^{ij} \frac{d^3}{dX^3} \right\} + \left\{ (\Gamma_r^{ij} + \Gamma_r^{ji}) u^r \frac{d}{dX} + \Gamma_r^{ij} u_X^r \right\} + \left\{ \eta^{ij} \frac{d}{dX} \right\} .$$

and the constant operator

$$\mathcal{H}_1^{ij} = \eta^{ij} \frac{d}{dX} .$$

Question:

Dispersive Miura transformations

Example

There exists a dispersive Miura transformation between the third-order Hamiltonian operator

$$\mathcal{H}_2^{ij} = \left\{ \eta_2^{ij} \frac{d^3}{dX^3} \right\} + \left\{ (\Gamma_r^{ij} + \Gamma_r^{ji}) u^r \frac{d}{dX} + \Gamma_r^{ij} u_X^r \right\} + \left\{ \eta^{ij} \frac{d}{dX} \right\}.$$

and the constant operator

$$\mathcal{H}_1^{ij} = \eta^{ij} \frac{d}{dX}.$$

Question: What is it?

mKdV and (modified)-Camassa-Holm bi-Hamiltonian structures

Apply Miura map to third KdV Hamiltonian structure to get second Hamiltonian structure of mKdV:

$$\mathcal{H}_2^{mKdV} = D^3 - DvD^{-1}vD\,.$$

mKdV and (modified)-Camassa-Holm bi-Hamiltonian structures

Apply Miura map to third KdV Hamiltonian structure to get second Hamiltonian structure of mKdV:

$$\mathcal{H}_2^{mKdV} = D^3 - DvD^{-1}vD\,.$$

Proposition

In the *associative* case:

$$\begin{aligned} \left(\mathcal{H}_1^{mKdV}\right)^{ij} &= \eta^{ij} D\,, \\ \left(\mathcal{H}_2^{mKdV}\right)^{ij} &= \eta^{ij} D^3 - c_p^{ij} c_{mn}^p D v^m D^{-1} v^n D\,. \end{aligned}$$

$\mathcal{A}$-valued KdV and mKdV equations can now easily be constructed.

Here we construct $\mathcal{A}$-valued modified Camassa-Holm equations.

Example

One may apply the standard tri-Hamiltonian 'tricks' to obtain the $\mathcal{A}$-valued bi-Hamiltonian pair:

$$\begin{aligned} \mathcal{C}_1^{ij} &= \eta^{ij}(D^3 + D)\,, \\ \mathcal{C}_2^{ij} &= c_p^{ij} c_{mn}^p D v^m D^{-1} v^n D\,. \end{aligned}$$

Multi-component modified Camassa-Holm equation

$$\begin{aligned} v_T + v_{XXT} = & \frac{1}{2} v_{XXX} \circ v_X \circ v_X + v_{XX} \circ v_{XX} \circ v_x \\ & + \frac{1}{2} v_{XXX} \circ v \circ v + 2 v_{XX} \circ v_X \circ v + \frac{1}{2} v_X \circ v_X \circ v_X \\ & + \frac{3}{2} v_X \circ v \circ v\,. \end{aligned}$$

Here we construct $\mathcal{A}$-valued modified Camassa-Holm equations.

Example

One may apply the standard tri-Hamiltonian 'tricks' to obtain the $\mathcal{A}$-valued bi-Hamiltonian pair:

$$\begin{aligned} \mathcal{C}_1^{ij} &= \eta^{ij}(D^3 + D)\,, \\ \mathcal{C}_2^{ij} &= c_p^{ij} c_{mn}^p D v^m D^{-1} v^n D\,. \end{aligned}$$

Multi-component modified Camassa-Holm equation

$$\begin{aligned} v_T + v_{XXT} = & \frac{1}{2} v_{XXX} \circ v_X \circ v_X + v_{XX} \circ v_{XX} \circ v_x \\ & + \frac{1}{2} v_{XXX} \circ v \circ v + 2 v_{XX} \circ v_X \circ v + \frac{1}{2} v_X \circ v_X \circ v_X \\ & + \frac{3}{2} v_X \circ v \circ v\,. \end{aligned}$$

Question:

Here we construct $\mathcal{A}$-valued modified Camassa-Holm equations.

Example

One may apply the standard tri-Hamiltonian 'tricks' to obtain the $\mathcal{A}$-valued bi-Hamiltonian pair:

$$\begin{aligned} \mathcal{C}_1^{ij} &= \eta^{ij}(D^3 + D)\,, \\ \mathcal{C}_2^{ij} &= c^{ij}_p c^p_{mn} D v^m D^{-1} v^n D\,. \end{aligned}$$

Multi-component modified Camassa-Holm equation

$$\begin{aligned} v_T + v_{XXT} = &\ \frac{1}{2} v_{XXX} \circ v_X \circ v_X + v_{XX} \circ v_{XX} \circ v_x \\ &+ \frac{1}{2} v_{XXX} \circ v \circ v + 2 v_{XX} \circ v_X \circ v + \frac{1}{2} v_X \circ v_X \circ v_X \\ &+ \frac{3}{2} v_X \circ v \circ v\,. \end{aligned}$$

Question: Novikov-algebra valued Camassa-Holm equation?