Universality in Random Tiling Models

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Joint work with Mark Adler and Kurt Johansson:

Mark Adler, Kurt Johansson & P. van Moerbeke, *Tilings of non-convex Polygons, skew-Young Tableaux and determinantal Processes*, Comm Math Phys, **364**, 287-342 (2018)(arXiv: 1609.06995).

Mark Adler, Kurt Johansson & P. van Moerbeke, *Lozenge tilings of hexagons with cuts and asymptotic fluctuations: a new universality class*, Math Phys Anal Geom **21** 1-53. (2018)(arXiv:1706.01055).

Mark Adler & P. van Moerbeke, *Probability distributions related to tilings of non-convex Polygons*, Journal of Mathematical Physics, **59**, 1-21. Special volume in memory of Ludvig Faddeev. (2018)

Mark Adler, Kurt Johansson & Pierre van Moerbeke: *Asymptotics for domino tilings of Aztec rectangles and the discrete tacnode kernel.* (2019)

Domino and Lozenge tilings

Domino or lozenge tilings of large geometric shapes: they are a rich source of new statistical phenomena near critical points.

They have sufficient complexity to have interesting features and yet are simple enough to be tractable!

Models have been studied having the three phases:

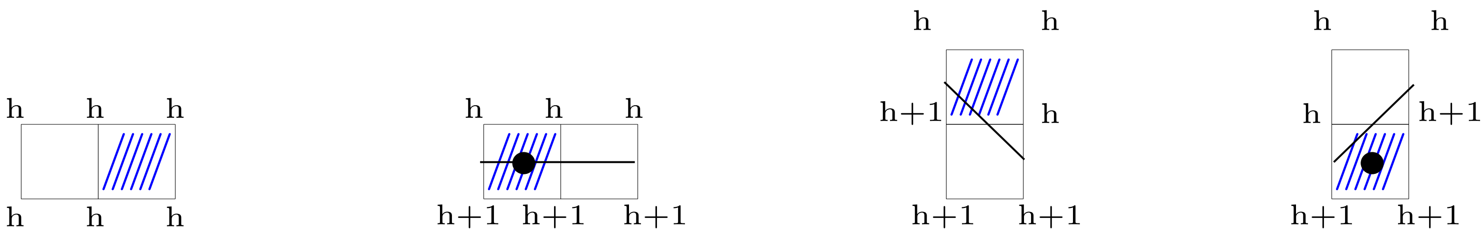
- Solid phase

- liquid phase: correlations are decaying **polynomially** with distance.

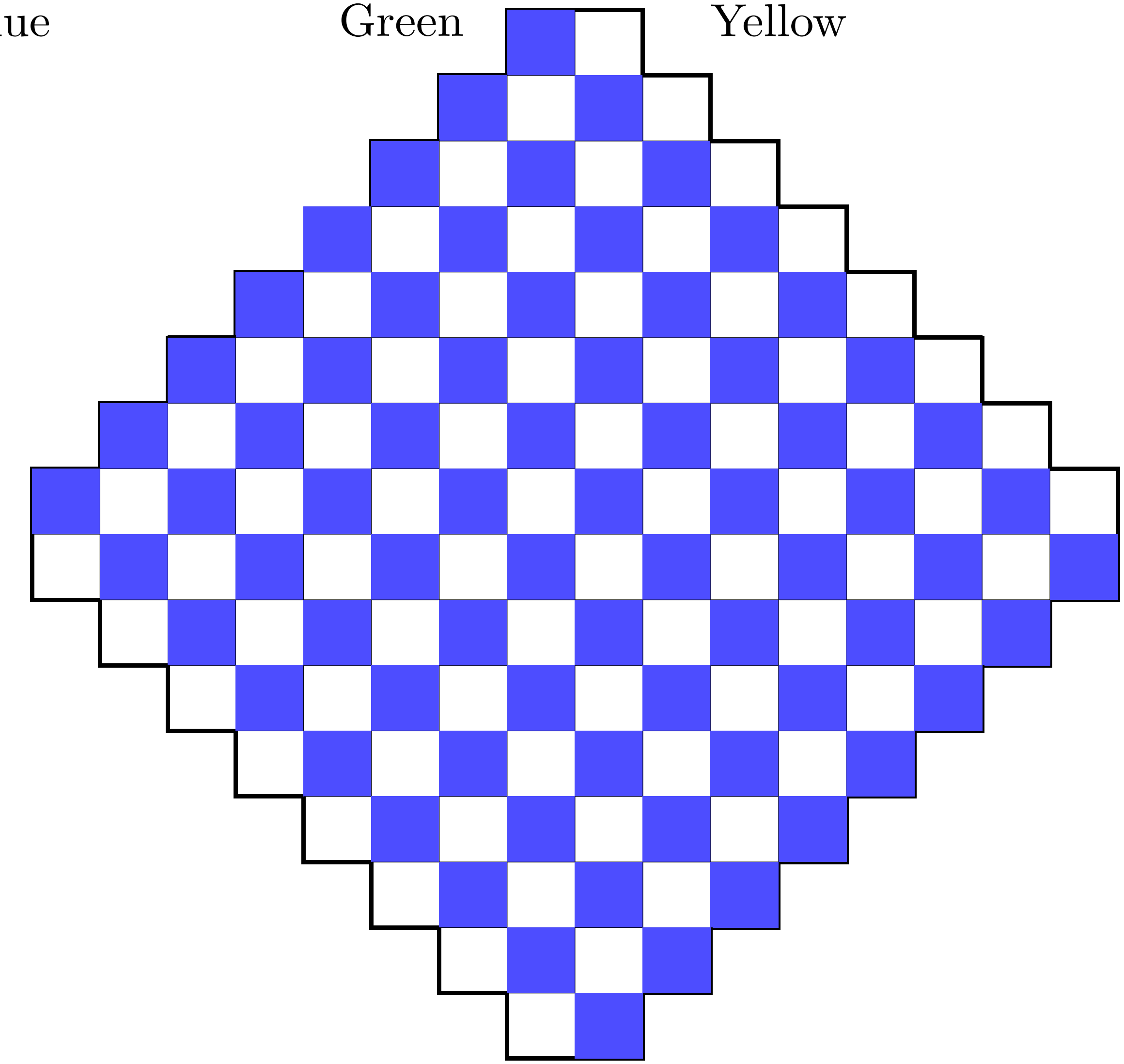
- Gas phase: correlations are decaying **exponentially** with distance.

In this talk, we discuss models with **solid and liquid phases** only.

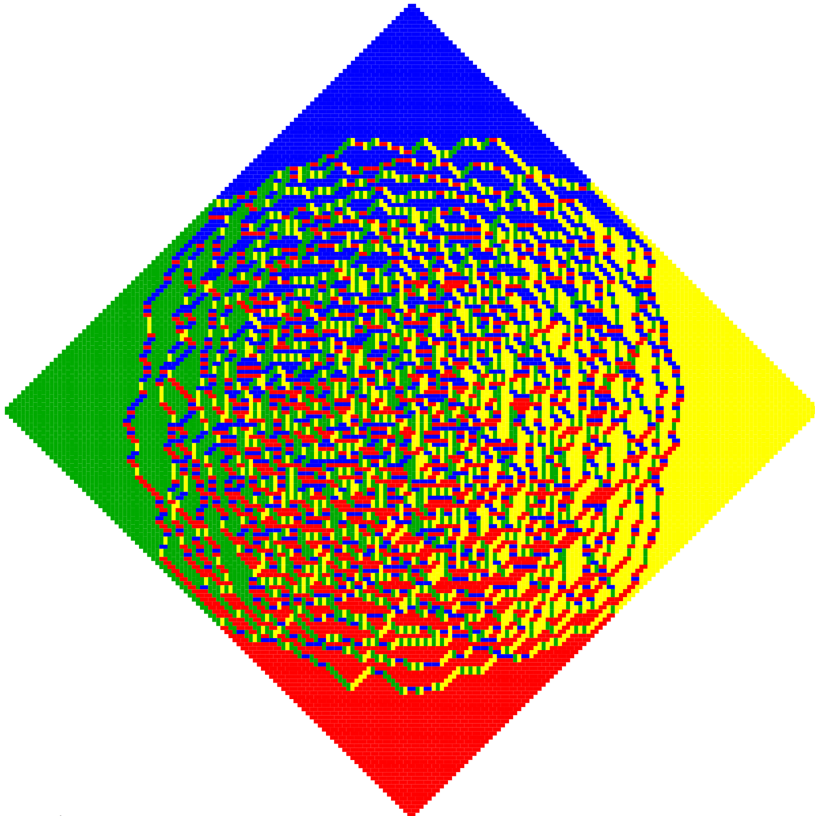
1. Domino tilings of Aztec diamonds of size n



Red Blue Green Yellow

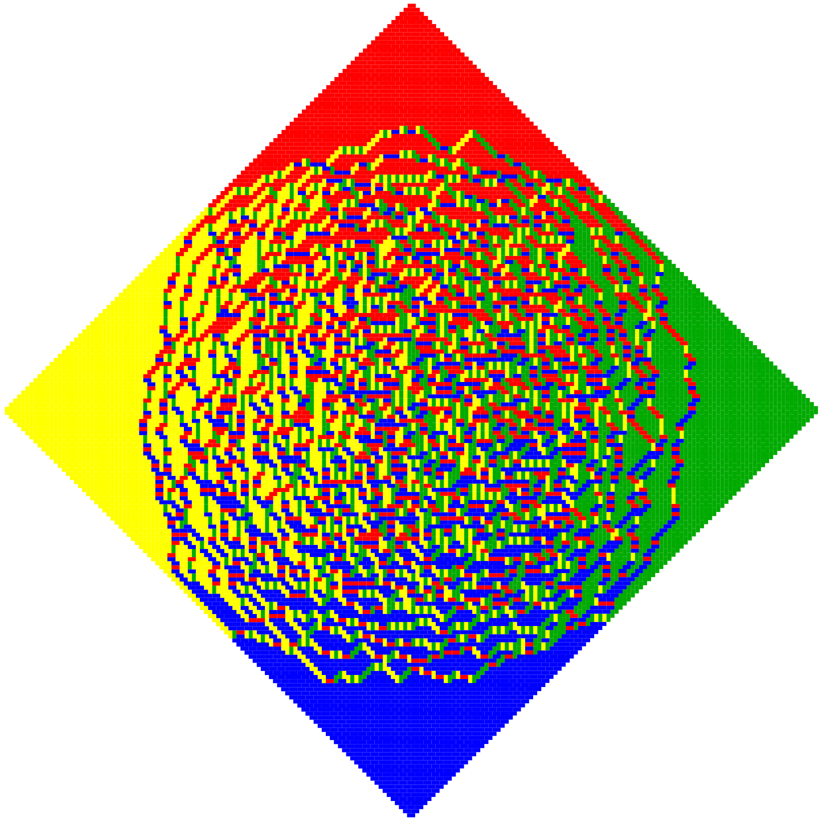


For $n \to \infty$?

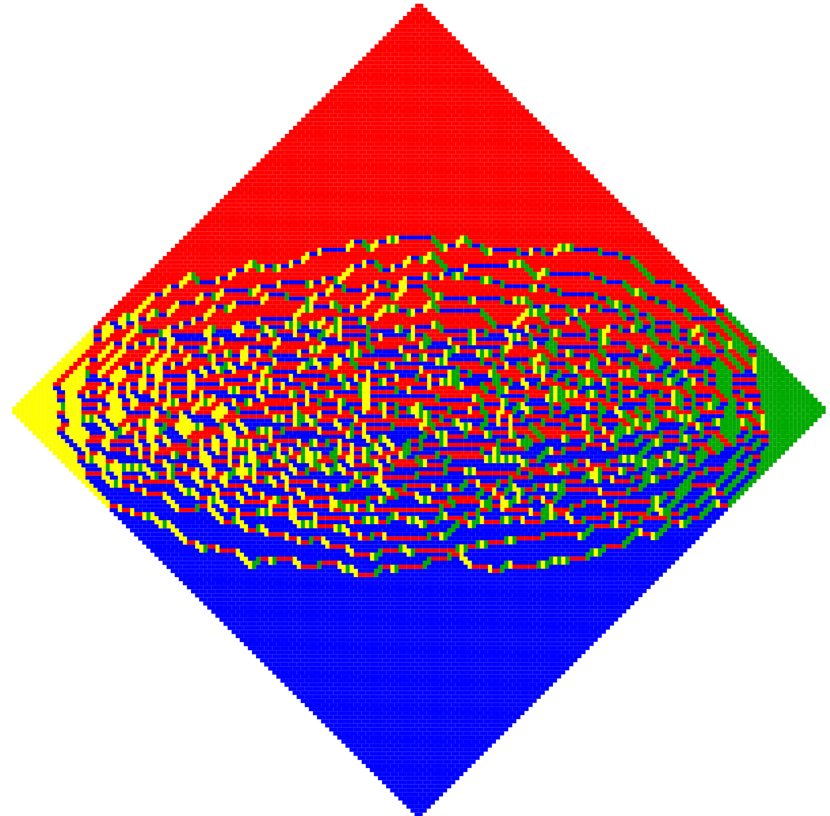


$a = 1$
Arctic circle (n=100)

- Airy Process (stationary process version of the Tracy-Widom distrib.)
- GUE-minor process (interlacing spectra of the minors of GUE-matrix)
- Gaussian Free Field (courtesy Sunil Chhita)



$a = 1$
Arctic circle (n=100)



$a = 1/2$
Arctic ellipse (n=100)

- Airy Process (stationary process version of the Tracy-Widom distrib.)

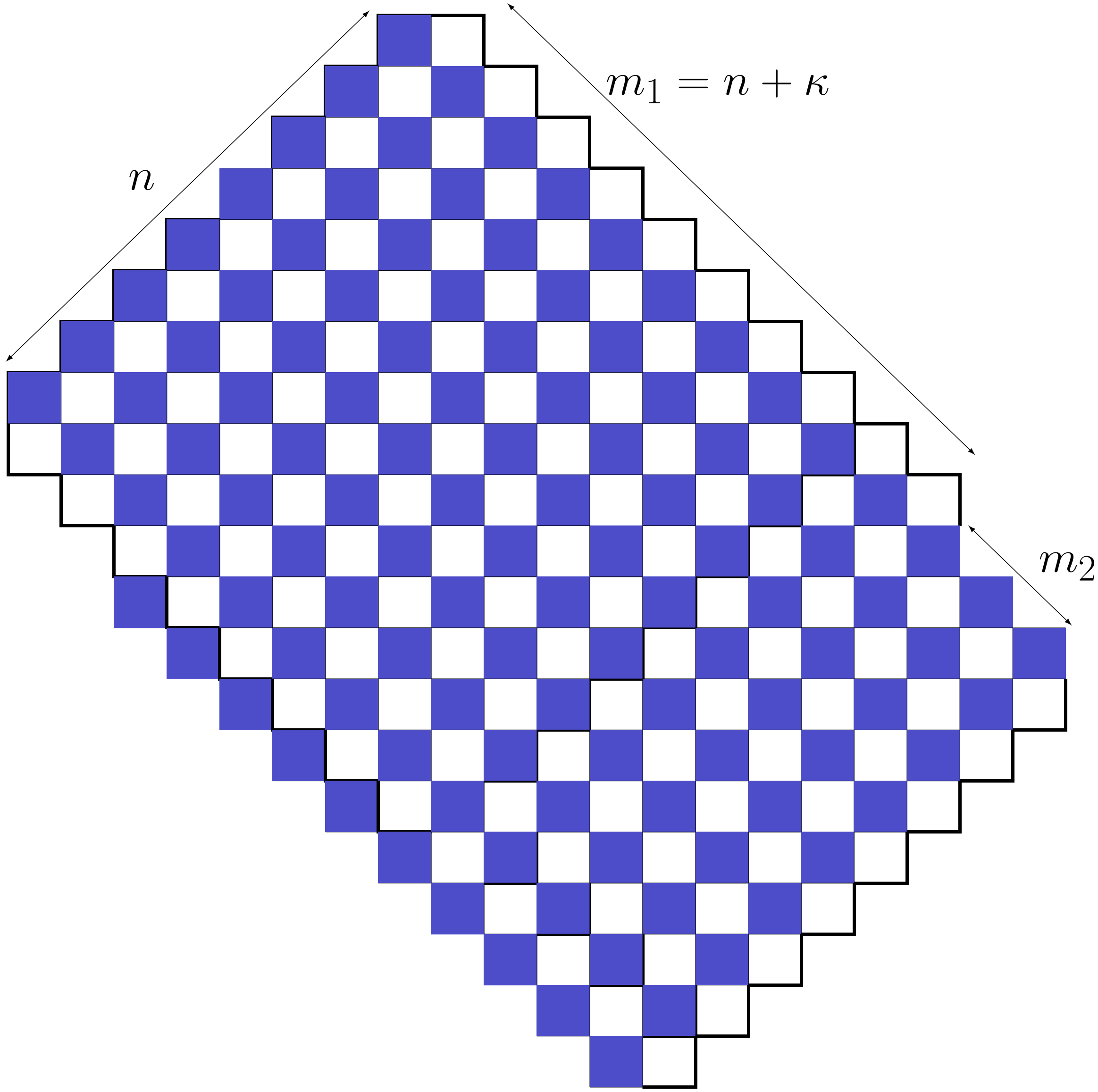
- GUE-minor process (interlacing spectra of the minors of GUE-matrix)

- Gaussian Free Field

(courtesy Sunil Chhita)

Other geometries:

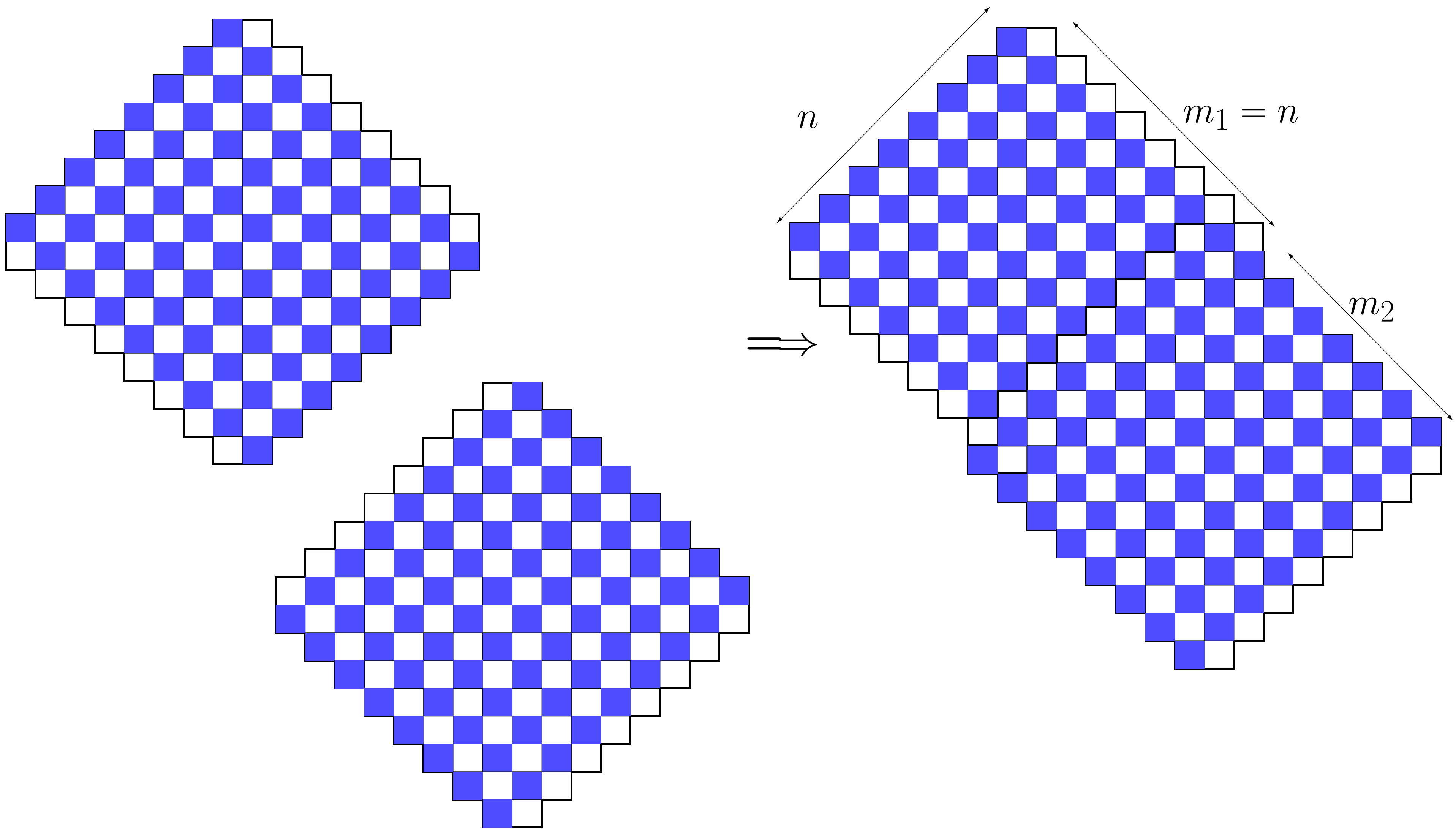
Rectangular Aztec diamond : Covering with domino's

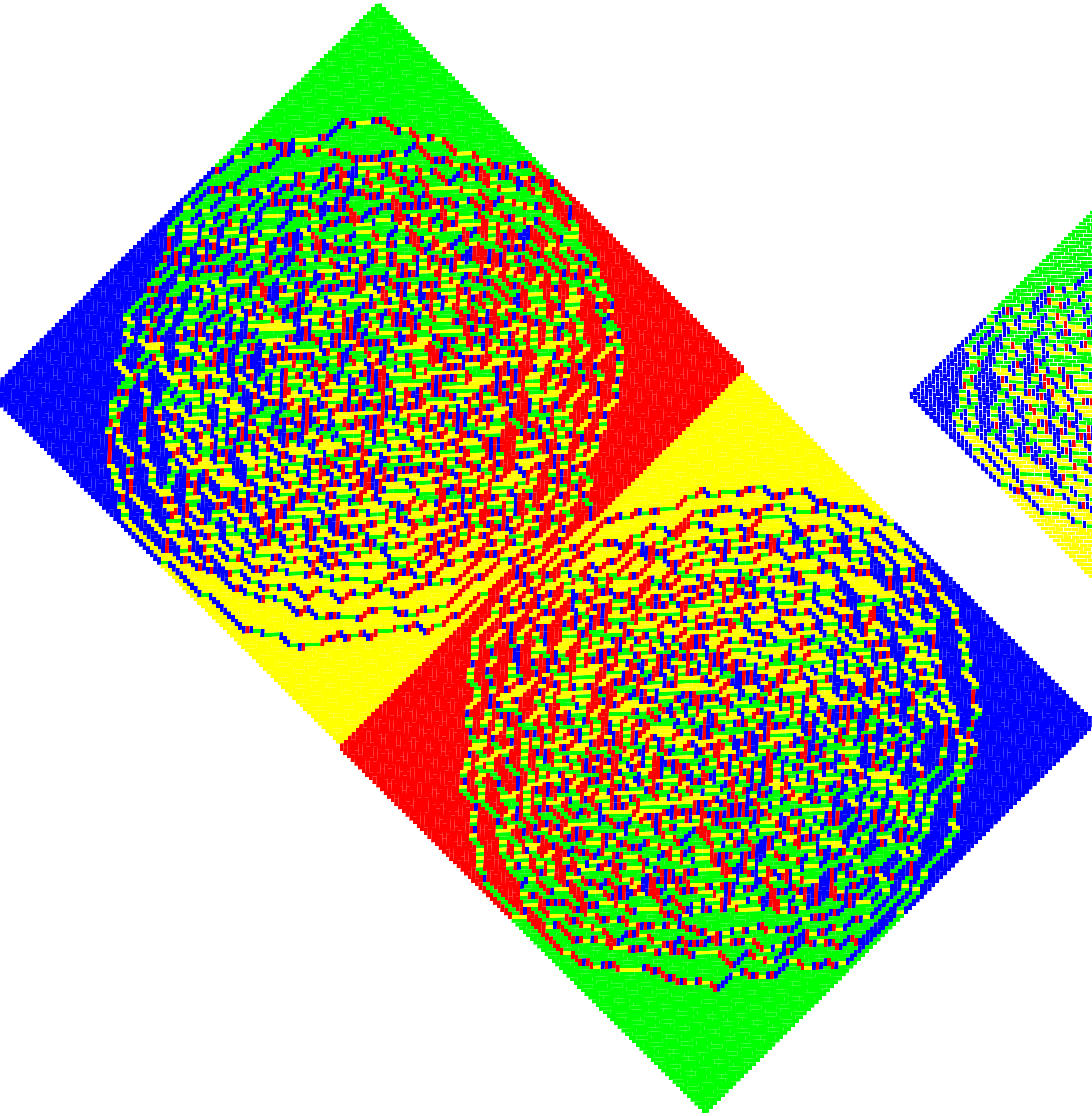


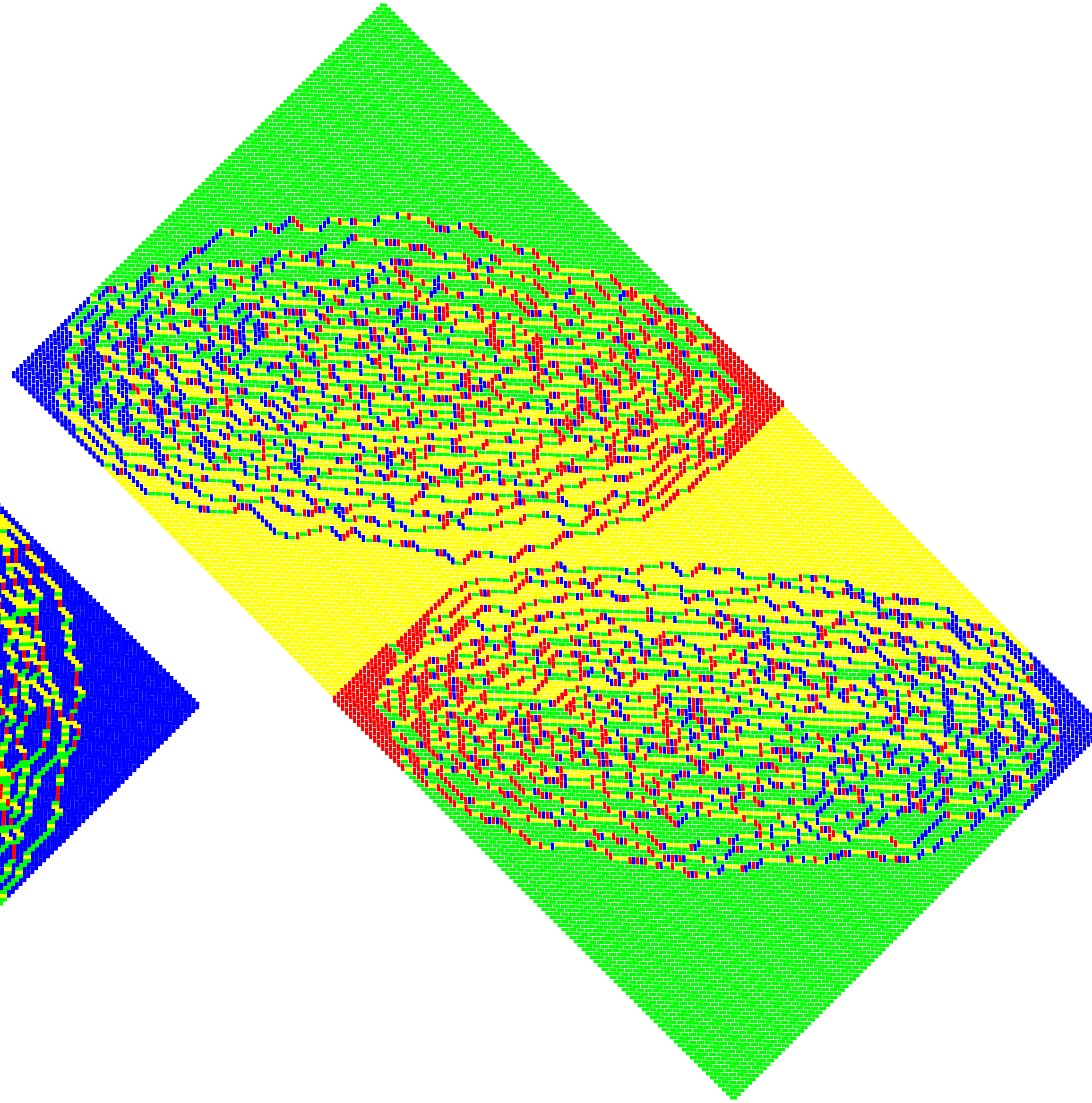
$$n = 8,\ m_1 = 10,\ m_2 = 3,\ \kappa = 2.$$

Imagine n, m_1, m_2 very large, keeping κ finite!

Example: two overlapping Aztec diamonds:
(creating such a geometry)





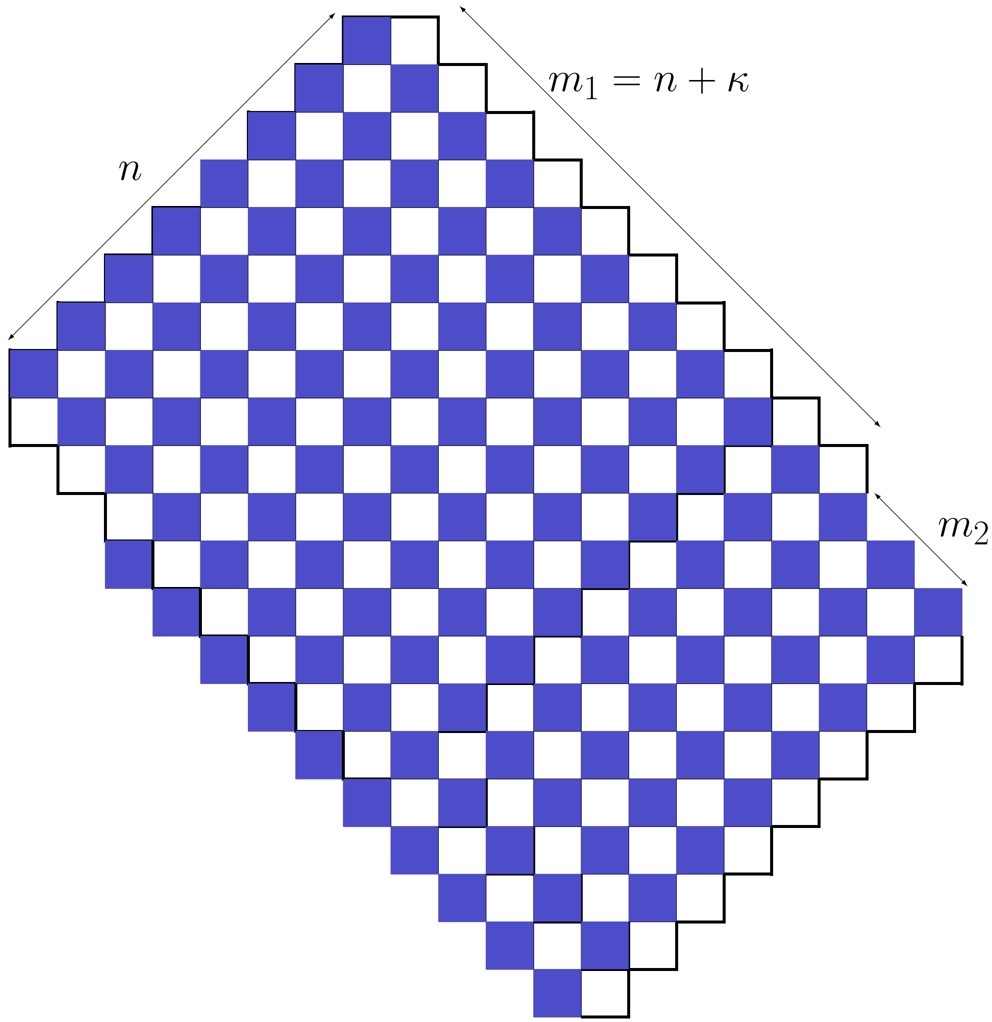


⇑ ↗

discrete tacnode process (Small overlap)

tacnode process (Big overlap)

(*Pearcey process*) (*even bigger overlap*)



Proposition: an Aztec rectangle is tilable if

$$m_1 \geq 0, \quad m_2 \leq n + 1$$

There are two cases: either $m_1 \geq n$ or $m_1 < n$. **In this talk assume $m_1 \geq n$!**

- Put a Weight a:

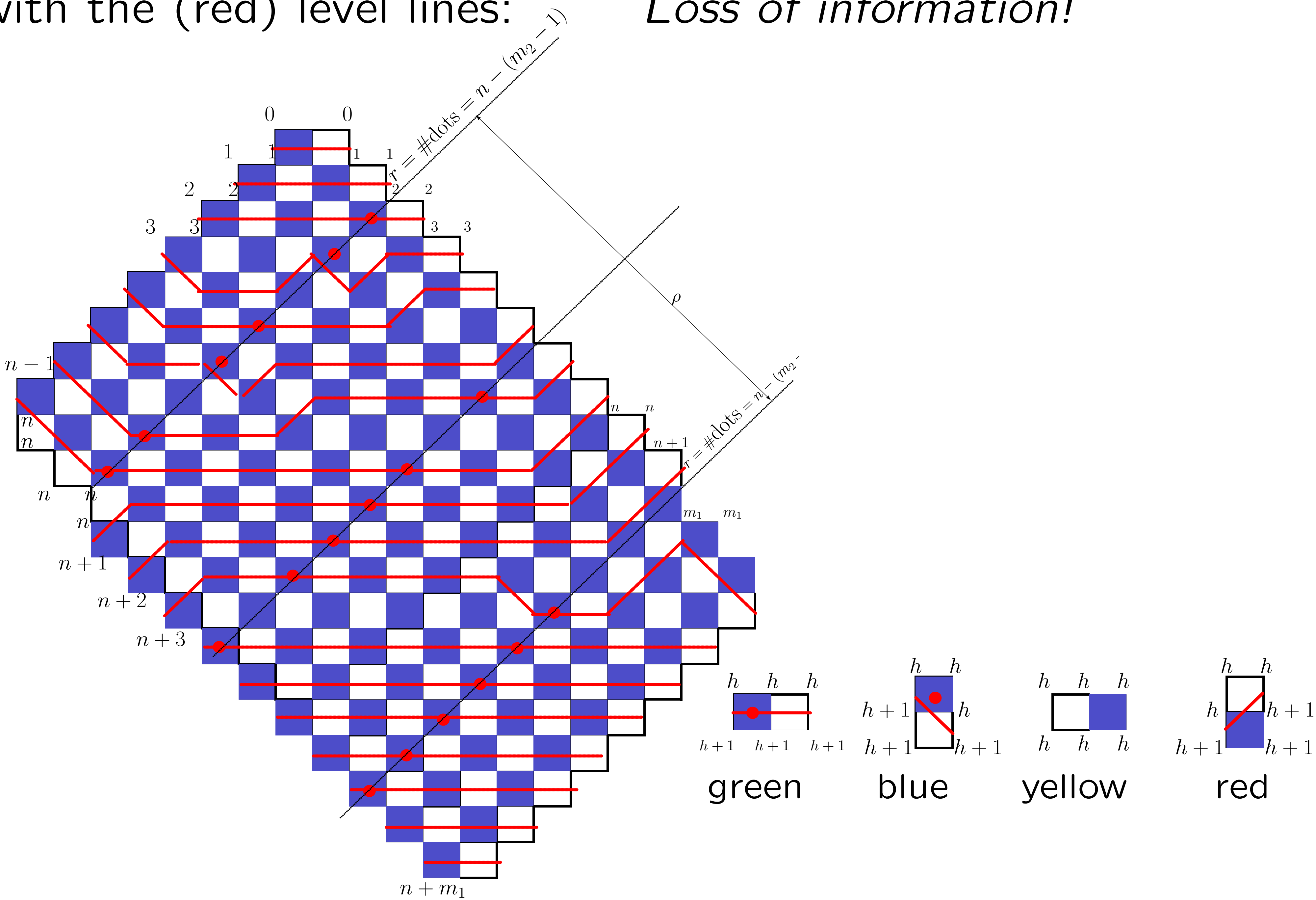
– vertical domino's have weight $0 < a \leq 1$
– horizontal domino's have weight 1

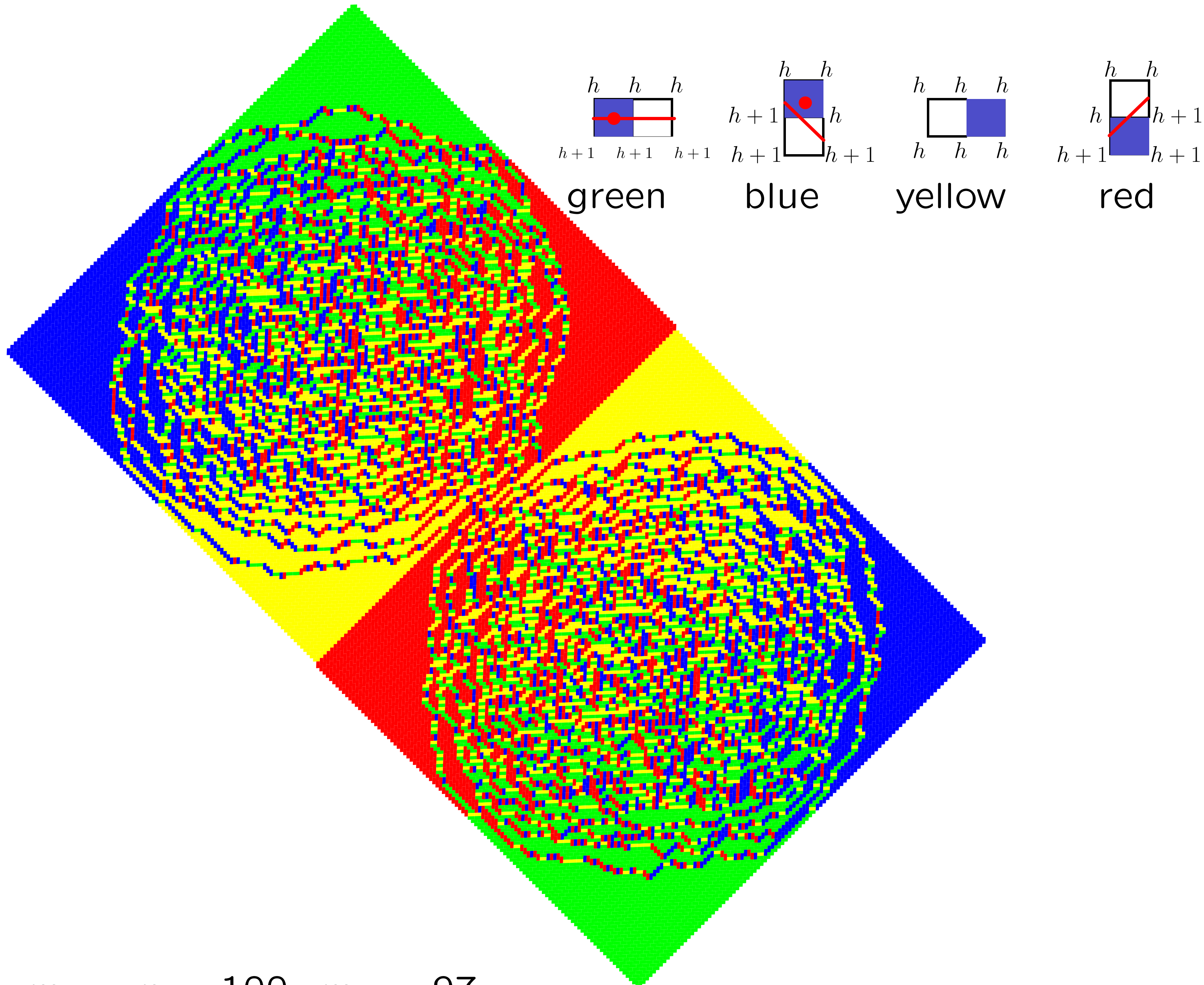
- Probability on tilings:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}}$$

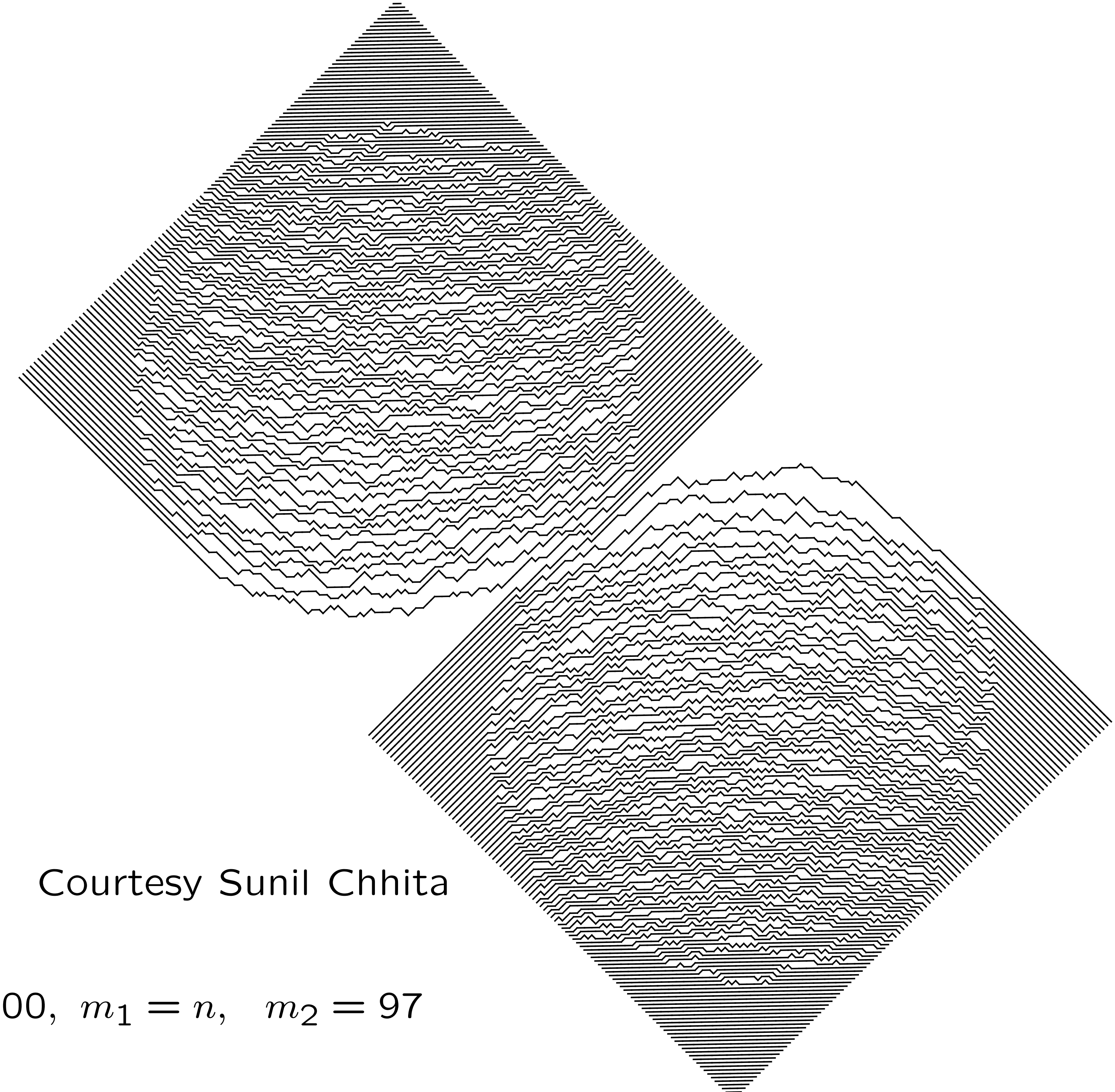
• **Step 1**. Random surface and its level lines, corresponding to levels 1/2, 3/2, ..., $n + 1/2$, etc...$\Leftrightarrow$ *Tiling*.

• The **point process** of the intersections of the oblique lines with the (red) level lines: *Loss of information!*



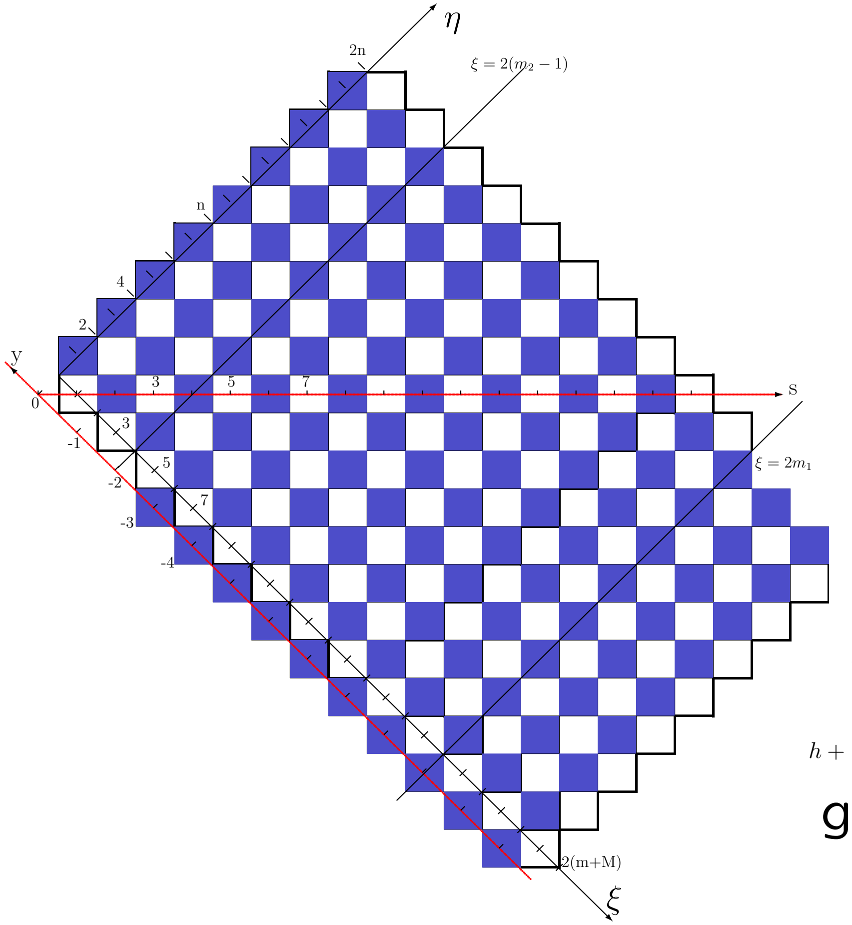


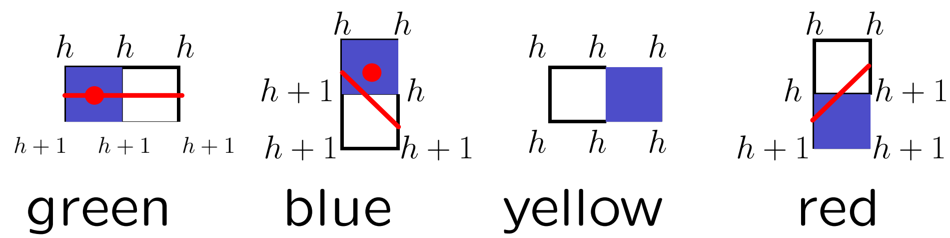
$n = 100,\ m_1 = n = 100,\ m_2 = 97,$



Courtesy Sunil Chhita

$n = 100,\ m_1 = n,\ \ m_2 = 97$





- GOAL: find the correlation kernel $\mathbb{K}^{red}$ of the (red) point process :

$$\mathbb{K}^{red}(\xi_1, \eta_1; \xi_2, \eta_2), \quad \xi_i \in 2\mathbb{Z},\ \eta_i \in 2\mathbb{Z}+1$$

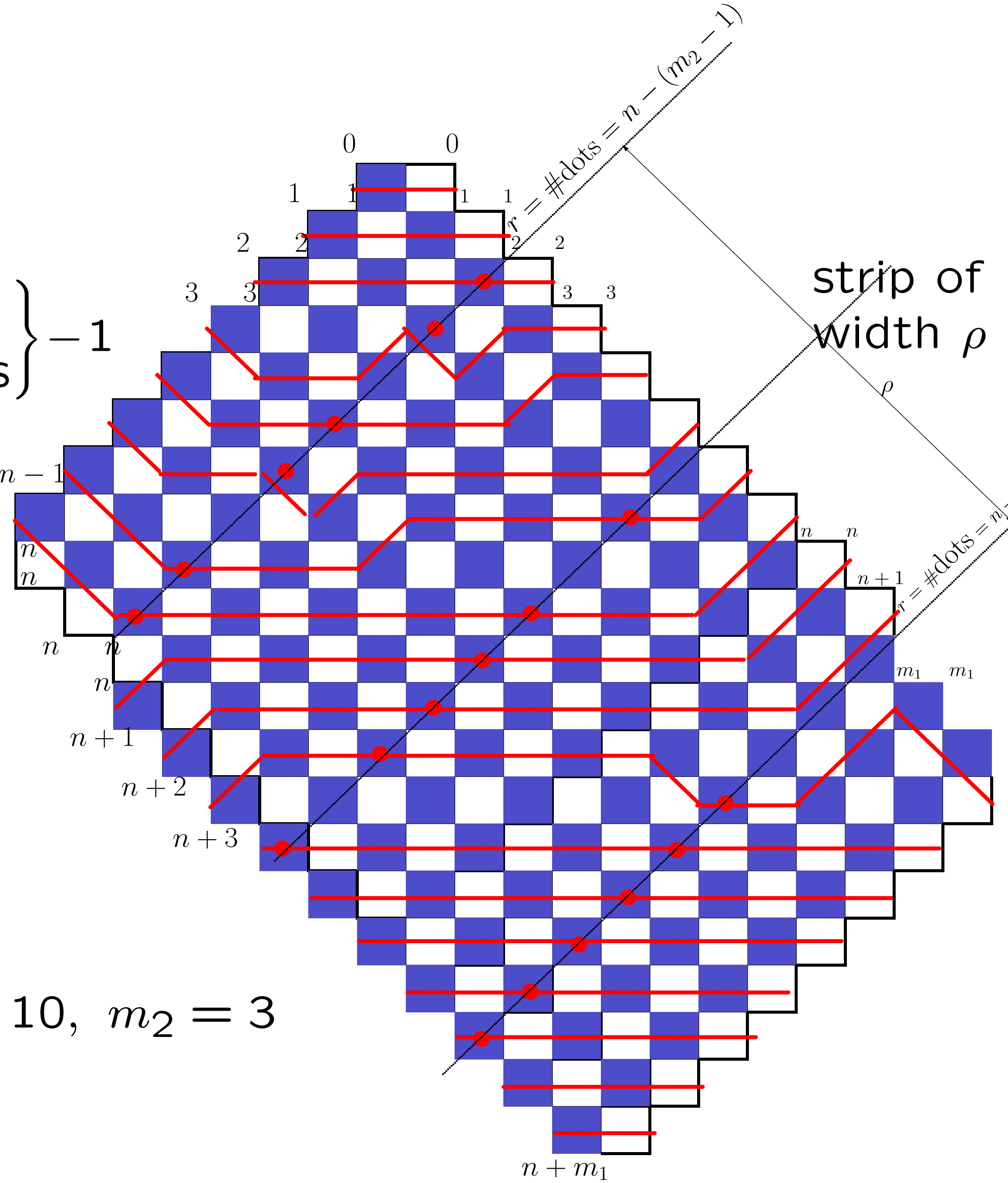
The correlation kernel enables one to compute the following probabilities:

$$\mathbb{P}\left(\begin{array}{l} \text{a green or blue tile at } \ (\xi_1, \eta_1) \\ \text{\& a green or blue tile at } \ (\xi_2, \eta_2) \end{array}\right)$$

$$= \det\begin{pmatrix} \mathbb{K}^{red}(\xi_1, \eta_1; \xi_1, \eta_1) & \mathbb{K}^{red}(\xi_1, \eta_1; \xi_2, \eta_2) \\ \mathbb{K}^{red}(\xi_2, \eta_2; \xi_1, \eta_1) & \mathbb{K}^{red}(\xi_2, \eta_2; \xi_2, \eta_2) \end{pmatrix}$$

Random surface and its level lines: two important numbers ρ and $\mathfrak{r}$.

$$\rho = \#\begin{Bmatrix}\text{oblique lines with} \\ \text{minimal \# of red dots}\end{Bmatrix} - 1$$
$$= m_1 - m_2 + 1$$
$$= \text{width of strip} = 8$$



In this example $n = 8,\ m_1 = 10,\ m_2 = 3$

Random surface and its level lines: two important numbers ρ and $\mathfrak{r}$.

$$\rho = \#\left\{\begin{matrix}\text{oblique lines with} \\ \text{minimal } \# \text{ of red dots}\end{matrix}\right\} - 1$$

$$= m_1 - m_2 + 1$$

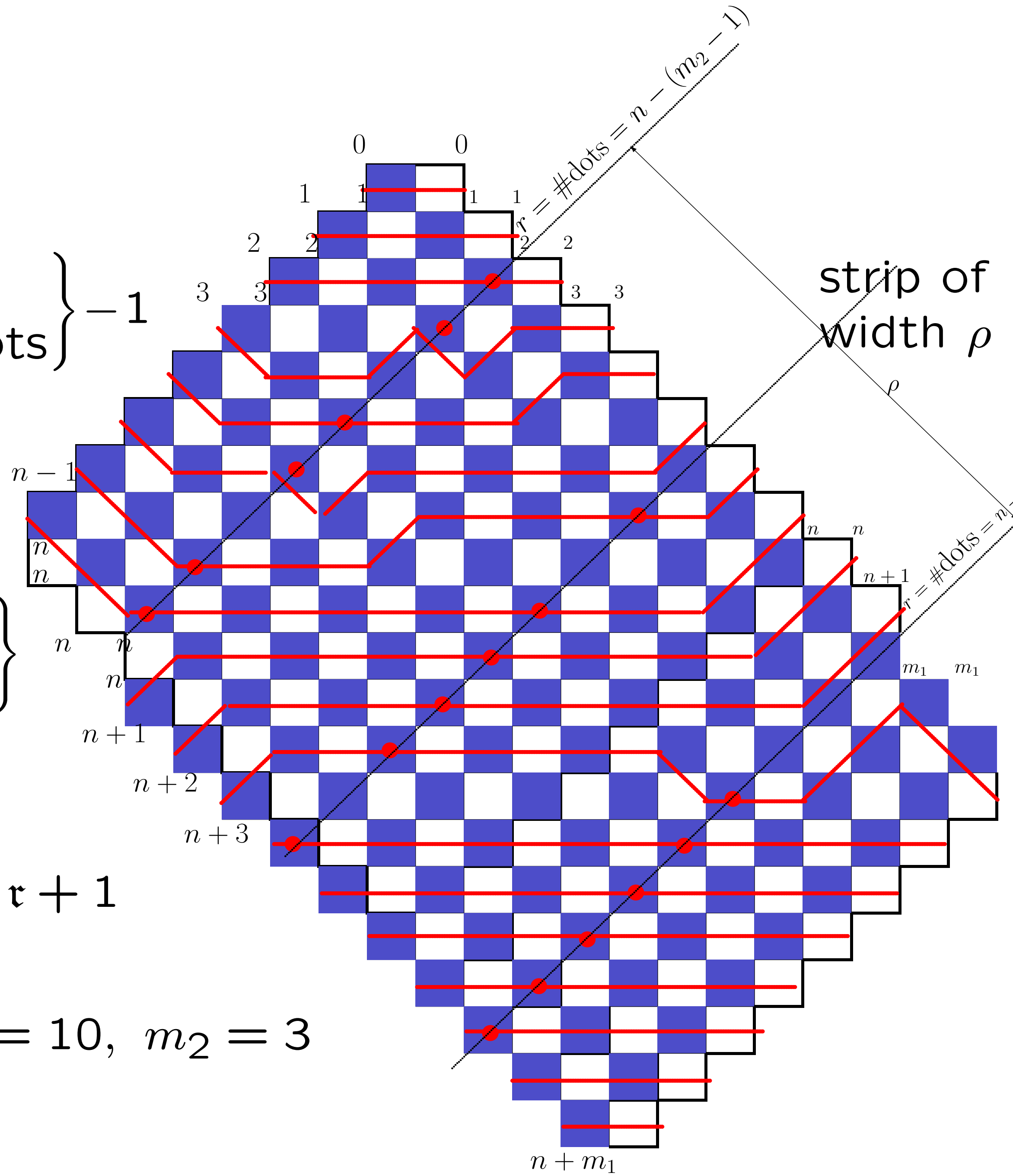
$$= \text{width of strip} = 8$$

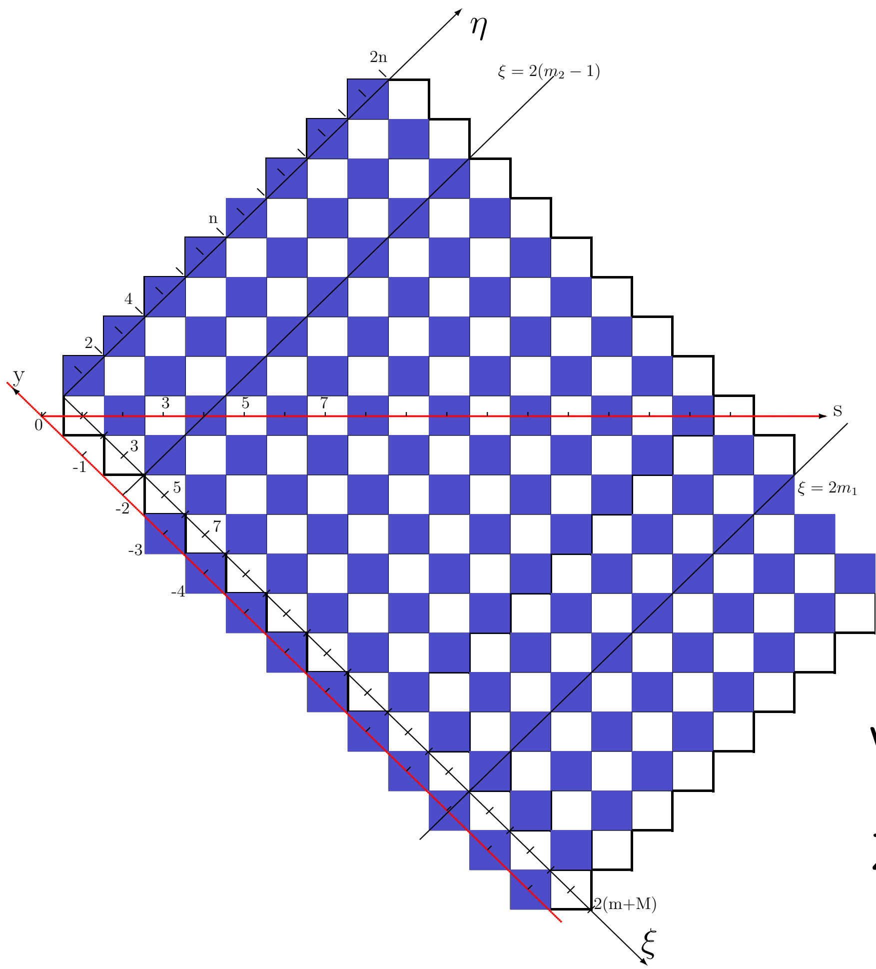
$$\mathfrak{r} = \#\left\{\begin{matrix}\text{red dots along the} \\ \rho + 1 \text{ oblique lines}\end{matrix}\right\}$$

$$= n - m_2 + 1 = \rho - \kappa = 6,$$

$$m_1 = n - \mathfrak{r} + \rho, \quad m_2 = n - \mathfrak{r} + 1$$

In this example $n = 8$, $m_1 = 10$, $m_2 = 3$





Keep $\mathfrak{r}$ and ρ finite.

Asymptotics for $n,\ m_1,\ m_2 \to \infty$

with $m_1 = n + \rho - \mathfrak{r}, \quad m_2 = n + 1 - \mathfrak{r}$

ZOOM ABOUT THE STRIP !!

STEP 2. SCALING: For $n \to \infty$, $x_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$,

new set of variables: $(\xi_i, \eta_i) \to (\tau_i, y_i) \in \mathbb{Z} \times \mathbb{R}$

$$\xi_i := 2m_1 - 2\tau_i, \quad \eta_i := n - 1 + y_i\sqrt{2n},\ a = 1 + \frac{\beta}{\sqrt{n}}, \quad \frac{\Delta\eta}{2} = dy\sqrt{\frac{n}{2}}$$

Step 3. Limiting kernel = **"Discrete Tacnode Kernel"** $\mathbb{L}^{\textbf{dTac}}$

For $\tau_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$ and $n = \frac{1}{t^2} \to \infty$:

$$\boxed{\lim_{t\to 0} \frac{\sqrt{2}}{1+a^2}(-a)^{\frac{\eta_1-\eta_2}{2}}(at)^{\frac{\xi_1-\xi_2}{2}}\mathbb{K}_n^{red}(\xi_1,\eta_1;\xi_2,\eta_2)\frac{\Delta\eta_2}{2} = \mathbb{L}^{\text{dTac}}(\tau_1,y_1;\tau_2,y_2)dy_2}$$

For $x_i, \tau_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$: (recall $\rho = m_1 - m_2 + 1$, $\mathfrak{r} = n - m_2 + 1$, $a = 1 + \beta t$)

$$
\begin{aligned}
&\mathbb{L}^{\mathrm{dTac}}(\tau_1, y_1; \tau_2, y_2) \\
&\quad := -\,\mathbb{H}^{\tau_1 - \tau_2}(\sqrt{2}(y_1 - y_2)) \\
&\qquad + \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} \frac{dZ}{Z - V} \frac{V^{\rho - \tau_1}}{Z^{\rho - \tau_2}} \frac{e^{-\frac{V^2}{2} + (\beta + y_1\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta + y_2\sqrt{2})Z}} \frac{\Theta_{\mathfrak{r}}(V, Z)}{\Theta_r(0, 0)} \\
&\qquad + \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} \frac{dZ}{Z - V} \frac{V^{\tau_2}}{Z^{\tau_1}} \frac{e^{-\frac{V^2}{2} + (\beta - y_2\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta - y_1\sqrt{2})Z}} \frac{\Theta_{\mathfrak{r}}(V, Z)}{\Theta_r(0, 0)} \\
&\qquad + \oint_{\uparrow L_{0+}} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} dZ \frac{V^{-\tau_1}}{Z^{\rho - \tau_2}} \frac{e^{-\frac{V^2}{2} - (\beta - y_1\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta + y_2\sqrt{2})Z}} \frac{\Theta^{+}_{\mathfrak{r}-1}(V, Z)}{\Theta_{\mathfrak{r}}(0, 0)} \\
&\qquad - \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dZ \frac{V^{\rho - \tau_1}}{Z^{-\tau_2}} \frac{e^{-\frac{V^2}{2} + (\beta + y_1\sqrt{2})V}}{e^{\frac{Z^2}{2} - (\beta - y_2\sqrt{2})Z}} \frac{\Theta^{-}_{\mathfrak{r}+1}(V, Z)}{\Theta_{\mathfrak{r}}(0, 0)}
\end{aligned}
$$

where

$$\Gamma_0 = \text{small circle about } V = 0 \text{ or } Z = 0,$$

$$\uparrow L_{0+} = \text{upgoing vertical line in } \mathbb{C} \text{ to the right of } \Gamma_0$$

where

$$\mathbb{H}^m(z) := \frac{z^{m-1}}{(m-1)!}\mathbb{1}_{z\geq 0}\mathbb{1}_{m\geq 1}, \qquad \text{(Heaviside function)}$$

$$\Theta_{\mathfrak{r}}(V,Z) := \tfrac{1}{\mathfrak{r}!}\left[\prod_1^{\mathfrak{r}}\oint_{\uparrow L_{0+}}\frac{e^{W_\alpha^2-2\beta W_\alpha}dW_\alpha}{2\pi \mathrm{i} W_\alpha^\rho}\Big(\frac{Z-W_\alpha}{V-W_\alpha}\Big)\right]\Delta_{\mathfrak{r}}^2(W_1,\dots,W_r)$$

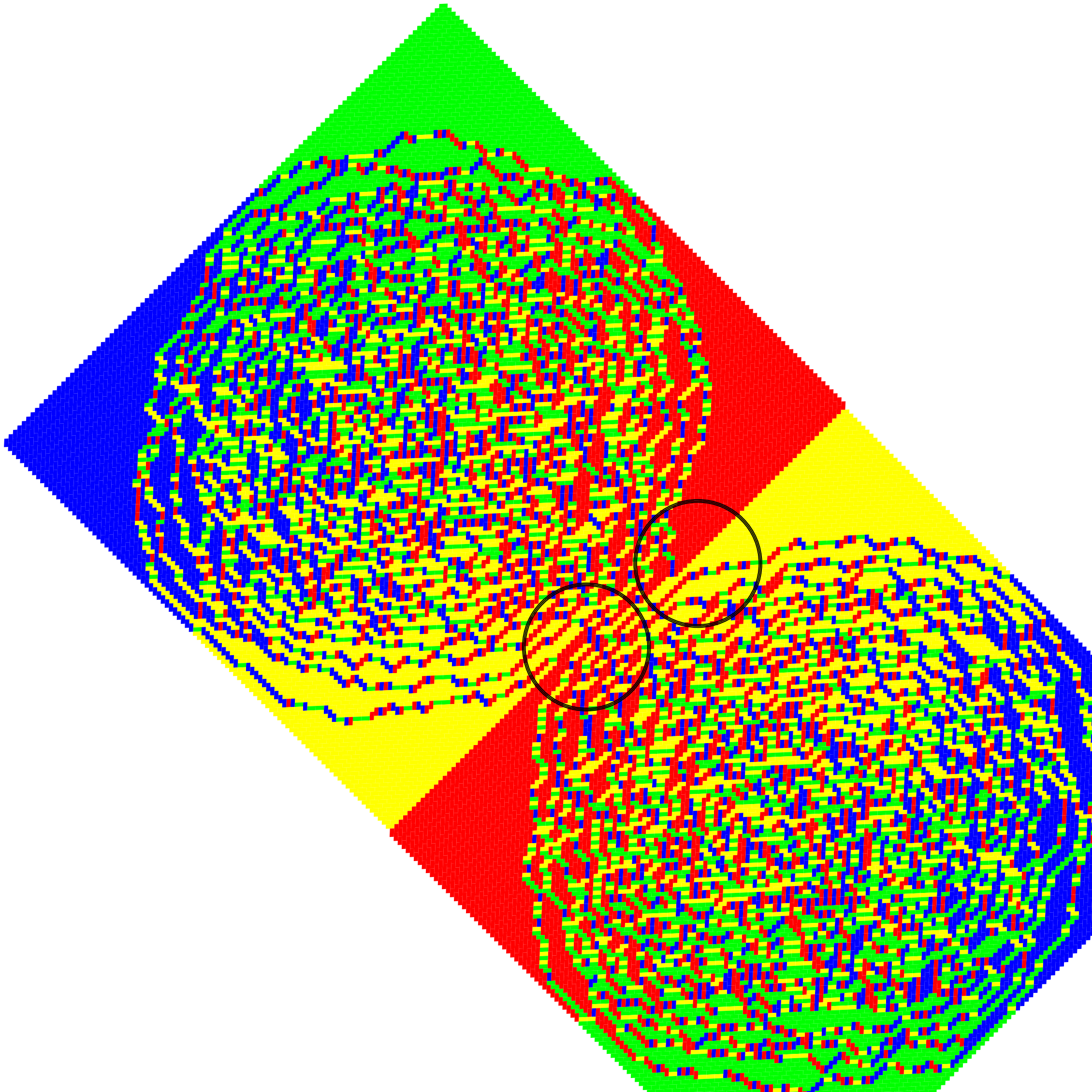
$$\begin{aligned}\Theta^{\pm}_{\mathfrak{r}\mp 1}(V,Z) := {} & \tfrac{1}{(\mathfrak{r}\mp 1)!} \\ & \times\left[\prod_1^{\mathfrak{r}\mp 1}\oint_{\uparrow L_{0+}}\frac{e^{W_\alpha^2-2\beta W_\alpha}dW_\alpha}{2\pi \mathrm{i} W_\alpha^\rho}\ \ ((Z-W_\alpha)\ (V-W_\alpha))^{\pm 1}\right] \\ & \qquad\qquad\qquad\qquad \times\ \Delta^2_{\mathfrak{r}\mp 1}(W_1,\dots,W_{\mathfrak{r}\mp 1}).\end{aligned}$$

$\Delta_{\mathfrak{r}}$ =Vandermonde

This enables one to compute:

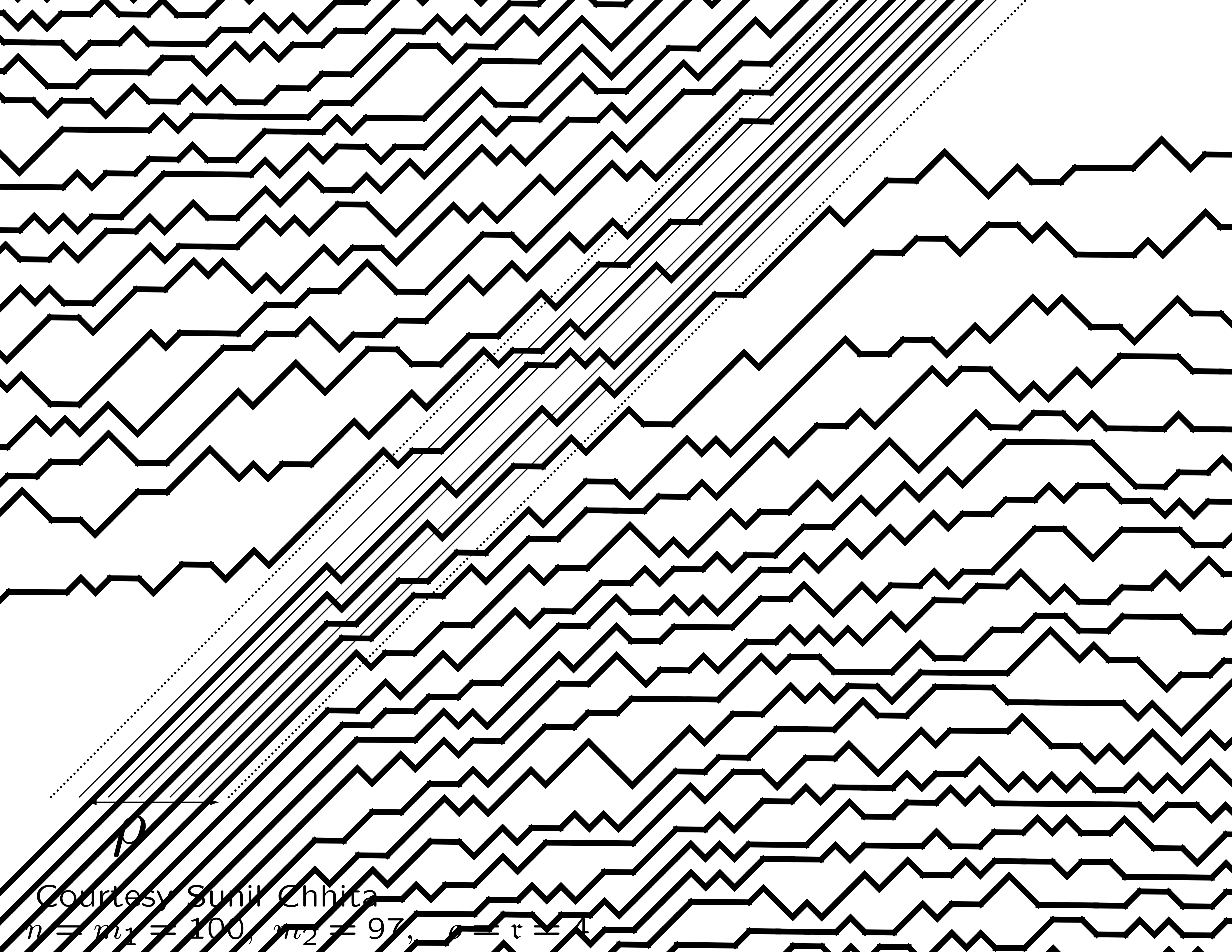
1. The distribution and joint distribution of blue and green tiles along oblique lines:

2. The cusp-Airy probability near the cusp with inside two colors of tiles



1. The distribution and joint distribution of blue and green tiles along oblique lines:


ρ
Courtesy Sunil Chhita
$n = m_1 = 100, \; m_2 = 97, \quad \rho = \mathfrak{r} = 4$



ρ
Courtesy Sunil Chhita
n = m1 = 100, m2 = 97, ρ = 𝔯 = 4

1. The distribution and joint distribution of blue or green tiles along oblique lines:

- $N_\tau = \#\{$blue or green tiles along the oblique line $\tau\} = (\tau-\rho)_{>0} + r$

- $\mathbf{x}^{(\tau)} \in \mathbb{R}^{N_\tau}$

Theorem (*distribution and joint distribution of the blue or green tiles*)

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau)} \in d\mathbf{x}\ \right) = D(\tau,\mathbf{x};\tau,\mathbf{x})d\mathbf{x}, \qquad \text{for } \tau \geq 0$$

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau_1)} \in d\mathbf{x} \text{ and } \mathbf{y}^{(\tau_2)} \in d\mathbf{y}\ \right)$$
$$= D(\tau_1,\mathbf{x};\tau_2,\mathbf{y})\mathrm{Vol}(\mathcal{C}(\tau_1,\mathbf{x};\tau_2,\mathbf{y}))d\mathbf{x}d\mathbf{y}$$
$$\text{for } 0 \leq \tau_1 < \tau_2 \leq \rho \quad \text{or} \quad \rho < \tau_1 < \tau_2$$

$$\mathrm{Vol}(\mathcal{C}(\tau_1,\mathbf{x};\tau_2,\mathbf{y})) = \text{Volume of polytope (Gibbs property!)}$$

where $(\prec =$:interlacing)

$$\mathcal{C}(\tau_1,\mathbf{x};\tau_2,\mathbf{y}) := \left\{\ \mathbf{x} = \mathbf{z}^{(\tau_1)} \prec \mathbf{z}^{(\tau_1+1)} \prec \ldots \prec \mathbf{z}^{(\tau_2-1)} \prec \mathbf{z}^{(\tau_2)} = \mathbf{y},\ \right\}$$

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$$\text{for } 0 \leq \tau_1 < \tau_2 \leq \rho \quad \text{or} \quad \rho < \tau_1 < \tau_2$$

where

- $$D(\tau_1,\mathbf{x}^{(\tau_1)};\tau_2,\mathbf{y}^{(\tau_2)}) := \begin{cases} C\widetilde{\Delta}_{N_{\tau_1},\tau_1}(\mathbf{x}+\frac{\beta}{2})\Delta_{N_{\tau_2}}(\mathbf{y})\left(\prod_{i=1}^{N_{\tau_2}} \frac{e^{-y_i^2}}{\sqrt{\pi}}\right) \\ \qquad\qquad \text{for } \rho < \tau_1 < \tau_2 \\ \\ C'\widetilde{\Delta}_{N_{\tau_1},\tau_1}(\mathbf{x}+\frac{\beta}{2})\widetilde{\Delta}_{N_{\tau_2},\rho-\tau_2}(-\mathbf{y}) \\ \\ \qquad\qquad \text{for } 0 \leq \tau_1 \leq \tau_2 \leq \rho \end{cases}$$

- $\Delta_{N_\tau}(y)$ = Vandermonde of size $N_\tau = \#$(blue or green tiles $\in$ level τ)

- For $N := (\tau - \rho)_{>0} + r$, define:

$$\widetilde{\Delta}_{N,\tau}(x) = \begin{cases} \det \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_N \\ \vdots & & \vdots \\ x_1^{\tau-\rho-1} & \dots & x_N^{\tau-\rho-1} \\ \Phi_{\tau-1}(x_1) & \dots & \Phi_{\tau-1}(x_N) \\ \vdots & & \\ \Phi_{\tau-r}(x_1) & \dots & \Phi_{\tau-r}(x_N) \end{pmatrix}, & \text{for } \rho < \tau \\ \\ \det \begin{pmatrix} \Phi_{\tau-1}(x_1) & \dots & \Phi_{\tau-1}(x_N) \\ \vdots & & \\ \Phi_{\tau-r}(x_1) & \dots & \Phi_{\tau-r}(x_N) \end{pmatrix}, & \text{for } 0 \leq \tau \leq \rho \end{cases}$$

with

$$\Phi_k(\eta) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{e^{v^2+2\eta v}}{v^{k+1}} dv$$

$$= \frac{e^{-\eta^2}}{2^{-k}\sqrt{\pi}} \times \begin{cases} \displaystyle\int_0^\infty \frac{\xi^k}{k!} e^{-\xi^2+2\xi\eta} d\xi & , k \geq 0 \\ \\ H_{-k-1}(-\eta) \quad , & k \leq -1 \end{cases}$$

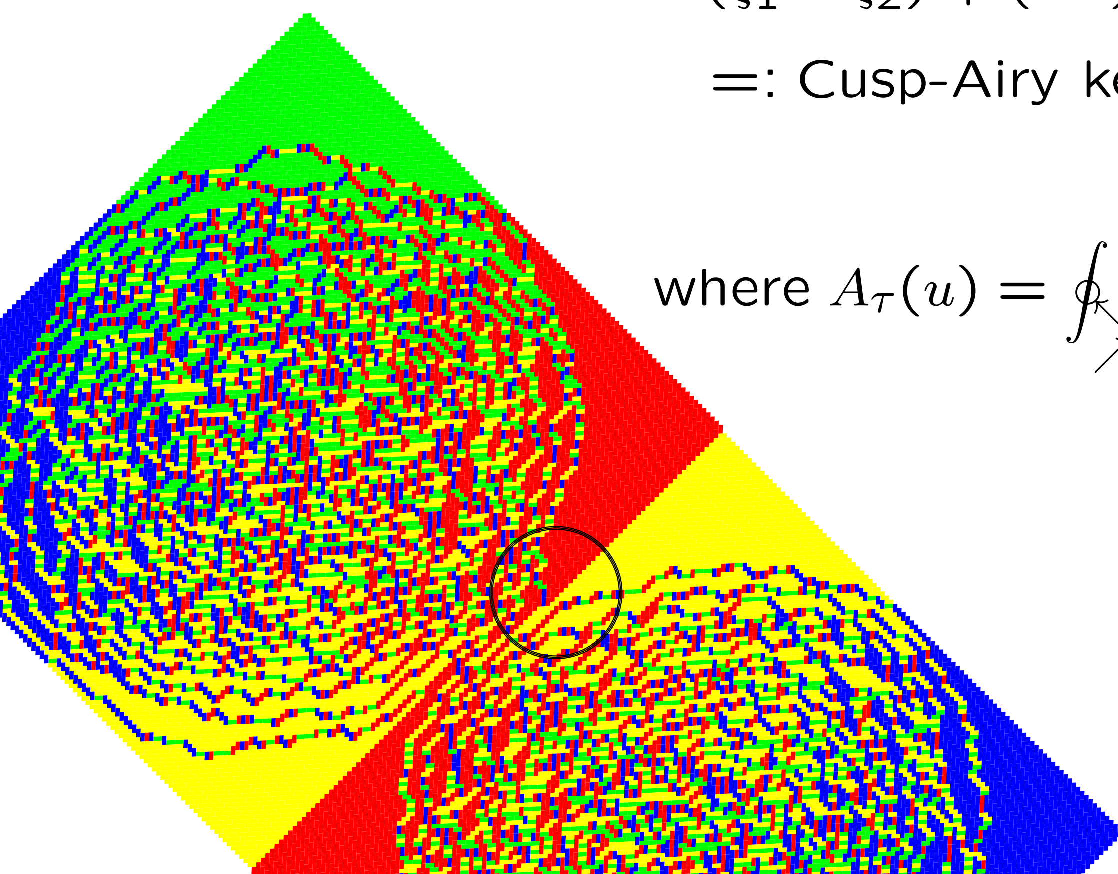
2. The cusp-Airy probability near a cusp bordering a frozen region with two colors of tiles for $\rho = 0$, obtained as a scaling limit of the discrete tacnode kernel $\mathbb{L}^{\mathrm{dTac}}$

The cusp-Airy kernel is given by: for all $\tau_1,\ \tau_2 \in \mathbb{Z}$, except for $\tau_1 \geq 0$ and $\tau_2 \leq 0$

$$\lim_{r\to\infty} (-r^{\frac{\tau_1-\tau_2}{6}}) \mathbb{L}^{\mathrm{dTac}}(\tau_1,\theta_1;\tau_2,\theta_2)\Big|_{\theta_i = 2\sqrt{r}+\frac{\xi_i}{r^{1/6}}}$$

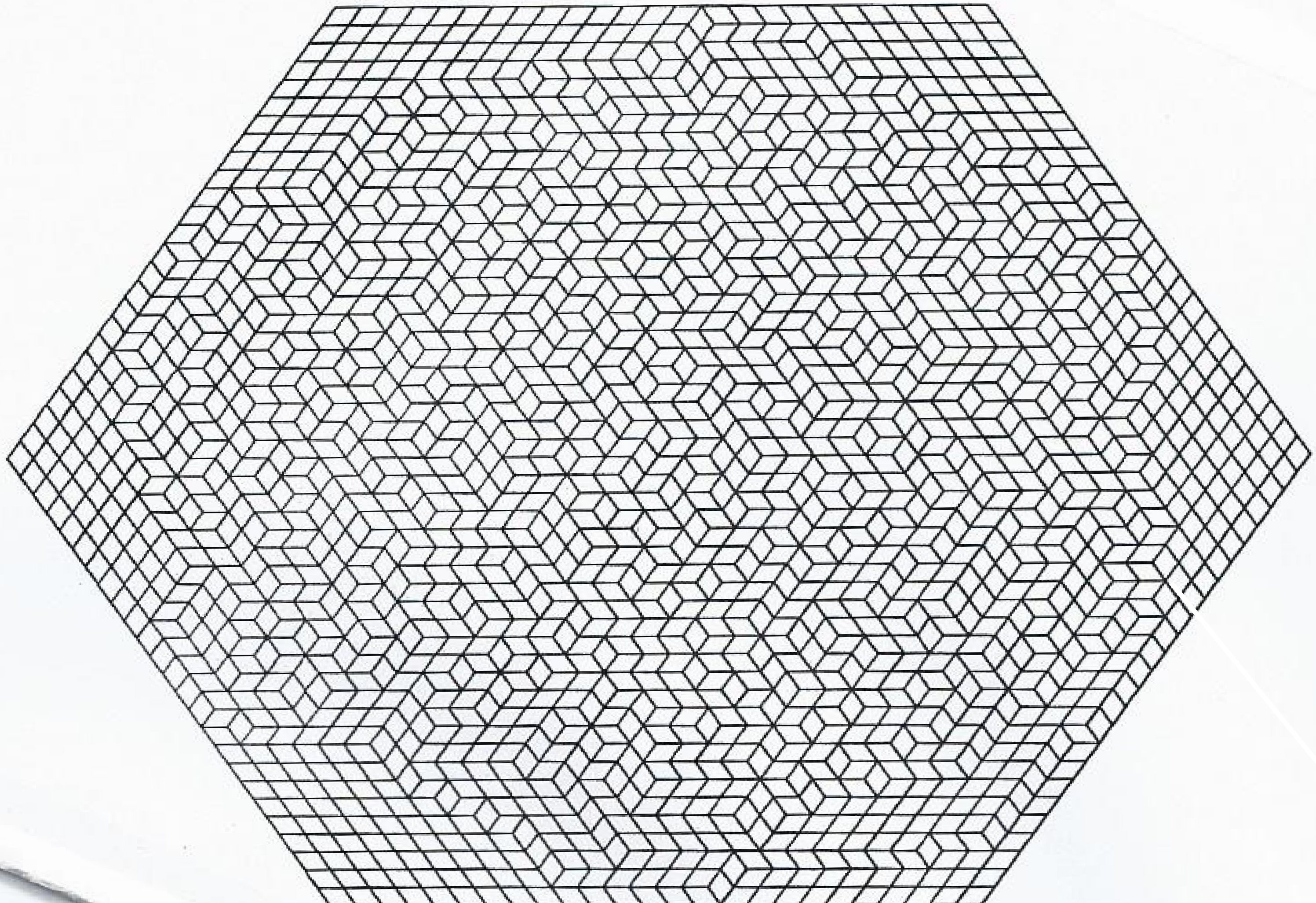
$$= -\mathbb{H}^{\tau_1-\tau_2}(\xi_1-\xi_2) + (-1)^{\tau_2}\int_0^\infty A_{-\tau_1}(\xi_1+\lambda)A_{-\tau_2}(\xi_2+\lambda)d\lambda$$

$$=: \text{Cusp-Airy kernel}$$

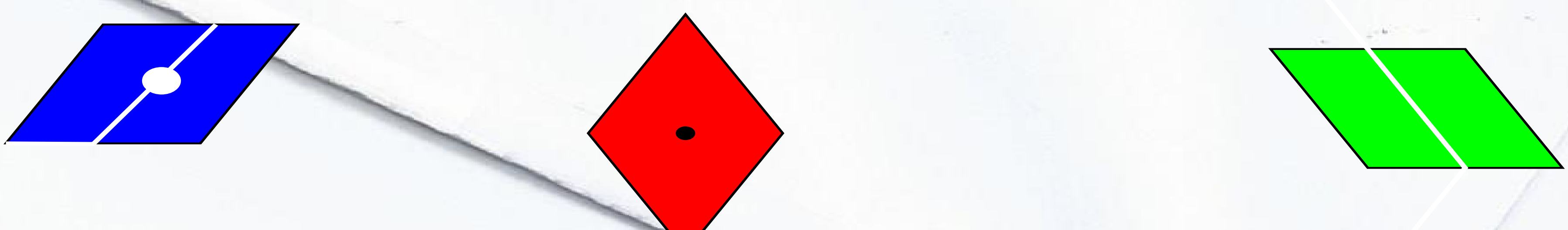


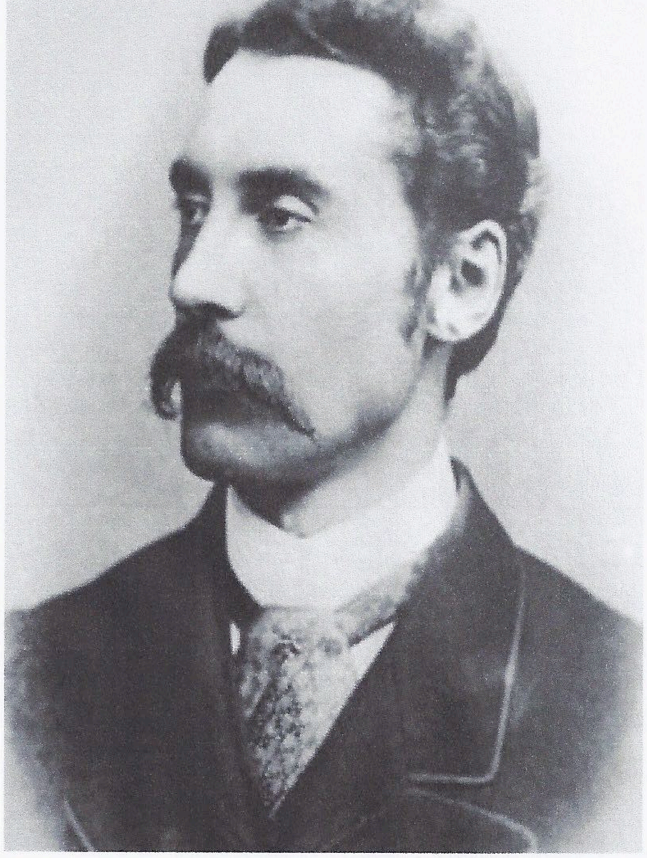
where $A_\tau(u) = \oint \frac{z^\tau dz}{2\pi \mathrm{i}} e^{-\frac{z^3}{3}+uz},\ \ \tau \in \mathbb{Z}$ (angles $= \frac{\pi}{3}$)

II. Tiling of a hexagon with Lozenges



Three different types of Lozenges:





Percy Alexander MacMahon (1854, Malta -1929, UK)

P.A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge UP, 1916
reprinted by Chelsea, New York, 1960.

MacMahon's formula for the number of domino tilings an hexagon with sides a, b, c: (other formula by MacDonald)

$$P(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}$$

where

$$H(n) = (n-1)!(n-2)!\dots 1!0!$$

MacMahon's formula for the number of domino tilings an hexagon with sides a, b, c: (other formula by MacDonald)

$$P(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}$$

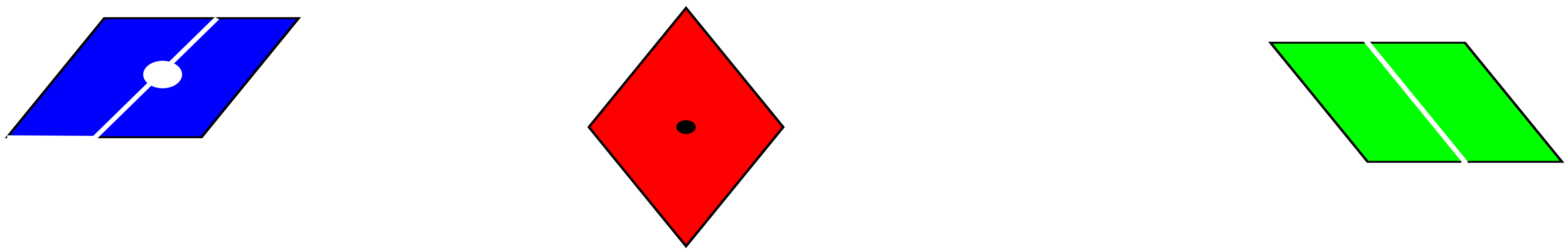
Statistical questions for large sizes ? Everything is known:

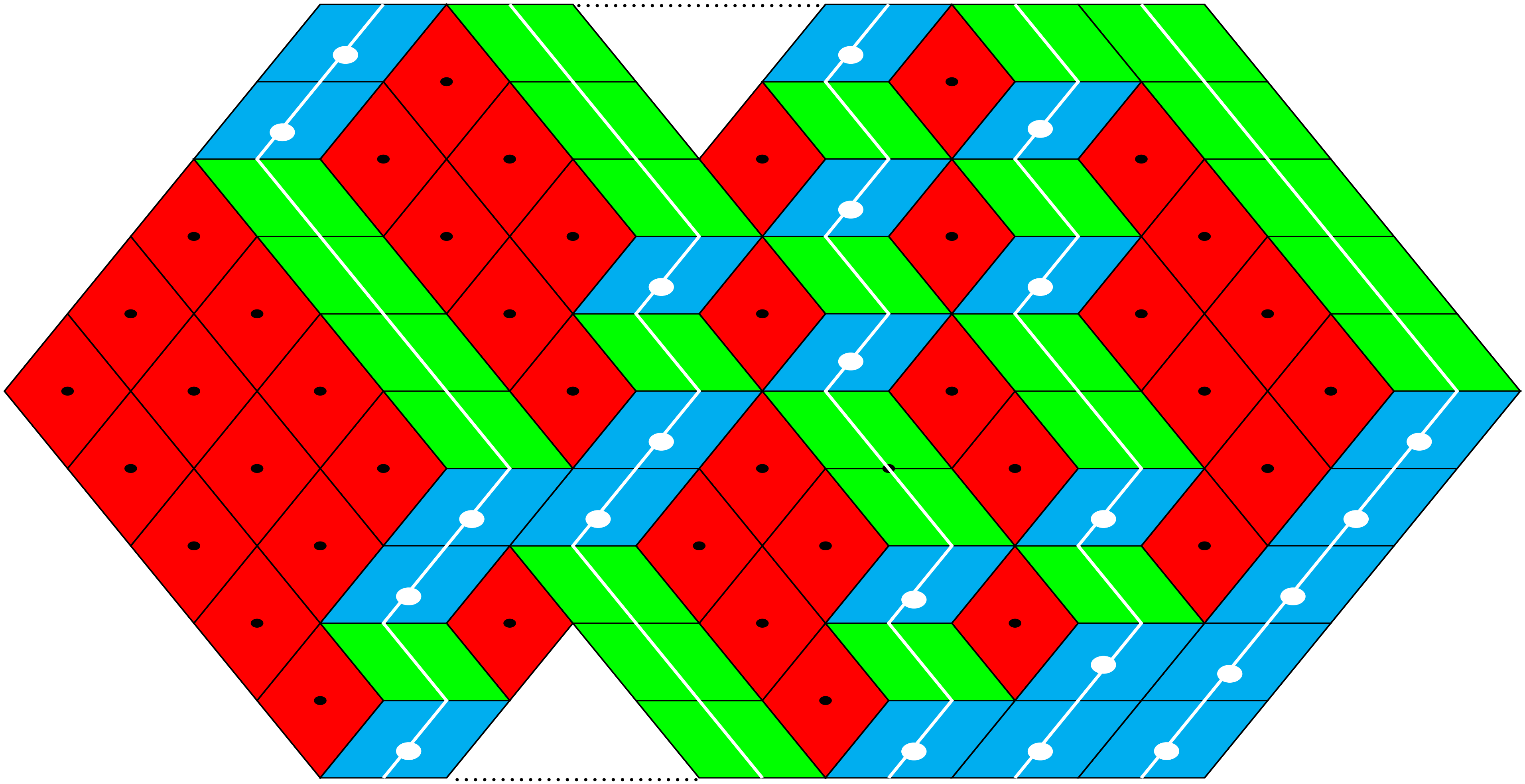
$\Rightarrow$ Arctic circle separates the frozen and liquid regions (as for the Aztec diamonds)

- **Gaussian Free Field** in the liquid region

- **Airy Process** along the arctic circle

- **GUE-minor process** at the tangency points of the arctic circle

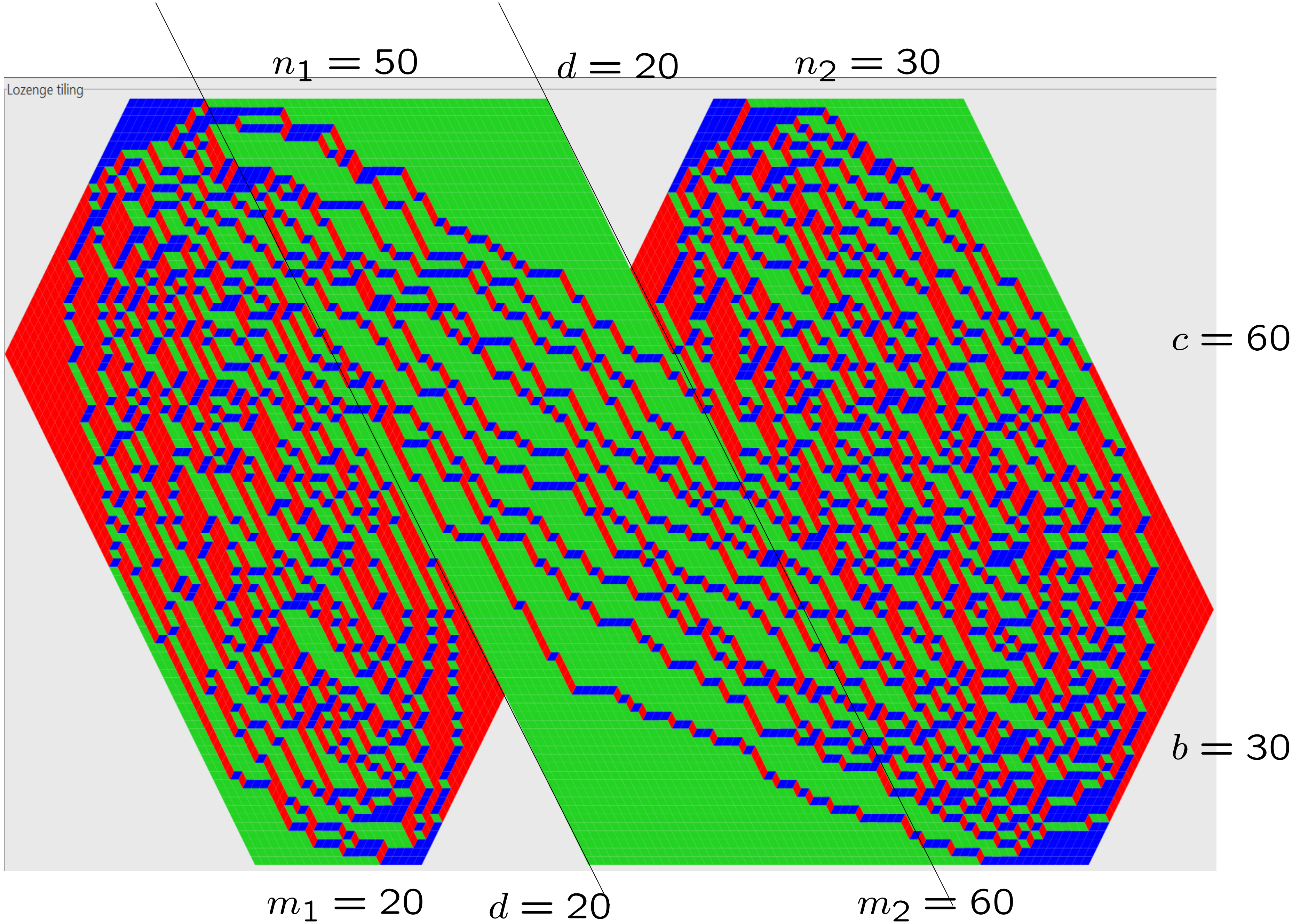




(Courtesy of Antoine Doeraene)

What are the statistical fluctuations of blue tiles in the region between the two cuts, when the size of the polygon and the cuts tends to infinity ?

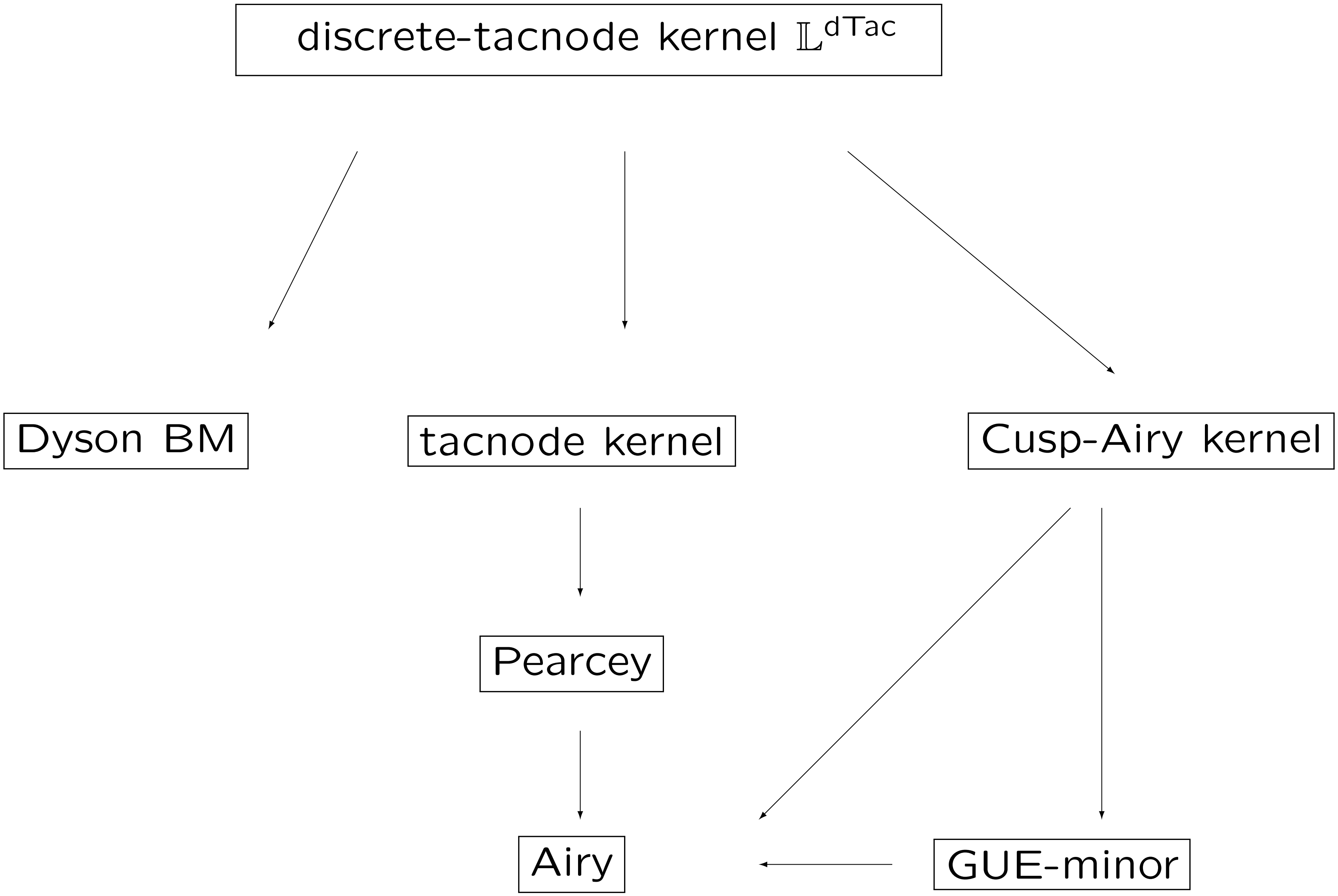
CLAIM: same statistics as in domino tilings of Aztec rectangles:
The discrete tacnode process



width of strip $= \rho := n_1 - m_1 + b - d = 40$

$\#\{$paths connecting the 2 arctic ellipses $\} = r := b - d = 10$

$\mathbb{L}^{\mathsf{dTac}}(\tau_1, y_1; \tau_2, y_2)$ is a master kernel, besides being universal



THANK YOU !