On some (non integrable) Kadomtsev-Petviashvili type equations

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Motivation

- The (formal) derivation (Kadomtsev-Petviashvili 1970) of the KP equation is independent of the dispersion in x and concerns only the transport part of the KdV equation

$$u_t + u_x + uu_x + (\frac{1}{3} - T)u_{xxx} = 0 \tag{1}$$

where $T \geq 0$ is the surface tension parameter.

- More precisely, it consists in looking for a weakly transverse perturbation of the one-dimensional transport equation

$$u_t + u_x = 0. \tag{2}$$

- This perturbation is (formally) obtained by a Taylor expansion of the dispersion relation $\omega(k_1, k_2) = \sqrt{k_1^2 + k_2^2}$ of the two-dimensional linear wave equation assuming $|k_1| \ll 1$ and $\frac{|k_2|}{|k_1|} \ll 1$.
- Namely, one writes formally

$$\omega(k_1, k_2) \sim \pm k_1 \left(1 + \frac{k_2^2}{2k_1^2}\right)$$

which amounts, coming back to the physical variables, to adding a nonlocal term to the transport equation,

$$u_t + u_x + \frac{1}{2}\partial_x^{-1} u_{yy} = 0. \quad (3)$$

Here the operator ∂_x^{-1} is defined via Fourier transform,

$$\widehat{\partial_x^{-1} f}(\xi) = \frac{1}{i\xi_1}\widehat{f}(\xi), \text{ where } \xi = (\xi_1, \xi_2).$$

- Assuming that the transverse dispersive effects are of the same order as the x-dispersive and nonlinear terms, yields the KP equation

$$u_t + u_x + uu_x + (\frac{1}{3} - T)u_{xxx} + \frac{1}{2}\partial_x^{-1}u_{yy} = 0. \qquad (4)$$

- The KP II equation is obtained when $T < \frac{1}{3}$, the KP I when $T > \frac{1}{3}$.
- Note that the KP I case is irrelevant for water waves since $T > \frac{1}{3}$ occurs for a very thin layer of water and them one should take viscous effects into account... (but KP I appears as a long wave limit of the Gross-Pitaevskii or other dispersive equations for instance).

- Formal derivation of KP-II in the context of water waves : Ablowitz-Segur 1983.
- Rigorous derivation : David Lannes, 2002 with a bad error estimate for

$$u_t + u_x + \epsilon u u_x + \epsilon u_{xxx} + \epsilon \partial_x^{-1} u_{yy} = 0, \quad \epsilon \ll 1,$$

$$||u_{\text{Euler}} - u_{KP}|| = o(1).$$

- This is the price to pay for the singularity at $\xi_1 = 0$.
- Optimal error estimate for the KdV equation

$$u_t + u_x + \epsilon u u_x + \epsilon u_{xxx} = 0,$$

$$||u_{\text{Euler}} - u_{KdV}|| = O(\epsilon^2 t).$$

FIGURE – Interaction of line solitons. Oregon coast

- It is thus quite natural to apply this formal process to any KdV type equation, in particular to the fractional (fKdV) equation

$$u_t + (u_x) + uu_x \pm D_x^\alpha u_x = 0, \quad \widehat{D_x^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi), \tag{5}$$

- ($\alpha = 2$: KdV ; $\alpha = 1$: Benjamin-Ono)

  to get the fractional KP (fKP) equations

$$u_t + (u_x) + uu_x - D_x^\alpha u_x + \kappa \partial_x^{-1} u_{yy} = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad u(\cdot, 0) = u_0, \tag{6}$$

  where $\kappa = \pm 1$ and $\alpha > -1$ (but we will focus on $0 < \alpha < 2$).

- The sign of the x-dispersive term in (5) will determine the fKP I or fKP II types.

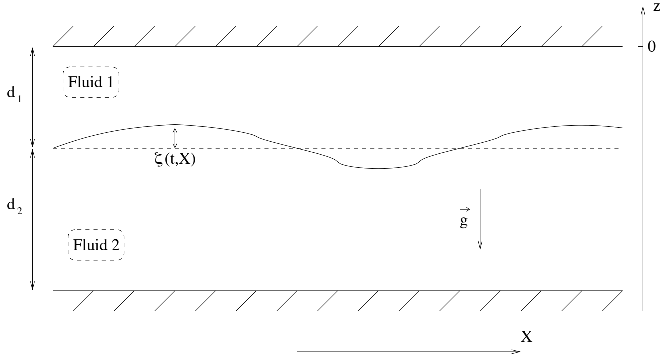
- fKP with $\alpha = 1$ is the relevant KP version of the Benjamin-Ono equation (KP-BO). Very similar to the KP-BO equation is the KP version of the intermediate long wave equation (ILW) :

$$u_t + uu_x - \mathcal{L}_{ILW} u_x = 0, \tag{7}$$

where $\widehat{\mathcal{L}_{ILW} f}(\xi) = p_{ILW}(\xi)\hat{f}(\xi), \quad p_{ILW}(\xi) = \xi \coth(\delta\xi) - \frac{1}{\delta}, \delta > 0.$

- The ILW equation is a model for long, weakly nonlinear internal waves, δ being proportional to the depth of the bottom layer. See Bona-Lannes-S, 2008 and below for a rigorous derivation (in the sense of consistency) of the ILW and related equations.
- The Benjamin–Ono equation is obtained in the infinite depth limit, $\delta \to +\infty$.
- In this context we are in the fKP II case (adding $\partial_x^{-1} u_{yy}$).
- See Ablowitz-Segur (1980) and Camassa-Choï (1999) for a formal derivation of KP-II-ILW and KP-II BO for internal waves.
- One can justify (in the sense of consistency with the full internal waves system) the KP-BO or KP-ILW equations following the procedure for the usual KP equation (see Lannes 2002)

The two fluid system with rigid lid



$$\gamma = \frac{\rho_1}{\rho_2} < 1.$$

- Many regimes depending on amplitudes, wavelengths,..
- The range of validity of the various regimes is summarized in the following table.

	$\varepsilon = O(1)$	$\varepsilon \ll 1$
$\mu = O(1)$	Full equations	$\delta \sim 1$: FD/FD eq'ns
$\mu \ll 1$	$\delta \sim 1$: SW/SW eq'ns $\delta^2 \sim \mu \sim \varepsilon_2^2$: SW/FD eq'ns	$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon$: B/FD eq'ns $\mu \sim \varepsilon$ and $\delta \sim 1$: B/B eq'ns $\delta^2 \sim \mu \sim \varepsilon^2$: ILW eq'ns $\delta = 0$ and $\mu \sim \varepsilon^2$: BO eq'ns

$$\gamma = \frac{\rho_1}{\rho_2}, \delta = \frac{d_1}{d_2}, \epsilon = \frac{a}{d_1}, \mu = \frac{d_1^2}{\lambda^2}, \epsilon_2 = \frac{a}{d_2} = \epsilon\delta, \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

- One starts from the following "ILW" system that is consistant with the full internal wave system (see Bona-Lannes-S 2008)

$$\begin{cases} \partial_t \zeta + \dfrac{1}{\gamma}\nabla \cdot ((1-\epsilon\zeta)\mathbf{v}) \\ -\dfrac{\sqrt{\mu}}{\gamma^2}|D|\coth(\sqrt{\mu_2}|D|)\nabla \cdot \mathbf{v} = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta - \dfrac{\epsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0, \end{cases} \tag{8}$$

where $\mathbf{v}$ is the velocity field, ζ the elevation, $\gamma = \frac{\rho_1}{\rho_2} < 1$ the ratio of densities and $\mu \sim \epsilon^2 \ll 1$. μ_2 is a "large parameter related to the depth of the lower layer. When $\mu_2 \to +\infty$ one get a Benjamin-Ono type system.

$$\begin{cases} \partial_t \zeta + \frac{1}{\gamma}\nabla \cdot ((1-\varepsilon\zeta)\mathbf{v}) - \frac{\sqrt{\mu}}{\gamma^2}|D|\nabla \cdot \mathbf{v} = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0. \end{cases} \tag{9}$$

- One deduces ILW and BO in the 1D case after the one-way propagation assumption.
- Then one looks for solutions satisfying the KP scaling via a suitable ansatz.

Why fKP equations ?

- Contain some physically relevant equations (KP-BO, KP-ILW,,...).
- To study the influence of dispersion on the space of resolution, on the lifespan, the possible blow-up and on the dynamics of solutions to the Cauchy problem for "weak" dispersive perturbations of hyperbolic quasilinear equations or systems.
- See Linares-Pilod-S 2014 and Klein-S 2014 for fKdV equations as weakly dispersive perturbations of the Burgers equation.
- In this, context it is more natural to keep the quadratic nonlinearity and weaken the dispersion than to keep the KdV dispersion and increase the nonlinearity...
- None of the fKP equations seems to be completely integrable except of course the classical ones, $\alpha = 2$...

The Khokhlov-Zabolotskaya (KZ) equation (1969)

$$u_t + uu_x + \partial_x^{-1}\Delta_\perp u = 0, \quad \Delta_\perp = \partial_{yy} \text{ or } \partial_{yy} + \partial_{zz}, \quad \text{(nonlinear acoustics)}$$

- Also derived by Alinhac (1996) to describe the blow-up of solutions to some quasilinear hyperbolic equations.
- Easy : LWP in $H^s_{-1}(\mathbb{R}^2)$, $s > 2$ where

$$H^s_{-1}(\mathbb{R}^2) = \{f \in H^s(\mathbb{R}^2) \ : \ \mathcal{F}^{-1}(\frac{\widehat{f}(\xi,\eta)}{\xi}) \in H^s(\mathbb{R}^2)\}.$$

- But ill posed in $H^2_{-1}(\mathbb{R}^2)$.
- "Hyperbolic nature" : Alinhac (1996), blow-up of $\sup_{x,y} \partial_x u(x,y,t)$ in finite time. See also Rozanova-Pierrat 2008 (also with a dissipative term) , Alterman-Rauch 2001.
- Look at fKP with small α (possibly negative) to understand the effect of a weak dispersion in x on the KZ dynamics.

General facts on fKP equations

- In addition to the L^2 norm, the fKP equation (6) conserves formally the energy (Hamiltonian)

$$H_\alpha(u) = \int_{\mathbb{R}^2} (\frac{1}{2}|D_x^{\frac{\alpha}{2}} u|^2 - \kappa \frac{1}{2}|\partial_x^{-1} u_y|^2 - \frac{1}{6}u^3). \tag{10}$$

The corresponding energy space is

$$Y_\alpha = \{u \in L^2(\mathbb{R}^2) \ : \ D_x^{\frac{\alpha}{2}} u, \ \partial_x^{-1} u_y \in L^2(\mathbb{R}^2)\}.$$

- One checks readily that the transformation

$$u_\lambda(x,y,t) = \lambda^\alpha u(\lambda x, \lambda^{\frac{\alpha+2}{2}} y, \lambda^{\alpha+1} t)$$

leaves (6) invariant.

- Moreover, $|u_\lambda|_2 = \lambda^{\frac{3\alpha-4}{4}} |u|_2$, so that $\alpha = \frac{4}{3}$ is the L^2 critical exponent. Note that the BO-KP and the ILW-KP equations are L^2 supercritical.

- Recall the fractional Gagliardo-Nirenberg inequality :

Lemma

Let $\frac{4}{5} \leq \alpha < 1$. For any $f \in Y_\alpha$ one has

$$|f|_3^3 \leq c|f|_2^{\frac{5\alpha-4}{\alpha+2}} \|f\|_{H_x^{\frac{\alpha}{2}}}^{\frac{18-5\alpha}{2(\alpha+2)}} |\partial_x^{-1} f_y|_2^{\frac{1}{2}} ,$$

where $\|\cdot\|_{H_x^{\frac{\alpha}{2}}}$ denotes the natural norm on the space

$$H_x^{\frac{\alpha}{2}}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \ : \ D_x^{\frac{\alpha}{2}} f \in L^2(\mathbb{R}^2)\}.$$

- Lemma 1 implies obviously the embedding $Y_\alpha \hookrightarrow L^3(\mathbb{R}^2)$ if $\alpha \geq \frac{4}{5}$.
- The energy critical value $\frac{4}{5}$ of α is related to the non existence of localized solitary waves. By Pohozaev type arguments :
- Assume that $0 < \alpha \leq \frac{4}{5}$ when $\kappa = -1$ or that α is arbitrary when $\kappa = 1$.Then (6) does not possess non trivial solitary waves $u(x-ct, y)$ in the space $Y_\alpha \cap L^3(\mathbb{R}^2)$.

An easy result

- Viewing (6) as a skew-adjoint perturbation of the Burgers equation, one easily establishes the elementary local well-posedness result $(H^s_{-1}(\mathbb{R}^2) = \{f \in H^s(\mathbb{R}^2), \mathcal{F}^{-1}(\frac{\hat{f}}{\xi_1}) \in H^s(\mathbb{R}^2)\})$.

Theorem

Let $u_0 \in H^s_{-1}(\mathbb{R}^2), s > 2$. Then there exists $T = T(||u_0||_s > 0$ and a unique solution $u \in C([0,T]; H^s_{-1}(\mathbb{R}^2))$, $u_t \in C([0,T]; H^{s-3}(\mathbb{R}^2))$. Furthermore, the map $u_0 \mapsto u$ is continuous from $H^s_{-1}(\mathbb{R}^2)$ to $C([0,T]; H^s_{-1}(\mathbb{R}^2))$. Moreover $|u(.,t)|_2$ and $H_\alpha(u(.,t))$ are conserved on $[0,T]$.

- This result does not depend on the dispersion (and the exponent 2 is the "hyperbolic" one...). A goal of this talk is to see how dispersion can help to enlarge the space of resolution.

- For the fKP-I equation with $\alpha > \frac{3}{2}$ one obtains the global existence of a weak solution $u \in L^\infty(\mathbb{R}; Y^\alpha)$, where

$$Y^\alpha = \{f \in L^2(\mathbb{R}^2), D_x^{\alpha/2} f \in L^2(\mathbb{R}^2), \partial_x^{-1}\partial_y f \in L^2(\mathbb{R}^2)\}$$

is the energy space.

- This results from a standard compactness method, using the Gagliardo-Nirenberg inequality.

The linear group

$$u_t - D^\alpha u_x \pm \partial_x^{-1} u_{yy} = 0. \tag{11}$$

- The linear part in (6) defines, for any $\alpha > -1$, a unitary group $U_\alpha(t)$ in $L^2(\mathbb{R}^2)$ and all $H^s(\mathbb{R}^2)$ Sobolev spaces, unitarily equivalent via Fourier transform to the Fourier multiplier

$$e^{it(|\xi_1|^\alpha \xi_1 \mp \xi_2^2/\xi_1)}.$$

- Local smoothing for the linear fKP II (see JCS 1993 for the usual KP II equation) :

Proposition

Let $\alpha > \frac{1}{2}$ and $s \geq 0, u_0 \in H^s_{-1}(\mathbb{R}^2)$. Then the solution u of (11) with the $+$ sign satisfies for any $R > 0$ and $T > 0$

$$|D_1|^{\alpha/2} D_1^{s_1} D_2^{s_2} u, \quad D_1^{s_1} D_2^{s_2} \partial_x^{-1} u_y \in L^2((-T,T)\times(-R,R)\times\mathbb{R}), \quad s_1 + s_2 = s.$$

Previous work on KP and fKP

1. "PDE" methods for the Cauchy problem

- Classical KP II : globally well-posed in $L^2(\mathbb{R}^2)$ (Bourgain 1993), improved by Takaoka-Tzvetkov (2003). Hadac-Herr-Koch (2009) : LWP in the critical space $H^{-1/2,0}(\mathbb{R}^2)$.
- Classical KP I : globally well-posed in a suitable Sobolev class (Molinet-JCS-Tzvetkov 2002). Best result : GWP in the energy space (Ionescu-Kenig-Tataru 2008).
- M. Hadac (2008) : local-well-posedness of the fKP-II equation in the L^2-subcritical case $\alpha > \frac{4}{3}$ in the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2), s_1 > \max(1-\frac{3}{4}\alpha, \frac{1}{4}-\frac{3}{8}\alpha), s_2 \geq 0$.

- No much is known concerning the long time behavior of global solutions of the classical KP-I and KP-II equations :
- Ifrim-Tataru (2015) : scattering of small solutions of KP-I in a galilean invariant space.
- For KP-II, scattering of solutions is expected but still open (see *eg* simulations in Klein-S 2012).

2. IST methods for the Cauchy problem

- The only known rigorous result are for small initial data in suitable spaces of regular functions :
- Wickerhauser (1987), for KP-II and Zhou (1990) for KP-I.
- To get ride of the smallness conditions (and to provide an asymptotics of solutions) is a challenging open problem.

Solitary waves

- KP-I possesses an explicit, finite energy, localized solitary wave (lump) :

$$\phi_c(x-ct,y) = \frac{8c(1-\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2)}{(1+\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2)^2}. \tag{12}$$

- No such solution exists for KP-II (de Bouard-S 1997).
- Ground states. We set

$$E_{KP}(\psi) = \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x\psi)^2 + \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x^{-1}\partial_y\psi)^2 - \frac{1}{2(p+2)}\int_{\mathbb{R}^2}\psi^3,$$

and we define the action

$$S(N) = E_{KP}(N) + \frac{c}{2}\int_{\mathbb{R}^2} N^2.$$

- A ground state is a solitary wave N which minimizes the action S among all finite energy non-constant solitary waves of speed c.
- Ground states exist if and only if $c > 0$. Moreover, the ground states are minimizers of the Hamiltonian E_{KP} with prescribed mass (L^2 norm) (de Bouard-S 1997).
- The set of ground states is orbitally stable (de Bouard-S 1997). Uniqueness (up to obvious symmetries is open).
- One does not know whether or not the lump is a ground state (presumably yes..)

- Recent nondegeneracy result (Yong Liu,-Junchang Wei, ARMA 2019) :
  Let Q be the lump and ϕ a smooth solution of
  $$\partial_x^2(\partial_x^2\phi - \phi + 6Q\phi) - \partial_y^2\phi = 0.$$
  Assume
  $$\phi(x, y) \to 0, \quad \text{as} \quad x^2 + y^2 \to +\infty.$$
  Then
  $\phi = c_1\partial_x Q + c_2\partial_y Q$, for some c_1, c_2.

Semilinear versus quasilinear

- The distinction between semilinear and quasilinear is not obvious for dispersive equations.
- Roughly speaking : Semilinear when the flow map is smooth (the Cauchy problem can be solved by Picard iteration on the Duhamel formulation). Quasilinear when the flow map is just (locally) continuous (cannot solve the Cauchy problem by Picard iteration on the Duhamel formulation).
- The usual KP-II is semilinear (Bourgain 1993). The usual KP-I is quasilinear (Molinet-JCS-Tzvetkov 2002).
- The fKdV is quasilinear when $\alpha < 2$ (including Benjamin-Ono, and ILW), Molinet-JCS-Tzvetkov 2001.
- For the fKP equations, the situation is a bit surprising (Linares-Pilod-S 2018) :

The fKP-II case

Theorem

Assume $\kappa = 1$ (fKP-II). Let $\alpha \in (0, \frac{4}{3})$ and $(s_1, s_2) \in \mathbb{R}^2$ (resp. $s \in \mathbb{R}$). Then, there exists no $T > 0$ such that (6) admits a unique local solution defined on the time interval $[0, T]$ and such that its flow-map

$$S_t : u_0 \mapsto u(t), \quad t \in (0, T]$$

is C^2 differentiable at zero from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$, (resp. from $X^s(\mathbb{R}^2)$ to $X^s(\mathbb{R}^2)$).

- The ILW-KP-II and BO-KP-II equations are thus quasilinear (cannot be solved by Picard iteration).
- The Hadac result on fKP-II for $\alpha > 4/3$ is thus sharp (he uses Picard iteration in a Bourgain spaces framework so that fKP-II is semilinear when $\alpha > 4/3$).

The fKP-I case

Theorem

Assume $\kappa = -1$ (fKP-I). Let $\alpha \in (0,2]$ and $(s_1, s_2) \in \mathbb{R}^2$ (resp. $s \in \mathbb{R}$). Then, there exists no $T > 0$ such that (6) admits a unique local solution defined on the time interval $[0, T]$ and such that its flow-map

$$S_t : u_0 \mapsto u(t), \quad t \in (0, T]$$

is C^2 differentiable at zero from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$, (resp. from $X^s(\mathbb{R}^2)$ to $X^s(\mathbb{R}^2)$).

- We consider only the cases where $0 < \alpha \leq 2$, but our result for the fKPI case probably holds for $0 < \alpha < \alpha_0$, for some $\alpha_0 > 2$.
- The fifth order ($\alpha = 4$) KP-I equation is semilinear (S-Tzvetkov 2000).
- The proofs follow the lines of Molinet-S-Tvetkov 2002 (KP I) and of Ribaud-Vento 2017 (fZK).

fKP-II case.

- Goal : to prove that the inequality

$$\left\| \int_0^t U_\alpha(t-t')(U_\alpha(t')\phi_1 \partial_x U_\alpha(t')\phi_2)(x,y)\, dt' \right\|_{H^{s_1,s_2}} \lesssim \|\phi_1\|_{H^{s_1,s_2}} \|\phi_2\|_{H^{s_1,s_2}}, \tag{13}$$

does not hold for any $\phi_1, \phi_2 \in H^{s_1,s_2}(\mathbb{R}^2)$ and any $s_1, s_2 \in \mathbb{R}$. This in particular implies that the data-solution map is not C^2.

- Construct sequences of functions $\phi_{i,N}$, $i = 1, 2$, such that for any $s_1, s_2 \in \mathbb{R}$ it holds

$$\|\phi_{i,N}\|_{H^{s_1,s_2}} \leq C \tag{14}$$

and

$$\lim_{N\to\infty} \left\| \int_0^t U_\alpha(t-t')(U_\alpha(t')\phi_{1,N} \partial_x U_\alpha(t')\phi_{2,N})(x,y)\, dt' \right\|_{H^{s_1,s_2}} = +\infty. \tag{15}$$

- Analysis of a resonant function linked to the symbol.

Improved well-posedness

- For any $s \geq 0$, we define the space $X^s(\mathbb{R}^2)$, which is well-adapted to (fKP), by the norm

$$\|f\|_{X^s} := \left(\|J_x^s f\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_x^{-1}\partial_y f\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}},$$

where

$$\widehat{J_x^s f}(\xi, \eta) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi, \eta),$$

and

$$X^\infty(\mathbb{R}^2) = \bigcap_{s \geq 0} X^s(\mathbb{R}^2).$$

- The main result states that for any $\alpha \in (0, 2]$, fKP is LWP in $X^s(\mathbb{R}^2)$, $s > 2 - \frac{\alpha}{4}$.
- We do not distinguish between fKP-I and fKP-II.

Theorem

(Linares-Pilod-S, SIMA 2018)
Let $0 < \alpha \leq 2$*. Define* $s_\alpha := 2 - \frac{\alpha}{4}$ *and assume that* $s > s_\alpha$*. Then, for any* $u_0 \in X^s$*, there exist a positive time* $T = T(\|u_0\|_{X^s})$ *(which can be chosen as a nondecreasing function of its argument) and a unique solution* u *to the IVP* (6) *in the class*

$$C([0,T] : X^s(\mathbb{R}^2)) \cap L^1((0,T) : W^{1,+\infty}(\mathbb{R}^2)) . \tag{16}$$

Moreover, for any $0 < T' < T$*, there exists a neighborhood* $\mathcal{U}$ *of* u_0 *in* $X^s(\mathbb{R}^2)$ *such that the flow map data solution*

$$S^s_{T'} : \mathcal{U} \to C([0,T'] : X^s(\mathbb{R}^2)), \quad u_0 \mapsto u \,,$$

is continuous.

- The proofs works (with some extra technicalities) for more general non-homogeneous symbols, for instance :

For $\alpha > 0$, let $\mathcal{L}_{\alpha+1}$ be the Fourier multiplier defined by

$$\mathcal{F}(\mathcal{L}_{\alpha+1}f)(\xi,\eta) = iw_{\alpha+1}(\xi)\mathcal{F}(f)(\xi,\eta)\,,$$

where $w_{\alpha+1}$ is an odd real-valued function belonging to $C^1(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$ satisfying

$$|w_{\alpha+1}(\xi)| \lesssim 1\,, \quad \forall\,|\xi| \leq \xi_0\,, \tag{17}$$

and

$$|\partial^\beta w_{\alpha+1}(\xi)| \sim |\xi|^{\alpha+1-\beta}, \quad \forall\,|\xi| \geq \xi_0, \quad \forall\,\beta = 0,1,2\,, \tag{18}$$

for some fixed $\xi_0 > 0$.

The following symbols satisfy the conditions (17) and (18) :

- *pure power symbol* $w_{\alpha+1}(\xi) = |\xi|^\alpha \xi$ corresponding to the fractional dispersive operator $\mathcal{L}_{\alpha+1} = D_x^\alpha \partial_x$ with $\alpha > 0$.
- *Whitham with surface tension symbol* $\left(\frac{\tanh \xi}{\xi}\right)^{\frac{1}{2}}(1+b\xi^2)^{\frac{1}{2}}\xi$, with $b > 0$ corresponding to $\alpha = \frac{1}{2}$.
- *intermediate long wave symbol* $\coth(\xi)|\xi|\xi$ corresponding to $\alpha = 1$.

Ideas on the (technical) proof

- The fKP is quasilinear (at least for small α) and one cannot use a fixed point argument on the Duhamel formulation but a compactness method (thus uniqueness and continuity of the flow need an extra argument). We use the strategy of C. Kenig (2004) for the KP-I equation, (see also Koch-Tzvetkov 2003 for the BO equation).
- Technical tools : various commutator and interpolation estimates (Kenig-Ponce-Vega, Muscalu-Pipher-Tao-Thiele,....).
- Linear estimates.
- Estimates on the nonlinear terms.
- Uniqueness and continuity of the flow map.

We just indicate some linear estimates

- Consider the linear IVP

$$\begin{cases} \partial_t u - D_x^\alpha \partial_x u - \kappa \partial_x^{-1} \partial_y^2 u = 0, \quad (x,y) \in \mathbb{R}^2, t > 0 \\ u(x,y,0) = u_0(x,y) \end{cases} \tag{19}$$

where $\kappa = \pm 1$ and whose solution is given by

$$u(x,y,t) = U_\alpha(t) u_0(x,y) := \left(e^{it(|\xi|^\alpha \xi + \kappa \frac{\eta^2}{\xi})} \widehat{u}_0(\xi,\eta) \right)^\vee (x,y). \tag{20}$$

- One has the decay estimate (generalizing JCS 1993 for $\alpha = 2$).

Lemma

For $\alpha \in (0,2]$, one has the decay estimate

$$\| D_x^{\frac{\alpha}{2}-1} U_\alpha(t) \phi \|_{L^\infty(\mathbb{R}^2)} \leq c\, |t|^{-1} \|\phi\|_{L^1}. \tag{21}$$

- By using the classical Stein-Thomas argument, we deduce Strichartz estimates .

Proposition

Let $0 < \alpha \leq 2$. Then, the following estimates hold

$$\| D_x^{\frac{1}{q}(\frac{\alpha}{2}-1)} U_\alpha(t)\phi \|_{L_t^q L_{xy}^r} \leq c \, \|\phi\|_{L^2_{xy}} \tag{22}$$

and

$$\| \int_0^t D_x^{\frac{2}{q}(\frac{\alpha}{2}-1)} U_\alpha(t-t')F(t')\,dt' \|_{L_T^q L_{xy}^r} \leq c \, \|F\|_{L_T^{q'} L_{xy}^{r'}} \tag{23}$$

for

$$\begin{cases} 2 \leq r < \infty \\ 2 \leq q \leq \infty \end{cases} \quad \textit{satisfying} \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{2} \quad \textit{and} \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1\,.$$

Related topics. Open questions
The Full-Dispersion KP equation

- Extension of the improved local well-posedness result to the full-dispersion KP equation ? (see Lannes 2013, Lannes-JCS 2013) :

$$\partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \qquad (24)$$

with

$$\tilde{c}_{WW}(\sqrt{\mu}k) = (1+\beta\mu k^2)^{\frac{1}{2}} \left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2},$$

where μ is a "small" parameter, $\beta \geq 0$ is a dimensionless coefficient measuring the surface tension effects and

$$|D^\mu| = \sqrt{D_1^2 + \mu D_2^2}, \quad D_1 = \frac{1}{i}\partial_x, \quad D_2 = \frac{1}{i}\partial_y.$$

- Idea : to weaken the bad behavior of the KP dispersion at low frequencies in x. The dispersion here is reminiscent of that of the water wave system.
- One gets formally the usual KP equation (KP-II if $\beta = 0$, KP-I if $\beta > 0$) in the long wave limit.
- When $\beta > 0$ Ehrnström and Groves (2018) have shown in this regime the existence of lump like traveling wave solutions of the FDKP equation, reminiscent of the KP-I lump solutions.

- In 1D the FDKP equation reduces to the Whitham equation, a fascinating object that links the KdV equation (in the long wave regime) to a Burgers like equation in the short wave limit.

$$u_t + \mathcal{L}_\epsilon u_x + \epsilon u u_x = 0, \quad \text{no surface tension, } \beta = 0 \tag{25}$$

$\mathcal{L}_\epsilon$ is related to the dispersion relation of the (linearized) water waves system :

$$\mathcal{L}_\epsilon = l(\sqrt{\epsilon}D) := \left(\frac{\tanh \sqrt{\epsilon}|D|}{\sqrt{\epsilon}|D|} \right)^{1/2} \quad \text{and} \quad D = -i\nabla = -i\frac{\partial}{\partial x}\,.$$

- The (small) parameter ϵ measures the comparable effects of nonlinearity and dispersion.
- See Klein-Linares-Pilod-S (2018) for connection with the KdV equation and other properties of the Whitham equation.
- Ehrnström-Groves-Wahlen (2012) : existence of stable solitary wave solutions of the Whitham equation in the long wave regime, close to the KdV soliton.

Global and blow-up issues :

- fKP I is focusing, fKP II defocusing.
- One expects finite blow-up for the fKP I in the L^2- supercritical case $\frac{4}{5} < \alpha < \frac{4}{3}$, as it is the case for the L^2- supercritical gKP I equation :

$$u_t + u^p u_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \quad p \geq \frac{4}{3}.$$

- One expects blow-up (of what type, but probably not a shock ?) in the energy critical or supercritical case $0 < \alpha \leq \frac{4}{5}$.
- $\alpha = 0$ corresponds to the Khokhlov-Zabolotskaya equation for which the existence of shock-like blow-up is known. This should be also the case for $-1 < \alpha < 0$. See Hur (2017) in the 1D case when $-1 < \alpha < -\frac{1}{3}$ and also for the Whitham equation without surface tension ($\beta = 0$).

- Global well-posedness for fKP II :
  Hadac result implies GWP when $\alpha > 4/3$. What happens when $\alpha < \frac{4}{3}$?
- Asymptotics of small solutions (see Harrop Griffiths-Ifrim-Tataru 2014 for the usual KP-I equation and Harrop Griffiths-Marzuola 2018 for the global existence of small solutions and scattering for the KP-II-ILW and KP-II-BO equations).

Systems

- There are system versions of the KP-II ILW or KP-II BO equations describing oblique interactions between internal solitary waves (Grimshaw-Zhu 1994, Matsuno 1998).
- Depending of the nature of the interaction, one can also get a system of two coupled KP-II equations. See S-Tzvetkov 2000 for a mathematical study. See Linares-Pilod-JCS (in progress) for the BO or ILW cases.

The KP version of the Benjamin equation

- When surface tension is not negligible, the Benjamin-Ono equation becomes the Benjamin equation :

$$u_t + uu_x - Hu_{xx} - \beta u_{xxx} = 0, \quad \beta > 0 \tag{26}$$

- The KP version (not known to be integrable)

$$u_t + uu_x - Hu_{xx} - \beta u_{xxx} + \partial_x^{-1} u_{yy} = 0 \tag{27}$$

is quite interesting since it has both focusing (KP-I type) and defocussing (KP-II type) aspects.

- Klein-Linares-Pilod-S (in progress)

Transverse stability issues for the line soliton of the fKdV equation (exist when $\alpha > \frac{1}{3}$).

- It is known (Zakharov, Rousset-Tzvetkov, Mizumachi) that the KdV soliton is transversally stable for KP II and unstable for KP I (at least for not too small velocity).
- What about fKP ? The case $\alpha = 1$ (KP II-BO) or KP II-ILW) is very relevant :
- Conjecture : the BO (or ILW) soliton is transversally stable for KP II-BO (or KP II-ILW). There are some formal considerations in Ablowitz-Segur 1980 and numerics (Klein 2019).
- At the level of the Cauchy problem, the transverse stability issues imply to work either in $\mathbb{R} \times \mathbb{T}$ (y-periodic perturbation, completely open) or in the context of a localized perturbation of the line soliton (probably ok). The KP-BO or KP-ILW cases are of real interest.

Control and stabilization issues for the KP-II-BO and KP-II-ILW equations (linear and non linear).

- See Linares-Ortega (2014) for the linear BO and Linares-Rosier (2015), Laurent-Linares-Rosier (2015) for the BO equation.
- Nothing seems to be known for the ILW equation.
- KP-II : Chenmin Sun-Ivonne Rivas (2017).
- Linear KP-I : Chenmin Sun (2018).
- Nothing seems to be known for the KP-BO and KP-ILW equations.

THANKS FOR YOUR ATTENTION !