On some (non integrable) Kadomtsev-Petviashvili type equations

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Classical and quantum integrability
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Motivation
The Cauchy problem
  General facts
  The linear group
Previous work on KP and fKP
  The Cauchy problem
Semilinear versus quasilinear
Improved well-posedness
  Ideas on the proof
Related topics. Work in progress. Open questions
  The Full-Dispersion KP equation
  Global and blow-up issues
  Systems
  The KP version of the Benjamin equation
  Transverse stability issues for the line soliton of the fKdV equation
Motivation

- The (formal) derivation (Kadomtsev-Petviashvili 1970) of the KP equation is independent of the dispersion in $x$ and concerns only the transport part of the KdV equation

\[
    u_t + u_x + uu_x + \left( \frac{1}{3} - T \right) u_{xxx} = 0 \tag{1}
\]

where $T \geq 0$ is the surface tension parameter.

- More precisely, it consists in looking for a weakly transverse perturbation of the one-dimensional transport equation

\[
    u_t + u_x = 0. \tag{2}
\]
This perturbation is (formally) obtained by a Taylor expansion of the dispersion relation \( \omega(k_1, k_2) = \sqrt{k_1^2 + k_2^2} \) of the two-dimensional linear wave equation assuming \( |k_1| \ll 1 \) and \( \frac{|k_2|}{|k_1|} \ll 1 \).

Namely, one writes formally

\[
\omega(k_1, k_2) \sim \pm k_1 \left(1 + \frac{k_2^2}{2k_1^2}\right)
\]

which amounts, coming back to the physical variables, to adding a nonlocal term to the transport equation,

\[
 u_t + u_x + \frac{1}{2} \partial_x^{-1} u_{yy} = 0. \tag{3}
\]

Here the operator \( \partial_x^{-1} \) is defined via Fourier transform,

\[
\hat{\partial_x^{-1} f}(\xi) = \frac{1}{i\xi_1} \hat{f}(\xi), \text{ where } \xi = (\xi_1, \xi_2).
\]
Assuming that the transverse dispersive effects are of the same order as the $x$-dispersive and nonlinear terms, yields the KP equation

$$u_t + u_x + uu_x + \left(\frac{1}{3} - T\right) u_{xxx} + \frac{1}{2} \partial_x^{-1} u_{yy} = 0.$$  \hspace{1cm} (4)

The KP II equation is obtained when $T < \frac{1}{3}$, the KP I when $T > \frac{1}{3}$.

Note that the KP I case is irrelevant for water waves since $T > \frac{1}{3}$ occurs for a very thin layer of water and then one should take viscous effects into account... (but KP I appears as a long wave limit of the Gross-Pitaevskii or other dispersive equations for instance).

Rigorous derivation: David Lannes, 2002 with a bad error estimate for

\[ u_t + u_x + \epsilon uu_x + \epsilon u_{xxx} + \epsilon \partial_x^{-1} u_{yy} = 0, \quad \epsilon \ll 1, \]

\[ ||u_{Euler} - u_{KP}|| = o(1). \]

This is the price to pay for the singularity at \( \xi_1 = 0 \).

Optimal error estimate for the KdV equation

\[ u_t + u_x + \epsilon uu_x + \epsilon u_{xxx} = 0, \]

\[ ||u_{Euler} - u_{KdV}|| = O(\epsilon^2 t). \]
Outline

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Figure – Interaction of line solitons. Oregon coast
It is thus quite natural to apply this formal process to any KdV type equation, in particular to the fractional (fKdV) equation

\[ u_t + (u_x) + uu_x \pm D_x^\alpha u_x = 0, \quad \hat{D}_x^\alpha f(\xi) = |\xi|^{\alpha} \hat{f}(\xi), \tag{5} \]

(\alpha = 2 : KdV ; \alpha = 1 : Benjamin-Ono)

to get the fractional KP (fKP) equations

\[ u_t + (u_x) + uu_x - D_x^\alpha u_x + \kappa \partial_x^{-1} u_{yy} = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad u(\cdot, 0) = u_0, \tag{6} \]

where \( \kappa = \pm 1 \) and \( \alpha > -1 \) (but we will focus on \( 0 < \alpha < 2 \)).

The sign of the x-dispersive term in (5) will determine the fKP I or fKP II types.
fKP with $\alpha = 1$ is the relevant KP version of the Benjamin-Ono equation (KP-BO). Very similar to the KP-BO equation is the KP version of the intermediate long wave equation (ILW):

$$u_t + uu_x - \mathcal{L}_{ILW} u_x = 0,$$

where $\mathcal{L}_{ILW} f(\xi) = p_{ILW}(\xi) \hat{f}(\xi)$, $p_{ILW}(\xi) = \xi \coth(\delta \xi) - \frac{1}{\delta}$, $\delta > 0$.

The ILW equation is a model for long, weakly nonlinear internal waves, $\delta$ being proportional to the depth of the bottom layer. See Bona-Lannes-S, 2008 and below for a rigorous derivation (in the sense of consistency) of the ILW and related equations.

The Benjamin–Ono equation is obtained in the infinite depth limit, $\delta \to +\infty$.

In this context we are in the fKP II case (adding $\partial_x^{-1} u_{yy}$).

See Ablowitz-Segur (1980) and Camassa-Choi (1999) for a formal derivation of KP-II-ILW and KP-II BO for internal waves.

One can justify (in the sense of consistency with the full internal waves system) the KP-BO or KP-ILW equations following the procedure for the usual KP equation (see Lannes 2002)
The two fluid system with rigid lid

\[ \gamma = \frac{\rho_1}{\rho_2} < 1. \]
Many regimes depending on amplitudes, wavelengths,..

The range of validity of the various regimes is summarized in the following table.

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<thead>
<tr>
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<th>$\varepsilon = O(1)$</th>
<th>$\varepsilon \ll 1$</th>
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<tbody>
<tr>
<td>$\mu = O(1)$</td>
<td>Full equations</td>
<td>$\delta \sim 1 : FD/FD$ eq’ns</td>
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<tr>
<td>$\mu \ll 1$</td>
<td>$\delta \sim 1 : SW/SW$ eq’ns</td>
<td>$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon : B/FD$ eq’ns</td>
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<td></td>
<td>$\delta^2 \sim \mu \sim \varepsilon^2 : SW/FD$ eq’ns</td>
<td>$\mu \sim \varepsilon$ and $\delta \sim 1 : B/B$ eq’ns</td>
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<td>$\delta^2 \sim \mu \sim \varepsilon^2 : ILW$ eq’ns</td>
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<td>$\delta = 0$ and $\mu \sim \varepsilon^2 : BO$ eq’ns</td>
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\[
\gamma = \frac{\rho_1}{\rho_2}, \quad \delta = \frac{d_1}{d_2}, \quad \varepsilon = \frac{a}{d_1}, \quad \mu = \frac{d_1^2}{\lambda^2}, \quad \varepsilon_2 = \frac{a}{d_2} = \varepsilon \delta, \quad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.
\]
One starts from the following "ILW" system that is consistent with the full internal wave system (see Bona-Lannes-S 2008)

\[
\begin{align*}
\partial_t \zeta + \frac{1}{\gamma} \nabla \cdot ((1 - \epsilon \zeta) \mathbf{v}) & - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2} |D|) \nabla \cdot \mathbf{v} = 0, \\
\partial_t \mathbf{v} + (1 - \gamma) \nabla \zeta - \frac{\epsilon}{2\gamma} \nabla |\mathbf{v}|^2 &= 0,
\end{align*}
\]

(8)

where \( \mathbf{v} \) is the velocity field, \( \zeta \) the elevation, \( \gamma = \frac{\rho_1}{\rho_2} < 1 \) the ratio of densities and \( \mu \sim \epsilon^2 \ll 1 \). \( \mu_2 \) is a "large parameter related to the depth of the lower layer. When \( \mu_2 \to +\infty \) one gets a Benjamin-Ono type system.

\[
\begin{align*}
\partial_t \zeta + \frac{1}{\gamma} \nabla \cdot ((1 - \epsilon \zeta) \mathbf{v}) - \frac{\sqrt{\mu}}{\gamma^2} |D| \nabla \cdot \mathbf{v} &= 0, \\
\partial_t \mathbf{v} + (1 - \gamma) \nabla \zeta - \frac{\epsilon}{2\gamma} \nabla |\mathbf{v}|^2 &= 0.
\end{align*}
\]

(9)

One deduces ILW and BO in the 1D case after the one-way propagation assumption.

Then one looks for solutions satisfying the KP scaling via a suitable ansatz.
Why fKP equations?

- Contain some physically relevant equations (KP-BO, KP-ILW,...).
- To study the influence of dispersion on the space of resolution, on the lifespan, the possible blow-up and on the dynamics of solutions to the Cauchy problem for “weak” dispersive perturbations of hyperbolic quasilinear equations or systems.
- In this context it is more natural to keep the quadratic nonlinearity and weaken the dispersion than to keep the KdV dispersion and increase the nonlinearity...
- None of the fKP equations seems to be completely integrable except of course the classical ones, $\alpha = 2$...
The Khokhlov-Zabolotskaya (KZ) equation (1969)

\[ u_t + uu_x + \partial_x^{-1} \Delta_\perp u = 0, \quad \Delta_\perp = \partial_{yy} \text{ or } \partial_{yy} + \partial_{zz}, \quad \text{(nonlinear acoustics)} \]

- Also derived by Alinhac (1996) to describe the blow-up of solutions to some quasilinear hyperbolic equations.
- Easy : LWP in \( H_{-1}^s(\mathbb{R}^2) \), \( s > 2 \) where
  \[
  H_{-1}^s(\mathbb{R}^2) = \{ f \in H^s(\mathbb{R}^2) : \mathcal{F}^{-1}\left(\frac{\hat{f}(\xi,\eta)}{\xi}\right) \in H^s(\mathbb{R}^2) \}. 
  \]
- But ill posed in \( H_{-1}^2(\mathbb{R}^2) \).
- "Hyperbolic nature" : Alinhac (1996), blow-up of \( \sup_{x,y} \partial_x u(x,y,t) \) in finite time. See also Rozanova-Pierrat 2008 (also with a dissipative term), Alterman-Rauch 2001.
- Look at fKP with small \( \alpha \) (possibly negative) to understand the effect of a weak dispersion in \( x \) on the KZ dynamics.
General facts on fKP equations

In addition to the $L^2$ norm, the fKP equation (6) conserves formally the energy (Hamiltonian)

$$H_\alpha(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |D_x^{\alpha/2} u|^2 - \kappa \frac{1}{2} |\partial_x^{-1} u_y|^2 - \frac{1}{6} u^3 \right).$$

The corresponding energy space is

$$Y_\alpha = \{ u \in L^2(\mathbb{R}^2) : D_x^{\alpha/2} u, \partial_x^{-1} u_y \in L^2(\mathbb{R}^2) \}.$$

One checks readily that the transformation

$$u_\lambda(x, y, t) = \lambda^{\alpha} u(\lambda x, \lambda^{\alpha/2} y, \lambda^{\alpha+1} t)$$

leaves (6) invariant.

Moreover, $|u_\lambda|_2 = \lambda^{3\alpha-4} |u|_2$, so that $\alpha = \frac{4}{3}$ is the $L^2$ critical exponent. Note that the BO-KP and the ILW-KP equations are $L^2$ supercritical.
Recall the fractional Gagliardo-Nirenberg inequality:

**Lemma**

Let $\frac{4}{5} \leq \alpha < 1$. For any $f \in Y_\alpha$ one has

$$|f|^3_3 \leq c|f|^\frac{5\alpha-4}{\alpha+2} H^\alpha_{x^2} \left(\frac{18-5\alpha}{2(\alpha+2)} \right) \|f\|_{H^\alpha_{x^2}} \left(\frac{1}{2}\right) |\partial_x^{-1} f_y|^{\frac{1}{2}},$$

where $\|\cdot\|_{H^\alpha_{x^2}}$ denotes the natural norm on the space

$$H^\alpha_{x^2}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : D_x^\alpha f \in L^2(\mathbb{R}^2)\}.$$

- Lemma 1 implies obviously the embedding $Y_\alpha \hookrightarrow L^3(\mathbb{R}^2)$ if $\alpha \geq \frac{4}{5}$.

- The energy critical value $\frac{4}{5}$ of $\alpha$ is related to the non existence of localized solitary waves. By Pohozaev type arguments:

- Assume that $0 < \alpha \leq \frac{4}{5}$ when $\kappa = -1$ or that $\alpha$ is arbitrary when $\kappa = 1$. Then (6) does not possess non trivial solitary waves $u(x - ct, y)$ in the space $Y_\alpha \cap L^3(\mathbb{R}^2)$.
An easy result

- Viewing (6) as a skew-adjoint perturbation of the Burgers equation, one easily establishes the elementary local well-posedness result

\[(H^s_{-1}(\mathbb{R}^2) = \{ f \in H^s(\mathbb{R}^2), \mathcal{F}^{-1}(\hat{f}/\xi_1) \in H^s(\mathbb{R}^2) \}).\]

**Theorem**

Let \( u_0 \in H^s_{-1}(\mathbb{R}^2), s > 2. \) Then there exists \( T = T(\|u_0\|_s > 0) \) and a unique solution \( u \in C([0, T]; H^s_{-1}(\mathbb{R}^2)), u_t \in C([0, T]; H^{s-3}(\mathbb{R}^2)). \)

Furthermore, the map \( u_0 \mapsto u \) is continuous from \( H^s_{-1}(\mathbb{R}^2) \) to \( C([0, T]; H^s_{-1}(\mathbb{R}^2)). \) Moreover \( \|u(., t)\|_2 \) and \( H_\alpha(u(., t)) \) are conserved on \([0, T].\)

- This result does not depend on the dispersion (and the exponent 2 is the "hyperbolic" one...). A goal of this talk is to see how dispersion can help to enlarge the space of resolution.
For the fKP-I equation with $\alpha > \frac{3}{2}$ one obtains the global existence of a weak solution $u \in L^\infty(\mathbb{R}; Y^\alpha)$, where

$$Y^\alpha = \{ f \in L^2(\mathbb{R}^2), D_x^{\alpha/2} f \in L^2(\mathbb{R}^2), \partial_x^{-1} \partial_y f \in L^2(\mathbb{R}^2) \}$$

is the energy space.

This results from a standard compactness method, using the Gagliardo-Nirenberg inequality.
The linear group

\[ u_t - D^\alpha u_x \pm \partial_x^{-1} u_{yy} = 0. \] (11)

- The linear part in (6) defines, for any \( \alpha > -1 \), a unitary group \( U_\alpha(t) \) in \( L^2(\mathbb{R}^2) \) and all \( H^s(\mathbb{R}^2) \) Sobolev spaces, unitarily equivalent via Fourier transform to the Fourier multiplier

\[ e^{it(|\xi_1|^\alpha \xi_1 \mp \xi_2^2 / \xi_1)}. \]

- Local smoothing for the linear fKP II (see JCS 1993 for the usual KP II equation):

**Proposition**

Let \( \alpha > \frac{1}{2} \) and \( s \geq 0 \), \( u_0 \in H_{-1}^s(\mathbb{R}^2) \). Then the solution \( u \) of (11) with the + sign satisfies for any \( R > 0 \) and \( T > 0 \)

\[ |D_1|^{\alpha/2} D_1^{s_1} D_2^{s_2} u, \quad D_1^{s_1} D_2^{s_2} \partial_x^{-1} u_y \in L^2((-T, T) \times (-R, R) \times \mathbb{R}), \quad s_1 + s_2 = s. \]
Previous work on KP and fKP

1. ”PDE” methods for the Cauchy problem


- M. Hadac (2008): local-well-posedness of the fKP-II equation in the $L^2$–subcritical case $\alpha > \frac{4}{3}$ in the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > \max(1 - \frac{3}{4}\alpha, \frac{1}{4} - \frac{3}{8}\alpha)$, $s_2 \geq 0$. 
No much is known concerning the long time behavior of global solutions of the classical KP-I and KP-II equations:

- For KP-II, scattering of solutions is expected but still open (see eg simulations in Klein-S 2012).
2. IST methods for the Cauchy problem

▶ The only known rigorous result are for small initial data in suitable spaces of regular functions:


▶ To get rid of the smallness conditions (and to provide an asymptotics of solutions) is a challenging open problem.
Solitary waves

- KP-I possesses an explicit, finite energy, localized solitary wave (lump):

\[
\phi_c(x - ct, y) = \frac{8c(1 - \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)}{(1 + \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)^2}.
\] (12)

- No such solution exists for KP-II (de Bouard-S 1997).

- **Ground states.** We set

\[
E_{KP}(\psi) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \psi)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_y \psi)^2 - \frac{1}{2(p + 2)} \int_{\mathbb{R}^2} \psi^3,
\]

and we define the action

\[
S(N) = E_{KP}(N) + \frac{c}{2} \int_{\mathbb{R}^2} N^2.
\]
A **ground state** is a solitary wave $N$ which minimizes the action $S$ among all finite energy non-constant solitary waves of speed $c$.

Ground states exist if and only if $c > 0$. Moreover, the ground states are minimizers of the Hamiltonian $E_{KP}$ with prescribed mass ($L^2$ norm) (de Bouard-S 1997).

The set of ground states is orbitally stable (de Bouard-S 1997). Uniqueness (up to obvious symmetries is open).

One does not know whether or not the lump is a ground state (presumably yes..)
Recent nondegeneracy result (Yong Liu,-Junchang Wei, ARMA 2019):
Let $Q$ be the lump and $\phi$ a smooth solution of

$$\partial_x^2(\partial_x^2 \phi - \phi + 6Q\phi) - \partial_y^2 \phi = 0.$$ 

Assume

$$\phi(x, y) \to 0, \quad \text{as} \quad x^2 + y^2 \to +\infty.$$ 

Then

$$\phi = c_1 \partial_x Q + c_2 \partial_y Q,$$ 

for some $c_1, c_2$. 
Semilinear versus quasilinear

- The distinction between semilinear and quasilinear is not obvious for dispersive equations.

- Roughly speaking: **Semilinear** when the flow map is smooth (the Cauchy problem can be solved by Picard iteration on the Duhamel formulation). **Quasilinear** when the flow map is just (locally) continuous (cannot solve the Cauchy problem by Picard iteration on the Duhamel formulation).

- The usual KP-II is semilinear (Bourgain 1993). The usual KP-I is quasilinear (Molinet-JCS-Tzvetkov 2002).

- The fKdV is quasilinear when $\alpha < 2$ (including Benjamin-Ono, and ILW), Molinet-JCS-Tzvetkov 2001.

- For the fKP equations, the situation is a bit surprising (Linares-Pilod-S 2018):
The fKP-II case

Theorem
Assume $\kappa = 1$ (fKP-II). Let $\alpha \in (0, \frac{4}{3})$ and $(s_1, s_2) \in \mathbb{R}^2$ (resp. $s \in \mathbb{R}$). Then, there exists no $T > 0$ such that (6) admits a unique local solution defined on the time interval $[0, T]$ and such that its flow-map

$$S_t : u_0 \mapsto u(t), \quad t \in (0, T]$$

is $C^2$ differentiable at zero from $H^{s_1, s_2}(\mathbb{R}^2)$ to $H^{s_1, s_2}(\mathbb{R}^2)$, (resp. from $X^s(\mathbb{R}^2)$ to $X^s(\mathbb{R}^2)$).

- The ILW-KP-II and BO-KP-II equations are thus quasilinear (cannot be solved by Picard iteration).
- The Hadac result on fKP-II for $\alpha > 4/3$ is thus sharp (he uses Picard iteration in a Bourgain spaces framework so that fKP-II is semilinear when $\alpha > 4/3$).
The fKP-I case

Theorem

Assume $\kappa = -1$ (fKP-I). Let $\alpha \in (0, 2]$ and $(s_1, s_2) \in \mathbb{R}^2$ (resp. $s \in \mathbb{R}$). Then, there exists no $T > 0$ such that (6) admits a unique local solution defined on the time interval $[0, T]$ and such that its flow-map

$$S_t : u_0 \mapsto u(t), \quad t \in (0, T]$$

is $C^2$ differentiable at zero from $H^{s_1, s_2}(\mathbb{R}^2)$ to $H^{s_1, s_2}(\mathbb{R}^2)$, (resp. from $X^s(\mathbb{R}^2)$ to $X^s(\mathbb{R}^2)$).

- We consider only the cases where $0 < \alpha \leq 2$, but our result for the fKPI case probably holds for $0 < \alpha < \alpha_0$, for some $\alpha_0 > 2$.
- The fifth order ($\alpha = 4$) KP-I equation is semilinear (S-Tzvetkov 2000).
- The proofs follow the lines of Molinet-S-Tvetkov 2002 (KP I) and of Ribaud-Vento 2017 (fZK).
fKP-II case.

- **Goal**: to prove that the inequality

\[
\left\| \int_0^t U_\alpha(t-t')(U_\alpha(t')\phi_1 \partial_x U_\alpha(t')\phi_2)(x,y) \, dt' \right\|_{H^{s_1,s_2}} \lesssim \|\phi_1\|_{H^{s_1,s_2}} \|\phi_2\|_{H^{s_1,s_2}},
\]

(13)
does not hold for any \(\phi_1, \phi_2 \in H^{s_1,s_2}(\mathbb{R}^2)\) and any \(s_1, s_2 \in \mathbb{R}\). This in particular implies that the data-solution map is not \(C^2\).

- **Construct sequences of functions** \(\phi_{i,N}, i = 1, 2,\) such that for any \(s_1, s_2 \in \mathbb{R}\) it holds

\[
\|\phi_{i,N}\|_{H^{s_1,s_2}} \le C
\]

(14)

and

\[
\lim_{N \to \infty} \left\| \int_0^t U_\alpha(t-t')(U_\alpha(t')\phi_{1,N} \partial_x U_\alpha(t')\phi_{2,N})(x,y) \, dt' \right\|_{H^{s_1,s_2}} = +\infty.
\]

(15)

- **Analysis of a resonant function linked to the symbol.**
Improved well-posedness

For any \( s \geq 0 \), we define the space \( X^s(\mathbb{R}^2) \), which is well-adapted to (fKP), by the norm

\[
\| f \|_{X^s} := \left( \| J_x^s f \|_{L^2(\mathbb{R}^2)}^2 + \| \partial_x^{-1} \partial_y f \|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}},
\]

where

\[
\widehat{J_x^s f}(\xi, \eta) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi, \eta),
\]

and

\[
X^{\infty}(\mathbb{R}^2) = \bigcap_{s \geq 0} X^s(\mathbb{R}^2).
\]

The main result states that for any \( \alpha \in (0, 2] \), fKP is LWP in \( X^s(\mathbb{R}^2), s > 2 - \frac{\alpha}{4} \).

We do not distinguish between fKP-I and fKP-II.
Theorem  
(Linares-Pilod-S, SIMA 2018)  
Let $0 < \alpha \leq 2$. Define $s_\alpha := 2 - \frac{\alpha}{4}$ and assume that $s > s_\alpha$. Then, for any $u_0 \in X^s$, there exist a positive time $T = T(\|u_0\|_{X^s})$ (which can be chosen as a nondecreasing function of its argument) and a unique solution $u$ to the IVP (6) in the class  

$$C([0, T] : X^s(\mathbb{R}^2)) \cap L^1((0, T) : W^{1, +\infty}(\mathbb{R}^2)).$$  

(16)

Moreover, for any $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of $u_0$ in $X^s(\mathbb{R}^2)$ such that the flow map data solution  

$$S^s_{T'} : \mathcal{U} \to C([0, T'] : X^s(\mathbb{R}^2)), \quad u_0 \mapsto u,$$

is continuous.
The proofs work (with some extra technicalities) for more general non-homogeneous symbols, for instance:

For $\alpha > 0$, let $\mathcal{L}_{\alpha+1}$ be the Fourier multiplier defined by

$$\mathcal{F}(\mathcal{L}_{\alpha+1}f)(\xi, \eta) = iw_{\alpha+1}(\xi)\mathcal{F}(f)(\xi, \eta),$$

where $w_{\alpha+1}$ is an odd real-valued function belonging to $C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ satisfying

$$|w_{\alpha+1}(\xi)| \lesssim 1, \quad \forall |\xi| \leq \xi_0,$$

and

$$|\partial^\beta w_{\alpha+1}(\xi)| \sim |\xi|^{|\alpha+1-\beta|}, \quad \forall |\xi| \geq \xi_0, \quad \forall \beta = 0, 1, 2,$$

for some fixed $\xi_0 > 0$. The following symbols satisfy the conditions (17) and (18):

- **Pure power symbol** $w_{\alpha+1}(\xi) = |\xi|^{\alpha}\xi$ corresponding to the fractional dispersive operator $\mathcal{L}_{\alpha+1} = D_x^{\alpha}\partial_x$ with $\alpha > 0$.

- **Whitham with surface tension symbol** $\left(\frac{\tanh \xi}{\xi}\right)^{\frac{1}{2}} (1 + b\xi^2)^{\frac{1}{2}}\xi$, with $b > 0$ corresponding to $\alpha = \frac{1}{2}$.

- **Intermediate long wave symbol** $\text{coth}(\xi)|\xi|\xi$ corresponding to $\alpha = 1$. 

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Ideas on the (technical) proof

- The fKP is *quasilinear* (at least for small $\alpha$) and one cannot use a fixed point argument on the Duhamel formulation but a compactness method (thus uniqueness and continuity of the flow need an extra argument). We use the strategy of C. Kenig (2004) for the KP-I equation, (see also Koch-Tzvetkov 2003 for the BO equation).

- Technical tools: various commutator and interpolation estimates (Kenig-Ponce-Vega, Muscalu-Pipher-Tao-Thiele, ...).

- Linear estimates.

- Estimates on the nonlinear terms.

- Uniqueness and continuity of the flow map.
We just indicate some linear estimates

- Consider the linear IVP

\[
\begin{aligned}
&\partial_t u - D_x^\alpha \partial_x u - \kappa \partial_x^{-1} \partial_y^2 u = 0, \quad (x, y) \in \mathbb{R}^2, \ t > 0 \\
u(x, y, 0) = u_0(x, y)
\end{aligned}
\]  

where \( \kappa = \pm 1 \) and whose solution is given by

\[
u(x, y, t) = U_\alpha(t)u_0(x, y) := \left( e^{it(|\xi|^\alpha \xi + \kappa \frac{\eta^2}{\xi} )} \hat{u}_0(\xi, \eta) \right)^\vee (x, y). \tag{20}\]

- One has the decay estimate (generalizing JCS 1993 for \( \alpha = 2 \)).

**Lemma**

*For \( \alpha \in (0, 2] \), one has the decay estimate*

\[
\| D_x^{\frac{\alpha}{2} - 1} U_\alpha(t) \phi \|_{L^\infty(\mathbb{R}^2)} \leq c |t|^{-1} \| \phi \|_{L^1}. \tag{21}\]
By using the classical Stein-Thomas argument, we deduce Strichartz estimates.

**Proposition**

Let $0 < \alpha \leq 2$. Then, the following estimates hold

\[
\| D_x^{\frac{1}{2} (\frac{\alpha}{2} - 1)} U_\alpha(t) \phi \|_{L^q_t L^r_{xy}} \leq c \| \phi \|_{L^2_{xy}} \tag{22}
\]

and

\[
\| \int_0^t D_x^{\frac{2}{q} (\frac{\alpha}{2} - 1)} U_\alpha(t - t') F(t') \, dt' \|_{L^q_T L^r_{xy}} \leq c \| F \|_{L^{q'}_T L^{r'}_{xy}} \tag{23}
\]

for

\[
\left\{ \begin{array}{l}
2 \leq r < \infty \\
2 \leq q \leq \infty
\end{array} \right. \quad \text{satisfying} \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{2} \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1.
\]
Related topics. Open questions

The Full-Dispersion KP equation

- Extension of the improved local well-posedness result to the full-dispersion KP equation? (see Lannes 2013, Lannes-JCS 2013):

\[ \partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1 + \mu \frac{D_2^2}{D_1^2})^{1/2} u_x + \mu \frac{3}{2} uu_x = 0, \quad (24) \]

with

\[ \tilde{c}_{WW}(\sqrt{\mu}k) = (1 + \beta \mu k^2)^{1/2} \left( \frac{\tanh \sqrt{\mu}k}{\sqrt{\mu}k} \right)^{1/2}, \]

where \( \mu \) is a "small" parameter, \( \beta \geq 0 \) is a dimensionless coefficient measuring the surface tension effects and

\[ |D^\mu| = \sqrt{D_1^2 + \mu D_2^2}, \quad D_1 = \frac{1}{i} \partial_x, \quad D_2 = \frac{1}{i} \partial_y. \]
Idea: to weaken the bad behavior of the KP dispersion at low frequencies in $x$. The dispersion here is reminiscent of that of the water wave system.

One gets formally the usual KP equation (KP-II if $\beta = 0$, KP-I if $\beta > 0$) in the long wave limit.

When $\beta > 0$ Ehrnström and Groves (2018) have shown in this regime the existence of lump like traveling wave solutions of the FDKP equation, reminiscent of the KP-I lump solutions.
In 1D the FDKP equation reduces to the Whitham equation, a fascinating object that links the KdV equation (in the long wave regime) to a Burgers like equation in the short wave limit.

\[ u_t + \mathcal{L}_\epsilon u_x + \epsilon uu_x = 0, \quad \text{no surface tension, } \beta = 0 \]  

(25)

\[ \mathcal{L}_\epsilon \text{ is related to the dispersion relation of the (linearized) water waves system :} \]

\[ \mathcal{L}_\epsilon = l(\sqrt{\epsilon}D) := \left(\frac{\tanh \sqrt{\epsilon}|D|}{\sqrt{\epsilon}|D|}\right)^{1/2} \quad \text{and} \quad D = -i\nabla = -i \frac{\partial}{\partial x}. \]

The (small) parameter \( \epsilon \) measures the comparable effects of nonlinearity and dispersion.

See Klein-Linares-Pilod-S (2018) for connection with the KdV equation and other properties of the Whitham equation.

Ehrnström-Groves-Wahlen (2012) : existence of stable solitary wave solutions of the Whitham equation in the long wave regime, close to the KdV soliton.
Global and blow-up issues:

- fKP I is focusing, fKP II defocusing.
- One expects finite blow-up for the fKP I in the $L^2$—supercritical case $\frac{4}{5} < \alpha < \frac{4}{3}$, as it is the case for the $L^2$—supercritical gKP I equation:

$$u_t + u^p u_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \quad p \geq \frac{4}{3}.$$  

- One expects blow-up (of what type, but probably not a shock?) in the energy critical or supercritical case $0 < \alpha \leq \frac{4}{5}$.

- $\alpha = 0$ corresponds to the Khokhlov-Zabolotskaya equation for which the existence of shock-like blow-up is known. This should be also the case for $-1 < \alpha < 0$. See Hur (2017) in the 1D case when $-1 < \alpha < -\frac{1}{3}$ and also for the Whitham equation without surface tension ($\beta = 0$).
Global well-posedness for fKP II:
Hadac result implies GWP when $\alpha > 4/3$. What happens when $\alpha < \frac{4}{3}$?

Systems

- There are system versions of the KP-II ILW or KP-II BO equations describing oblique interactions between internal solitary waves (Grimshaw-Zhu 1994, Matsuno 1998).

- Depending of the nature of the interaction, one can also get a system of two coupled KP-II equations. See S-Tzvetkov 2000 for a mathematical study. See Linares-Pilod-JCS (in progress) for the BO or ILW cases.
The KP version of the Benjamin equation

When surface tension is not negligible, the Benjamin-Ono equation becomes the Benjamin equation:

$$u_t + uu_x - Hu_{xx} - \beta u_{xxx} = 0, \quad \beta > 0$$  \hspace{1cm} (26)

The KP version (not known to be integrable)

$$u_t + uu_x - Hu_{xx} - \beta u_{xxx} + \partial_x^{-1} u_{yy} = 0$$  \hspace{1cm} (27)

is quite interesting since it has both focusing (KP-I type) and defocussing (KP-II type) aspects.

Klein-Linares-Pilod-S (in progress)
Transverse stability issues for the line soliton of the fKdV equation (exist when $\alpha > \frac{1}{3}$).

- It is known (Zakharov, Rousset-Tzvetkov, Mizumachi) that the KdV soliton is transversally stable for KP II and unstable for KP I (at least for not too small velocity).

- What about fKP? The case $\alpha = 1$ (KP II-BO) or KP II-ILW) is very relevant:
  - Conjecture: the BO (or ILW) soliton is transversally stable for KP II-BO (or KP II-ILW). There are some formal considerations in Ablowitz-Segur 1980 and numerics (Klein 2019).
  - At the level of the Cauchy problem, the transverse stability issues imply to work either in $\mathbb{R} \times \mathbb{T}$ (y-periodic perturbation, completely open) or in the context of a localized perturbation of the line soliton (probably ok). The KP-BO or KP-ILW cases are of real interest.
Control and stabilization issues for the KP-II-BO and KP-II-ILW equations (linear and non linear).

- Nothing seems to be known for the ILW equation.
- Nothing seems to be known for the KP-BO and KP-ILW equations.
THANKS FOR YOUR ATTENTION!