On a $\partial, \overline{\partial}$ system with a large parameter

Johannes Sjöstrand
IMB, Université de Bourgogne

Joint work in progress with C. Klein and N. Stoilov,
Classical and quantum integrability
IMB, Dijon, 2–6 juin, 2019

1. Introduction

This talk is motivated by questions concerning the Davey Stewartson II equation and my own knowledge is very limited, so I will only speak about two problems, where I have been involved. In the DSII theory appears the following problem on $\mathbf{C} \simeq \mathbf{R}^2$:

$$\begin{cases} \overline{\partial}\phi_1 = \frac{q}{2} e^{\overline{kz}-kz}\phi_2, \\ \partial\phi_2 = \frac{\overline{q}}{2} e^{kz-\overline{kz}}\phi_1, \end{cases} \tag{1}$$

$$\phi_1(z) = 1 + o(1),\ \phi_2(z) = o(1),\ |z| \to \infty. \tag{2}$$

Here $k \in \mathbf{C}$ and q is a potential which is small near infinity,

$$\partial = \partial_z = \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right),\ z = x + iy.$$

Existence and uniqueness have been established by P. Perry [Pe16] A. Nachman, I. Regev, D. Tataru [NaReTa17]. Here we focus the asymptotic behaviour when $|k| \to \infty$.

Plan:

- The $\overline{\partial}$ operator with polynomial weights; Hörmander - Carleman approach.
- The convergence of a perturbation series solution when $|k| \to \infty$, provided $q = \mathcal{O}(\langle z \rangle^{-2})$ is smooth.
- The (would be) leading correction to ϕ_2 when $q = 1_\Omega$, $\Omega \Subset \mathbf{C}$ is strictly convex, $\partial\Omega \in C^\infty$.
- Some numerical illustrations.

2. $\overline{\partial}$ on **C** with polynomial weights.

This is very classical (Hörmander–Carleman). Put

$$\Phi(z) := \ln\langle z\rangle^2,\ \langle z\rangle := (1+z\overline{z})^{1/2}.$$

Let $\epsilon > 0$ and put

$$P_\epsilon := \langle\cdot\rangle^{-\epsilon}\circ\overline{\partial}\circ\langle\cdot\rangle^{\epsilon} = \overline{\partial} + \epsilon\overline{\partial}\Phi/2,$$

$$P_\epsilon^* = \langle\cdot\rangle^{\epsilon}\circ(-\partial)\circ\langle\cdot\rangle^{-\epsilon} = -\partial + \epsilon\partial\Phi/2,$$

$$[P_\epsilon, P_\epsilon^*] = \epsilon\partial\overline{\partial}\Phi = \epsilon\langle z\rangle^{-4} > 0.$$

$$P_\epsilon P_\epsilon^* \geq P_\epsilon P_\epsilon^* - P_\epsilon^* P_\epsilon = \frac{\epsilon}{\langle\cdot\rangle^4}$$

This implies:

- An apriori estimate for P_ϵ^*,
- An existence result for P_ϵ,
- An existence result for $\overline{\partial}$:

Proposition

Let $\epsilon > 0$. *For every* $v \in \langle\cdot\rangle^{\epsilon-2}L^2$, *there exists* $u \in \langle\cdot\rangle^{\epsilon}L^2$ *such that*

$$\overline{\partial}u = v \text{ and } \|\langle\cdot\rangle^{-\epsilon}u\| \leq \epsilon^{-1/2}\|\langle\cdot\rangle^{2-\epsilon}v\|.$$

Proposition

When $0 < \epsilon \leq 1$ *the solution is unique and given by*

$$u(z) = \frac{1}{\pi}\int \frac{v(w)}{z-w}L(dw), \; L(dw) \simeq d\Re w \wedge d\Im w = \frac{d\overline{w}\wedge dw}{2i}$$

Semi-classical point of view

Let $0 < h \ll 1$ and consider $h\overline{\partial}u = v$, i.e. $\overline{\partial}u = v/h$. In addition to loosing two powers of $\langle z \rangle$, we then also loose one power of h. However, if $\chi \in C_0^\infty(\mathbf{R}^2)$ is equal to 1 near 0, then from $h\overline{\partial}u = v$, we get $h\overline{\partial}(1-\chi(hD))u = (1-\chi(hD))v$, $hD = hD_{x,y}$. The symbol $i\overline{\zeta} := (i/2)(\xi + i\eta)$ of $\overline{\partial}$ is $\neq 0$ on $\mathrm{supp}\,(1-\chi(\xi,\eta))$ and one can show that

$$\|\langle\cdot\rangle^{2-\epsilon}(1-\chi(hD))u\| \leq \mathcal{O}(1)\|\langle\cdot\rangle^{2-\epsilon}(1-\chi(hD))v\|.$$

3. Application to the $\partial, \overline{\partial}$ system

Let $q \in C^\infty(\mathbf{C})$ with

$$\partial_x^\alpha \partial_y^\beta q = \mathcal{O}(\langle z \rangle^{-2}), \ \alpha, \beta \in \mathbf{N}. \tag{3}$$

Let $k = k_x + ik_y \in \mathbf{C}$ with $|k| \gg 1$ and write

$$kz - \overline{kz} = \frac{i}{h}\langle (x,y), \omega \rangle_{\mathbf{R}^2}, \quad h = \frac{1}{|k|}, \ \omega = 2i\frac{\overline{k}}{|k|}.$$

Writing $\widehat{\tau}_\omega u = e^{\frac{i}{h}\langle (x,y),\omega\rangle} u$ (translation by ω on the h Fourier transform side), the system (1) becomes

$$\begin{cases} h\overline{\partial}\phi_1 - \widehat{\tau}_{-\omega} h\frac{q}{2}\phi_2 = 0, \\ h\partial\phi_2 - \widehat{\tau}_\omega h\frac{\overline{q}}{2}\phi_1 = 0 \end{cases} \tag{4}$$

Trying $\phi_1^0 = 1$, $\phi_2^0 = 0$ gives

$$\begin{cases} h\overline{\partial}\phi_1^0 - \widehat{\tau}_{-\omega} h\frac{q}{2}\phi_2^0 = 0, \\ h\partial\phi_2^0 - \widehat{\tau}_\omega h\frac{\overline{q}}{2}\phi_1^0 = -\widehat{\tau}_\omega \frac{h\overline{q}}{2}\phi_1^0 \end{cases} \tag{5}$$

and to correct, we need to solve the inhomogeneous system

$$\begin{cases} h\overline{\partial}\phi_1 - \widehat{\tau}_{-\omega} h\frac{q}{2}\phi_2 = \psi_1, \\ h\partial\phi_2 - \widehat{\tau}_{\omega} h\frac{\overline{q}}{2}\phi_1 = \psi_2 \end{cases} \tag{6}$$

in $(\langle\cdot\rangle^{\epsilon} L^2)^2$ for $(\psi_1, \psi_2) \in (\langle\cdot\rangle^{\epsilon-2} L^2)^2$. For the right hand side in (5) to be in the right space, we add an assumption on q:

$$q \in \langle\cdot\rangle^{\epsilon_0 - 2} L^2 \text{ for some } \epsilon_0 \in]0, 1]. \tag{7}$$

Let $u = Ev$, $\widetilde{u} = Fv$ be the unique solutions in $\langle\cdot\rangle^{\epsilon_0} L^2$ of the equations $h\overline{\partial}u = v$ and $h\partial\widetilde{u} = v$, when $v \in \langle\cdot\rangle^{\epsilon_0 - 2} L^2$. Applying E and F to the two equations in (6) leads to the equivalent system

$$(1 - \mathcal{K}) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} E\psi_1 \\ F\psi_2 \end{pmatrix}, \tag{8}$$

$$\mathcal{K} := \begin{pmatrix} 0 & E\widehat{\tau}_{-\omega}\frac{hq}{2}, \\ F\widehat{\tau}_{\omega}\frac{h\overline{q}}{2} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{9}$$

We see that $\mathcal{K} = \mathcal{O}(1) : (\langle\cdot\rangle^{\epsilon_0} L^2)^2 \to (\langle\cdot\rangle^{\epsilon_0} L^2)^2$ which is not enough to invert $1 - \mathcal{K}$ without an extra smallness assumption on q.
However $\mathcal{K}^2$ is much smaller, cf. Lemma 3.2 in [Pe16]:

Proposition

$\mathcal{K}^2 = \mathcal{O}(h) : (\langle\cdot\rangle^{\epsilon_0} L^2)^2 \to (\langle\cdot\rangle^{\epsilon_0} L^2)^2.$

It follows that $1 - \mathcal{K}$ is bijective with inverse
$(1+\mathcal{K})(1-\mathcal{K}^2)^{-1} = (1-\mathcal{K}^2)^{-1}(1+\mathcal{K}) = 1 + \mathcal{K} + \mathcal{O}(h)$.
Proof of the proposition.

$$\mathcal{K}^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix},$$

$$AB = \frac{h^2}{4} E \widehat{\tau}_{-\omega} q F \widehat{\tau}_{\omega} \overline{q} = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

Let χ be as before and put $\chi^w = \chi(hD)$.

$$\mathrm{I} = \frac{h^2}{4} E \widehat{\tau}_{-\omega} q (1 - \chi^w) F \widehat{\tau}_{\omega} \overline{q} = \mathcal{O}(h),$$
$$\mathrm{II} = \frac{h^2}{4} E (1 - \chi^w) \widehat{\tau}_{-\omega} q \chi^w F \widehat{\tau}_{\omega} \overline{q} = \mathcal{O}(h),$$
$$\mathrm{III} = \frac{h^2}{4} E \chi^w \widehat{\tau}_{-\omega} q \chi^w F \widehat{\tau}_{\omega} \overline{q}.$$

At first sight we only have $\mathrm{III} = \mathcal{O}(1)$ but $\chi^w \widehat{\tau}_{-\omega} q \chi^w = \mathcal{O}(h^\infty)$ by pseudodifferential calculus, so $\mathrm{III} = \mathcal{O}(h^\infty)$. □

NB: For I, II we only use that $q \in \langle \cdot \rangle^{-2} L^\infty$. If $\forall\, \epsilon > 0$, $\exists \widetilde{q}$ satisfying (3) such that $\| \langle \cdot \rangle^2 (q - \widetilde{q}) \|_{L^\infty} \leq \epsilon$, then $\mathrm{III} = o(1)$ and hence $\mathcal{K}^2 = o(1)$ when $h \to \infty$, allowing to invert $1 - \mathcal{K}$ and $1 - \mathcal{K}^2$.

Returning to (4), (5), cf. (8), we get

$$(1-\mathcal{K})\begin{pmatrix}\phi_1-\phi_1^0\\ \phi_2-\phi_2^0\end{pmatrix}=\begin{pmatrix}0\\ F\widehat{\tau}_\omega\left(\frac{h\overline{q}}{2}\right)\end{pmatrix}=\mathcal{O}(1)\text{ in }\langle\cdot\rangle^{\epsilon_0}L^2,$$

(actually $\mathcal{O}(h)$ if we use the smoothness of q) recalling that $\phi_1^0=1$,

$$\begin{pmatrix}\phi_1-1\\ \phi_2\end{pmatrix}=(1+\mathcal{K}+\mathcal{O}(h))\begin{pmatrix}0\\ F\widehat{\tau}_\omega\left(\frac{h\overline{q}}{2}\right)\end{pmatrix}.$$

Thus,

$$\phi_1-1=E\widehat{\tau}_{-\omega}\frac{hq}{2}F\widehat{\tau}_\omega\left(\frac{hq}{2}\right)+\underbrace{\mathcal{O}(h)}_{\mathcal{O}(h^2)}\text{ in }\langle\cdot\rangle^{\epsilon_0}L^2,$$

$$\phi_2=F\widehat{\tau}_\omega\left(\frac{h\overline{q}}{2}\right)+\underbrace{\mathcal{O}(h)}_{\mathcal{O}(h^2)}\text{ in }\langle\cdot\rangle^{\epsilon_0}L^2. \tag{10}$$

4. The leading correction term when $q = 1_\Omega$.

One can give weaker regularity assumptions for q that still imply the convergence of the Neumann series (preliminary result) but this convergence is still an open problem (to us) when $q = 1_\Omega$, $\Omega \Subset \mathbf{C}$ simply connected domain with smooth boundary. In this case we study the leading term in (10):

$$\widetilde{f}(z,k) := \frac{1}{2\pi}\int_\Omega \frac{1}{\overline{z}-\overline{w}} e^{kw-\overline{kw}} L(dw). \qquad (11)$$

Equivalently we can study

$$f(z,k) = \iint_\Omega \frac{e^{kw-\overline{kw}}}{z-w}\frac{d\overline{w}\wedge dw}{2i}. \qquad (12)$$

We have

$$d_w\left(\frac{1}{\overline{k}(w-z)}e^{kw-\overline{kw}}dw\right) = \left(\frac{1}{z-w} + \frac{\pi\delta_z(w)}{\overline{k}}\right)e^{kw-\overline{kw}}d\overline{w}\wedge dw.$$

Integration over Ω and Stokes' formula give

$$f(z,k) = \frac{1}{2i\overline{k}}\int_{\partial\Omega}\frac{1}{w-z}e^{kw-\overline{kw}}dw - \begin{cases} 0 \text{ if } z \notin \Omega \\ \frac{\pi}{\overline{k}}e^{kz-\overline{kz}} \text{ if } z \in \Omega. \end{cases} \tag{13}$$

Assume

$$\Omega \text{ is strictly convex and } \partial\Omega \text{ is real analytic.} \tag{14}$$

Parametrize: $t \mapsto \gamma(t) \in \partial\Omega$, $|\dot{\gamma}(t)| = 1$ with the positive orientation.
Write

$$e^{kw-\overline{kw}} = e^{iu_0(w,\kappa)} \text{ on } \partial\Omega,\ \ u_0(w,\kappa) = \Re(w\overline{\kappa}) = \langle w,\kappa\rangle_{\mathbf{R}^2},\ \ \kappa = 2i\overline{k}.$$

With $u_0(t) \simeq u(\gamma(t), \kappa)$, we have

$$\dot{u}_0(t) = \langle \dot{\gamma}(t), \kappa \rangle, \;\; \ddot{u}_0(t) = \langle \ddot{\gamma}(t), \kappa \rangle$$

Let

- $w_+ = w_+(\kappa) \in \partial\Omega$ be the North pole where the *exterior* unit normal is equal to $\kappa/|\kappa|$,
- w_- be the South pole defined the same way in terms of the *interior* unit normal.
- γ_+ be the open boundary segment from the South pole to the North pole and γ_- the one from the North to the South.

We have

$$\pm\partial_t u_0 > 0 \text{ on } \gamma_\pm, \;\; \partial_t u_0(w_\pm) = 0, \;\; \pm\partial_t^2 u_0(w_\pm) < 0$$

Let $u(w,\kappa)$ be the holomorphic extension of u_0 to $\mathrm{neigh}\,(\partial\Omega, \mathbf{C})$.
Then $\Im u > 0$ in $\mathrm{neigh}\,(\gamma_+) \cap \Omega$ and in $\mathrm{neigh}\,(\gamma_-) \cap (\mathbf{C} \setminus \overline{\Omega})$.
Assume first that

$$z \notin \mathrm{neigh}\,(\{w_+, w_-\}). \tag{15}$$

We then would like to replace the contour $\partial\Omega$ in the integral in (13) by a new contour obtained by deforming γ_+ inwards (into Ω) and γ_- outwards (towards the exterior of Ω). Such a small deformation can be chosen so that Γ also avoids a neighborhood of z, thanks to the assumption (15). If the deformation crosses z, then a residue term has to be added. The integral along Γ can be expanded with stationary phase – steepest descent:

$$\begin{aligned} I_\Gamma(k) :&= \frac{1}{2i\overline{k}} \int_\Gamma \frac{1}{w-z} e^{iu(w,\kappa)} dw \\ &= C_+ |k|^{-\frac{3}{2}} e^{i\langle w_+,\kappa\rangle} + C_- |k|^{-\frac{3}{2}} e^{i\langle w_-,\kappa\rangle} + \mathcal{O}(|k|^{-\frac{5}{2}}). \end{aligned} \tag{16}$$

$C_\pm = C_\pm(w_\pm)$ are explicit constants.

Taking into account the residue terms which may appear at the deformation, we get from (13) for $z \in \mathrm{neigh}\,(\gamma_+ \cup \gamma_-)$ when (15) holds:

$$f(z,k) = I_\Gamma(z,k) \begin{cases} +\frac{\pi}{k} e^{iu(z,\kappa)} & -\frac{\pi}{k} e^{i\langle z,\kappa\rangle},\ z \in \mathrm{neigh}\,(\gamma_+) \cap \Omega, \\ +0 & +0, \quad z \in \mathrm{neigh}\,(\gamma_+) \cap (\mathbf{C} \setminus \overline{\Omega}), \\ +0 & -\frac{\pi}{k} e^{i\langle z,\kappa\rangle},\ z \in \mathrm{neigh}\,(\gamma_-) \cap \Omega, \\ -\frac{\pi}{k} e^{iu(z,\kappa)} & +0, \quad z \in \mathrm{neigh}\,(\gamma_-) \cap (\mathbf{C} \setminus \Omega). \end{cases} \tag{17}$$

This result, including (16), still makes sense when the distance from z to the poles is small but $\gg |k|^{-1/2}$.

When the distance is even smaller we still have asymptotics, now in terms of the special function

$$G(z) := \int_{\mathbf{R}} \frac{1}{z-w} e^{-w^2/2} dw. \tag{18}$$

5. Numerics

$\Phi_2 = \phi_2$, $\widetilde{\Phi}_2 = \widetilde{f}$ (leading term in (10)).

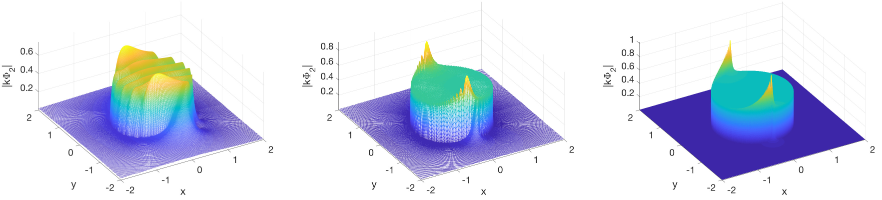

Figure: The solution Φ_2 for the characteristic function of the disk multiplied by k for $k = 10, 100, 1000$ from left to right.

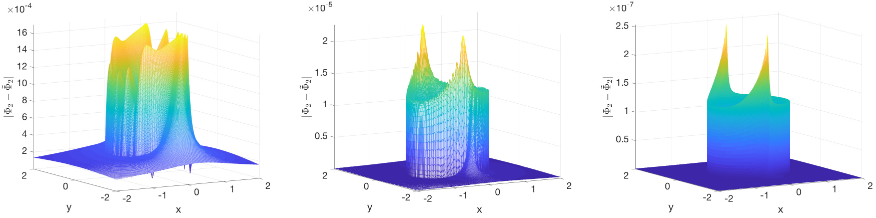

Figure: Difference between the solution Φ_2 for the characteristic function of the disk and $\tilde{\Phi}_2$ for $k = 10, 100, 1000$ from left to right.

$\Phi^I_{2,i(e)}$: approximation of $\widetilde{f}$ in the interior (exterior) without the residue term and without I_Γ.

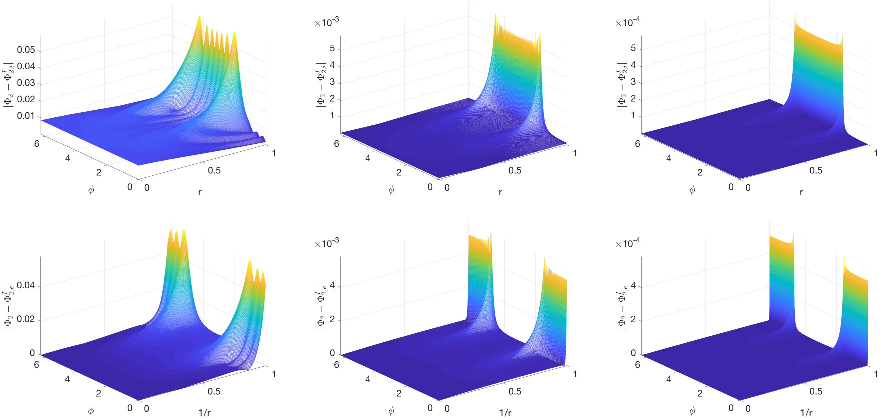

Figure: Difference between the solution Φ_2 for the characteristic function of the disk and $\Phi^I_{2,i}$ in the upper row and the difference between Φ_2 and $\Phi^I_{2,e}$ in the lower row, both for $k = 10, 100, 1000$ from left to right.

$\Phi_{2,i}^{\mathrm{II}}$ also with the residue term. Only the interior case (the other one looks very similar)

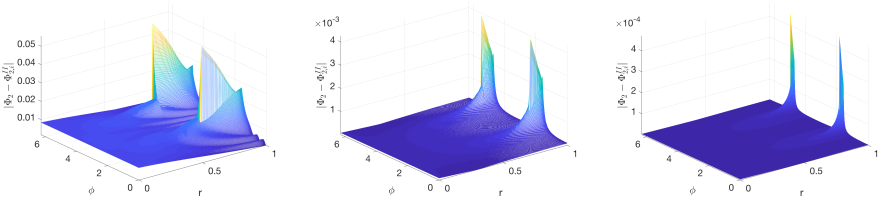

Figure: Difference between the solution Φ_2 for the characteristic function of the disk and $\Phi_{2,i}^{II}$ from (**??**) at the disk for $k = 10, 100, 1000$ from left to right.

We get a further moderate improvement by implementing the special function approach near the poles.

References I

P. Perry, *Global well-posedness and long-time asymptotics for the defocussing Davey–Stewartson II equation in* $H^{1,1}(\mathbb{C})$, J. Spectr. Theory 6 (2016), no. 3, 429–481.

A. Nachman, I. Regev, D. Tataru, *A nonlinear Plancherel theorem with applications to ...*, arXiv:1708.04759.