# Inverse Scattering for the Intermediate Long Wave Equation

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# Three Integrable PDE's

The following integrable PDE's describe the profile $u(x,t)$ for small-amplitude internal waves at a the interface of two fluids

Shallow depth: the KdV Equation

$$u_t + 2uu_x + (\delta/3)u_{xxx} = 0$$

Depth $\delta$: the Intermediate Long Wave (ILW) Equation

$$u_t + 2uux + T(u_{xx}) + \frac{1}{\delta}u_x = 0, \quad \mathcal{F}(Tf)(\xi) = i\coth(\delta\xi)\mathcal{F}f(\xi)$$

Infinite Depth: the Benjamin-Ono (BO) Equation

$$u_t + 2uu_x + Hu_{xx} = 0, \quad \mathcal{F}(Hu)(\xi) = i(\text{sgn}(\xi)u)(\xi)$$

All are solvable, in principle, by the inverse scattering method. The BO and ILW equations are *quasilinear*, while KdV is *semilinear*.

# From KdV to ILW to BO

$$u_t + 2uu_x + (\delta/3)u_{xxx} = 0 \qquad \text{KdV}$$

$$u_t + 2uux + T(u_{xx}) + \frac{1}{\delta}u_x + = 0, \quad \widehat{(Tf)}(\xi) = i\coth(\delta\xi)\widehat{f}(\xi) \qquad \text{ILW}$$

$$u_t + 2uu_x + Hu_{xx} = 0, \quad \widehat{(Hf)}(\xi) = i(\text{sgn}(\xi)\widehat{f})(\xi) \qquad \text{BO}$$

The linear part (in $x$) of ILW has symbol

$$i\left(-\xi^2\coth(\delta\xi) + \delta^{-1}\right) \sim \begin{cases} -i\dfrac{\delta\xi^3}{3} & \delta \to 0 \\ -i\xi^2\,\text{sgn}(\xi), & \delta \to \infty \end{cases}$$

In fact, if $u_\delta(x,t)$ solves ILW, then $v_\delta(x,t) = \frac{3}{\delta}u\left(x, \frac{3}{\delta}t\right)$ converges as $\delta \to 0$ to a solution of KdV, while $u_\delta(x,t)$ converges as $\delta \to \infty$ to a solution of BO (Abdelouhab, Bona, Felland, Saut 1989)

# Complete Integrability - KdV and BO

- The KdV equation ($\delta \to 0$) is an isospectral flow for the Schröinger operator:

$$(L\psi)(x) = -\psi''(x) + q(x)\psi(x)$$

- The BO equation ($\delta \to \infty$) is an isospectral flow for the operator

$$(L\psi)(x) = \frac{1}{i}\frac{d}{dx} - C_+(u(C_+\psi))$$

acting on

$$H^2_+(\mathbb{R}) = \{f \in L^2 : f \text{ extends analytically to } \mathbb{C}^+\}.$$

Here $C_+$ is the Cauchy projector onto $H^2_+(\mathbb{R})$.

# Complete Integrability - ILW

The ILW equation is an 'isospectral' flow for the operator

$$L\psi = T_+ \left( \frac{1}{i} \frac{d}{dx} \psi \right)(x) + iu(x) T_+ \psi$$

Here $T_+ = T + iI$ where

$$Tf = -\frac{1}{2\delta} \coth\left( \frac{\pi x}{2\delta} \right) * f.$$

Note that

$$\sigma(T_+)(\xi) = \frac{2i}{1 - e^{-2\delta\xi}}$$

which converges so $2i\chi_+(\xi)$ as $\delta \to \infty$.

We'll spend much of the talk understanding what this problem actually means.

# Motivations

Global well-posedness for the ILW is well-known by PDE methods (Saut 1979, Molinet-Vento 2015, Molinet-Pilod-Vento 2019). We develop the inverse scattering method because:

- ILW interpolates between KdV (whose inverse map is governed by a Riemann-Hilbert problem) and BO (whose inverse map is governed by a nonlocal Riemann-Hilbert problem)
- Spectral theory will illuminate the soliton solutions that occur
- Inverse scattering, if properly developed, can be used to prove soliton resolution and obtain complete asymptotics - see, for example
    - Borghese-Jenkins-McLaughlin (*Ann. I.H.P.*, 2018), focussing NLS
    - Jenkins-Liu-Perry-Sulem (*Comm. Math. Phys.*, 2019), DNLS

# Some History (IST)

For a comprehensive review of BO and ILW, see Jean-Claude Saut's arXiv survey

- Kubota, Ko, Dobbs (1978) - Introduced ILW as a model of weakly nonlinear flow in a stratified fluid that interpolates between KdV and BO
- Satsuma, Ablowitz, Kodama (1979) - Showed that ILW is the compatibility condition for a spectral problem and an evolution equation
- Joseph-Egri (1978) and Chen-Lee (1979) found multiple soliton solutions
- Kodama, Ablowitz, Satsuma (1981) gave a solution of ILW by inverse scattering
- Santini, Ablowitz, Fokas (1982) used IST to show that BO is the $\delta \to \infty$ limit of ILW

To date there are no rigorous results on inverse scattering for ILW, even for small data. Recently, Allen Wu has studied the direct problem for BO for initial data of arbitrary size and completely characterized the spectral and scattering data for the direct problem.

# Summary of the Results

We emphasize that the formal theory of direct and inverse scattering is already contained in the papers of Kodama, Ablowitz, and Satsuma and Santini, Ablowitz, and Fokas.

We will:

- Formulate the direct scattering problem as a Riemann-Hilbert problem on a strip
- Reformulate the RHP as an integral equation and solve it uniquely for small data
- Define a small-data direct scattering map and characterize its range
- Recall the nonlocal Riemann-Hilbert problem that inverts the direct scattering map

The technical core of our work is fine estimates on Green's function for the linear problem and its analytic extension in *both* the spatial and spectral variables.

# The Operator $T$ and Function Theory

The operator

$$(Tf)(x) = \text{PV}\left[\frac{1}{2\delta}\int \coth\left(\frac{\pi}{2\delta}(x-x')\right) f(x')\,dx'\right]$$

is closely related to the periodized Cauchy kernel that generates functions periodic under $z \mapsto z + 2i\delta$ and analytic in strips

$$S_n = \{z \in \mathbb{C} : 2(n-1)\delta < \operatorname{Im} z < 2n\delta\}$$

The Cauchy integral is

$$(\mathcal{C}f)(z) = \frac{1}{2\pi i}\int \frac{1}{s-z} f(s)\,ds$$

while the periodized Cauchy integral is

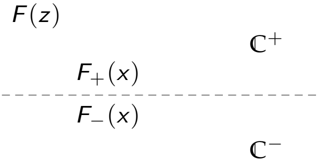
$$(\mathcal{C}_{\text{per}}\, f)(z) = \frac{1}{4i\delta}\int \coth\left(\frac{\pi}{2\delta}(s-z)\right) f(s)\,ds$$

# Analytic Functions on $\mathbb{C} \setminus \mathbb{R}$

A function $f \in L^2(\mathbb{R})$ generates

- an analytic function $F$ in $\mathbb{C} \setminus \mathbb{R}$
- boundary values $F_\pm$

by:




$$F(z) = (\mathcal{C}f)(z) = \frac{1}{2\pi i} \int \frac{1}{s-z} f(s)\, ds$$

$$F_\pm(x) = \lim_{\varepsilon \downarrow 0} (\mathcal{C}f)(x \pm i\varepsilon) = \mp \frac{1}{2} f(x) + (Hf)(x)$$

where

$$(Hf)(x) = \frac{1}{2\pi i} \,\mathrm{PV} \int \frac{1}{x-s} f(s)\, ds$$

# Analytic Functions on $S$

A function $f \in C_0^\infty(\mathbb{R})$ generates

- an analytic function $W_{\text{per}}$ in $\{\operatorname{Im} z \notin 2\delta\mathbb{Z}\}$
- boundary values $W_{\text{per}}^{\pm}$

by:

$W_{\text{per}}^{+}(x) \downarrow$

- - - - - - - - - - - $\operatorname{Im} z = 2\delta$

$W_{\text{per}}^{-}(x) \uparrow$

$W_{\text{per}}(z)$

$W_{\text{per}}^{+}(x) \downarrow$

- - - - - - - - - - - $\operatorname{Im} z = 0$

$W_{\text{per}}^{-}(x) \uparrow$

$$W_{\text{per}}(z) = (\mathcal{C}_{\text{per}} f)(z) = \frac{1}{4i\delta} \int \coth\left(\frac{\pi}{2\delta}(s-z)\right) f(s)\, ds$$

$$W_{\text{per}}^{\pm}(x) = \lim_{\varepsilon \downarrow 0} (\mathcal{C}_{\text{per}} f)(x \pm i\varepsilon) = \mp \frac{1}{2} f + \frac{1}{2i} Tf$$

where

$$(Tf)(x) = \frac{1}{2\delta}\, \text{PV} \int \coth\left(\frac{\pi}{2\delta}(x-s)\right) f(s)\, ds$$

# Analytic Functions on $S$

$$\lim_{\varepsilon\downarrow 0}(\mathcal{C}_{\mathrm{per}}\, f)(x+i\varepsilon) = -\frac{1}{2}f + \frac{1}{2i}Tf$$

$$\lim_{\varepsilon\downarrow 0}(\mathcal{C}_{\mathrm{per}}\, f)(x+i(2-\varepsilon)) = \frac{1}{2}f + \frac{1}{2i}Tf$$

where

$$(Tf)(x) = \frac{1}{2\delta}\,\mathrm{PV}\int \coth\left(\frac{\pi}{2\delta}(x-s)\right) f(s)\, ds$$

We define

$$(T_{\pm}f)(x) = \pm 2if(x) + (Tf)(x)$$

$$W_{\text{per}}^{+}(x) \downarrow$$

$$\text{- - - - - - - - - - - - - - - - - - - - - - - -} \quad \operatorname{Im} z = 2\delta$$

$$W_{\text{per}}^{-}(x) \uparrow$$

$$W_{\text{per}}(z)$$

$$W_{\text{per}}^{+}(x) \downarrow$$

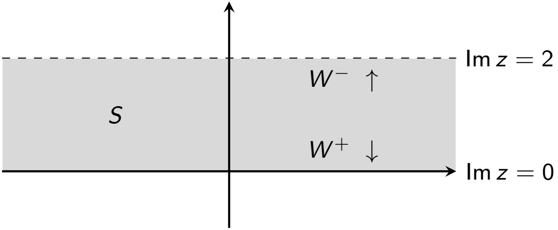
$$\text{- - - - - - - - - - - - - - - - - - - - - - - -} \quad \operatorname{Im} z = 0$$

$$W_{\text{per}}^{-}(x) \uparrow$$

For the rest of the talk, we will take $\delta = 1$

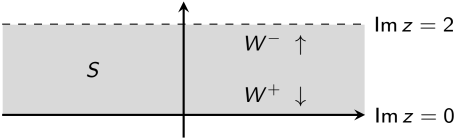
# Direct Scattering: The Problem

An equivalent form of the ILW direct scattering problem is the following Riemann-Hilbert Problem for a function $W$ in a strip $S$, depending on a spectral parameter $\zeta \in (0, \infty)$.




$$\frac{1}{i}\frac{d}{dx}W^+(x) - u(x)W^+(x) = \zeta\left(W^+(x) - W^-(x)\right).$$

# Direct Scattering: Fourier Analysis




$$(L_0 W)(x) = u(x) W^+(x)$$

where

$$(L_0 W)(x) = \frac{1}{i}\frac{d}{dx} W^+(x) - \zeta\left(W^+(x) - W^-(x)\right)$$

If

$$W(z) = \int e^{iz\xi}\,\widehat{f}(\xi)\,d\xi$$

then

$$\widehat{L_0 W}(\xi) = p(\xi,\zeta)\widehat{f}(\xi), \quad p(\xi,\zeta) = \xi - \zeta\left(1 - e^{-2\xi}\right)$$

# Direct Scattering: Reparameterization

$$(L_0 W)(x) = u(x) W^+(x)$$
$$\sigma(L_0) = \xi - \zeta\left(1 - e^{-2\xi}\right), \quad \zeta \in (0, \infty)$$

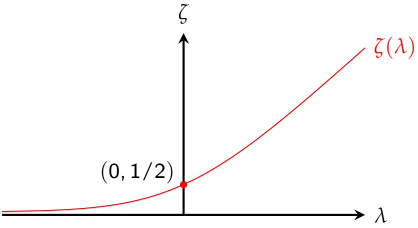
Let

$$\zeta(\lambda) = \frac{\lambda}{1 - e^{-2\lambda}}$$

The map

$$\zeta : (0, \infty) \to \mathbb{R}$$

is a diffeomorphism




We may write $\zeta = \zeta(\lambda)$ and factor

$$\sigma(L_0) = p(\xi, \lambda) = (\zeta(\xi) - \zeta(\lambda))(1 - e^{-2\xi})$$

so that $p(\xi, \lambda)$ has real zeros at $\xi = 0$ and $\xi = \lambda$

# Integral Equation: Green's Functions

$$(L_0 W)(x) = u(x) W^{+}(x)$$

$$p(\xi, \lambda) = [\zeta(\xi) - \zeta(\lambda)] \left[1 - e^{-2\xi}\right]$$

where $\zeta : \mathbb{R} \to (0, \infty)$ is a diffeomorphism

The symbol $p(\xi, \lambda)$ has real zeros at $\xi = 0$ and $\xi = \lambda$ corresponding so homogeneous solutions $W(x) = 1$ and $W(x) = e^{i\lambda x}$ of

$$\frac{1}{i} \frac{d}{dx} W - \zeta(\lambda)(W^{+}(x) - W^{-}(x)) = 0$$

To form a Green's function for $L_0$ via

$$G(x, \lambda) = \frac{1}{2\pi} \int_{\Gamma} \frac{e^{i\xi x}}{p(\xi, \lambda)} \, d\xi$$

we need to choose contours that go around these zeros

# Integral Equation: Green's Functions

We define, for $* = L, R$ and $\lambda \in \mathbb{R}$:

$$G_*(x,\lambda) = \frac{1}{2\pi}\int_{\Gamma_*} \frac{e^{ix\xi}}{p(\xi,\lambda)}\, d\xi, \qquad p(\xi,\lambda) = \xi - \zeta(\lambda)(1 - e^{-2\xi})$$

where


These Green's functions obey

- Symmetry: $G_R(x,\lambda) = \overline{G_L(-x,\lambda)}$
- Asymptotics: $\lim_{x\to-\infty} G_L(x,\lambda) = 0, \quad \lim_{x\to\infty} G_R(x,\lambda) = 0$
- Left-Right Relation:

$$G_L(x,\lambda) - G_R(x,\lambda) = \frac{i}{1-2\zeta(\lambda)} + \frac{i}{1-2\zeta(-\lambda)} e^{i\lambda x}$$

# Integral Equations: Jost Solutions

Define Jost solutions $M_1$, $M_e$, $N_1$, $N_e$ of

$$iW_x + \zeta\left(W^+ - W^-\right) = uW$$

by the asymptotics

$$\lim_{x\to-\infty} |M_1(x,\lambda) - 1| = 0 \qquad \lim_{x\to+\infty} |N_1(x,\lambda) - 1| = 0$$

$$\lim_{x\to-\infty} |M_e(x,\lambda) - e_\lambda(x)| = 0 \qquad \lim_{x\to+\infty} |N_e(x,\lambda) - e_\lambda(x)| = 0$$

where $e_\lambda(x) = e^{i\lambda x}$. Then

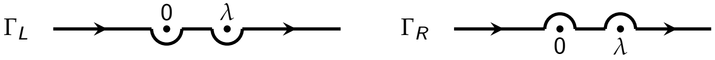
$$M_1 = 1 + G_L * (uM_1) \qquad N_1 = 1 + G_R * (uM_1)$$

$$M_e = e_\lambda + G_L * (uM_e) \qquad N_e = 1 + G_R * (uN_e)$$

# Green's Function Estimates

$$G_*(x,\lambda) = \frac{1}{2\pi}\int_{\Gamma_*} \frac{e^{ix\xi}}{p(\xi,\lambda)}\,d\xi, \qquad p(\xi,\lambda) = \xi - \zeta(\lambda)(1-e^{-2\xi})$$

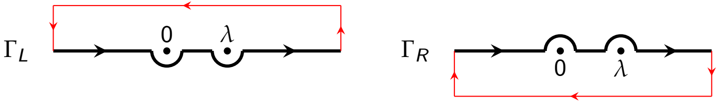
where



# Green's Function Estimates

$$G_*(x,\lambda) = \frac{1}{2\pi}\int_{\Gamma_*} \frac{e^{ix\xi}}{p(\xi,\lambda)}\, d\xi, \qquad p(\xi,\lambda) = \xi - \zeta(\lambda)(1-e^{-2\xi})$$

where




$$G_L(x,\lambda) - H_L(x,\lambda) = \begin{cases} \dfrac{i}{1-2\zeta(\lambda)} + \dfrac{i}{1-2\zeta(-\lambda)} e^{i\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

where $H_L(x,\lambda)$ is $\mathcal{O}\left(e^{-\pi|x|}\right)$ as $x \to \pm\infty$.

A similar statement holds for $G_R(x,\lambda)$ but with cases $x > 0$ and $x < 0$ reversed.

# Green's Function Estimates

$$G_L(x,\lambda) - H_L(x,\lambda) = \begin{cases} \dfrac{i}{1-2\zeta(\lambda)} + \dfrac{i}{1-2\zeta(-\lambda)} e^{i\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Note that

$$\lim_{\lambda \to 0} \frac{i}{1-2\zeta(\lambda)} + \frac{i}{1-2\zeta(-\lambda)} e^{i\lambda x} = ix$$

One finds (after quite a bit of work!)

$$|G_L(x,\lambda)| \lesssim 1 + |x| + \log_+(1/|x|)$$

with constants uniform in $\lambda$, where

$$\log_+(t) = \begin{cases} \log(t), & t > 1 \\ 0 & t \leq 1 \end{cases}$$

# Scattering Solutions

Let $X$ denote the space of measurable real-valued functions with

$$\|u\|_X := \left\| \langle x \rangle^2 u \right\|_{L^1} + \text{ess sup}_{x \in \mathbb{R}} \left( \int \log_+ \left( \frac{1}{|x-y|} \right) \langle y \rangle |u(y)| \, dy \right)$$

**Theorem**

There is a $c_0 > 0$ and a ball $B = \{u \in X : \|u\|_X < c_0\}$ so that there exist unique solutions to the integral equations

$$\begin{aligned} M_1 &= 1 + G_L * (uM_1) & \qquad N_1 &= 1 + G_R * (uN_1) \\ M_e &= e_\lambda + G_L * (uM_e) & \qquad N_e &= e_\lambda + G_R * (uN_e) \end{aligned}$$

in the space $\langle x \rangle L^\infty(\mathbb{R})$. If $\lambda \neq 0$, the solutions lie in $L^\infty(\mathbb{R})$. Moreover, the solution maps

$$u \mapsto M_1, \quad u \mapsto M_e, \quad u \mapsto N_1, \quad u \mapsto N_e$$

are Lipschitz continuous from $B$ into $\langle x \rangle L^\infty$.

# Integral Equation and Differential Equation

We can continue $M_1(x,\lambda)$ say to a function of $z$ in the strip by the formula:

$$M_1(x+iy,\lambda) = 1 + \int G_L(x+iy-x',\lambda)u(x')M(x',\lambda)\,dx'$$

**Theorem**

For each $\lambda \neq 0$ the solutions $M_1(x,\lambda)$ and $N_1(x,\lambda)$ continue to analytic functions $M_1(z,\lambda)$ and $N_1(z,\lambda)$ in $0 < \operatorname{Im} z < 2$ with pointwise boundary value

$$M_1^+(x,\lambda) = M_1(x,\lambda) = M_1(x+i0,\lambda)$$

and pointwise a.e. boundary value

$$M_1^-(x,\lambda) = M_1(x+2i-i0,\lambda).$$

Moreover, $M_1$ and $N_1$ are weak solutions of

$$\frac{1}{i}\frac{d}{dx}W^+ + \zeta(\lambda)\left[W^+ - W^-\right] = uW^+.$$

# Scattering Map: Scattering Asymptotics

From the integral equation

$$M_1 = 1 + G_L * (uM_1)$$

and the relation

$$G_L(x,\lambda) - G_R(x,\lambda) = \frac{i}{1-2\zeta(\lambda)} + \frac{i}{1-2\zeta(-\lambda)} e_\lambda(x)$$

we deduce

$$M_1(x,\lambda) \underset{x\to+\infty}{\sim} a(\lambda) + b(\lambda)e^{i\lambda x}$$

where

$$a(\lambda) = 1 + \frac{i}{1-2\zeta(\lambda)} \int u(x) M_1(x,\lambda)\, dx$$

$$b(\lambda) = \frac{i}{1-2\zeta(-\lambda)} \int e^{-i\lambda x} u(x) M_1(x,\lambda)\, dx$$

# Scattering Data

$$a(\lambda) = 1 + \frac{i}{1 - 2\zeta(\lambda)} \int u(x) M_1(x, \lambda)\, dx \qquad \text{"transmission"}$$

$$b(\lambda) = \frac{i}{1 - 2\zeta(-\lambda)} \int e^{-i\lambda x} u(x) M_1(x, \lambda)\, dx \qquad \text{"reflection"}$$

We'll analyze asymptotics of $a$ and $b$, and argue (not yet prove) that $a$ and $b$ are uniquely determined by the ratio

$$\rho(\lambda) = b(\lambda)/a(\lambda)$$

(up to a finite number of poles of $a$) via a *nonlocal* Riemann-Hilbert problem.

# Asymptotics of $a$, $b$

Let $I = \int u(x) M(x, 1/2)\, dx$. The potential $u$ is

- *generic* if $I \neq 0$
- *non-generic* if $I = 0$

The scattering data $a$ and $b$ have the following properties:

- $|a(\zeta)| \geq 1$, $|a(\zeta) - 1| = \mathcal{O}\left(\zeta^{-1}\right)$ as $\zeta \to \infty$
- $b(\zeta) \in L^2(0, \infty)$
- $\lim_{\zeta \to 1/2} -2i(\zeta - 1/2) a(\zeta) = 1$ if $u$ is generic
- $a(\zeta)$ and $b(\zeta)$ have finite limits if $u$ is nongeneric
- $\lim_{\zeta \to 1/2} b(\zeta)/a(\zeta) = -1$ if $u$ is generic, and finite if $u$ is non-generic

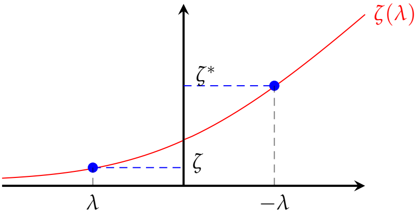
# Interlude: Nonlinear Reflection

In formulating the nonlocal RHP satisfied by $\rho$, we'll view $\rho$ as a function of $\zeta$. The nonlinear reflection $\zeta \mapsto \zeta^*$ uniquely defined by

$$\zeta^*(\lambda) = \zeta(-\lambda)$$

will play a role owing to the identity

$$G_L(x,\lambda) = e^{i\lambda x} G_L(x,-\lambda)$$



# Interlude: Green's Functions Again

The function $\zeta(\lambda)$ is uniquely invertible near $(0, \infty) \subset \mathbb{C}$. Let

$$G_L(x, \lambda) = G_L(x, \zeta + i0), \quad G_R(x, \lambda) = G_R(x, \zeta - i0)$$

By contour integration

$$G_L(x, \zeta) \sim$$

$$\begin{cases} \dfrac{i}{1-2\zeta}\chi_{\mathbb{R}^+}(x) + \dfrac{i}{1-2\zeta^*}e^{i\lambda(\zeta)x}\chi_{\mathbb{R}^+}(x) + \mathcal{O}\left(e^{-\varepsilon|x|}\right), & \mathrm{Im}(\zeta) > 0 \\ \dfrac{i}{1-2\zeta}\chi_{\mathbb{R}^+}(x) + \mathcal{O}\left(e^{-\varepsilon|x|}\right) & \mathrm{Im}(\zeta) < 0 \end{cases}$$

- $G_L(x, \zeta) \sim x$ as $\zeta \to 1/2, \mathrm{Im}\,\zeta > 0$
- $G_L(x, \zeta) \to \infty$ as $\zeta \to 1/2, \mathrm{Im}\,\zeta < 0$!

# Interlude: Green's Functions Again

- $G_L(x,\zeta)$ and $G_R(x,\zeta)$ extend to analytic functions in $\zeta \in \mathbb{C} \setminus (0,\infty)$.
- The following jump relations hold for $\zeta \neq 1/2$:

$$G_L(x,\zeta+i0) - G_L(x,\zeta-i0) = \frac{i}{1-2\zeta^*} e^{i\lambda x}$$

$$G_R(x,\zeta+i0) - G_R(x,\zeta-i0) = \frac{i}{1-2\zeta^*} e^{i\lambda x}$$

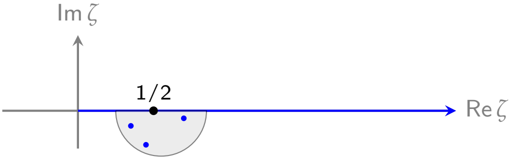
- $G_L(x,\zeta-i0)$ and $G_R(x,\zeta+i0)$ have a singularity at $\zeta = 1/2$ as

# Meromorphic Extension of $a(\zeta)$

The function

$$a(\zeta) = 1 + \frac{1}{1-2\zeta} \int u(x) M_1(x,\zeta)\, dx$$

extends to a *meromorphic* function of $\zeta \in \mathbb{C} \setminus (0, \infty)$




The function $a(\zeta)$ may have finitely many poles even for "small data" in a half-disc below $\zeta = 1/2$ due to the fact that $G_L(x, \zeta)$ blows up as $\zeta \to 1/2$, $\operatorname{Im} \zeta < 0$

# Meromorphic Extension of $M_1(x, \zeta)$

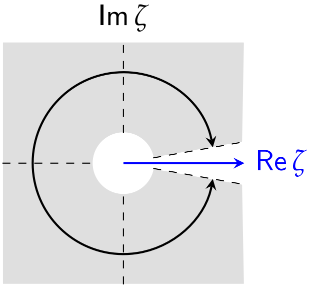
$$\| G_L(\,\cdot\,, \zeta) \|_{L^2} \lesssim 1/\sqrt{|\zeta|}$$

uniformly in subsectors of $\mathbb{C} \setminus (0, \infty)$ with a ball removed,

Hence, $M_1(x, \zeta)$ extends to an analytic function of $\zeta$ with

$$\lim_{\zeta \to \infty} M_1(x, \zeta) = 1$$

with the same uniformity



# $\rho$ probably determines $a$

Extend $a$ to a meromorphic function on $\mathbb{C} \setminus (0, \infty)$. One can show that

$$a(\zeta + i0) = [1 - \rho(\zeta)\rho(\zeta^*)]\ a(\zeta - i0)$$

Let $\{\zeta_k\}_{k=1}^{N}$ be the poles of $a$, counted with multiplicity.

One *should* be able to recover $a$ from $\rho$ and the poles of $a$ but we don't yet know how to prove this.

# Nonlocal Riemann-Hilbert Problem

The next step is to show that the nonlocal Riemann-Hilbert problem for $N_1$:

$$N_1(x+iy,\zeta) = 1 + \frac{1}{2\pi i}\int_0^\infty \frac{\rho(\zeta)N_1(x+iy,s^*)e^{-y\lambda(s)}e^{ix\lambda(s)}}{\zeta - s + i0}\, ds$$

together with the reconstruction formula

$$u(x) = \frac{1}{2\pi i}\int_0^\infty \rho(\zeta)N_1(x,\zeta^*)e^{ix\lambda(\zeta)}d\zeta - \frac{1}{2\pi i}\int_0^\infty \rho(\zeta)N_1(x+2i,\zeta^*)e^{ix\lambda(\zeta)}\, d\zeta$$

invert the direct scattering transform.

# Now, A Word From Our Sponsor

This fall, Springer-Verlag will publish a volume in the *Fields Communications in Mathematics* on IST and PDE. The volume will include six original research papers in the area and surveys by:

- Percy Deift (Coxeter Lectures)
- Patrick Gérard (Wave Turbulence and Complete Integrability)
- Jean-Claude Saut (BO and ILW equations, IST and PDE)
- Peter Perry (Inverse scattering and GWP in 1 and 2 dimensions)
- Momar Dieng, Ken McLaughlin, Peter Miller (Dispersive asymptotics for integrable equations by the $\overline{\partial}$ steepest descent method)