Integrable coupled sigma-models from affine Gaudin models

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Classical and Quantum Integrability Conference
Institut de Mathématiques de Bourgogne, Dijon
September 4, 2019

Collaboration with François Delduc (CNRS, ENS de Lyon), Marc Magro (ENS de Lyon) and Benoit Vicedo (University of York):
[1811.12316] and [1903.00368]
Introduction
**Introduction: integrable $\sigma$-models**

- $\sigma$-models: 2d fields theories
- Applications to high energy physics, condensed matter, ...

**Integrable $\sigma$-models**
- Principal chiral model, symmetric space $\sigma$-model, ...
- Superstring on $\text{AdS}_5 \times S^5$ (AdS/CFT)
- Integrable deformations

- Common algebraic structure behind their integrability
  $\rightarrow$ affine Gaudin models
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  - Principal chiral model, symmetric space $\sigma$-model, ...
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Introduction: affine Gaudin models

- Gaudin models: integrable systems associated with Lie algebras \( \mathfrak{g} \)
  - See talk by Benoit Vicedo
  - Finite Gaudin models \( \leftrightarrow \) \( \mathfrak{g} \) semi-simple finite dimensional
    \( \rightarrow \) integrable spin chains
  - Affine Gaudin models (AGM) \( \leftrightarrow \) \( \mathfrak{g} \) affine Kac-Moody algebra
    \( \rightarrow \) integrable 2d field theories
    \( \rightarrow \) contain integrable \( \sigma \)-models
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    $\rightarrow$ contain integrable $\sigma$-models

- Application of seeing integrable $\sigma$-models as AGM ?

- Quantisation: circumvent non-ultralocality issues

- Previously known integrable $\sigma$-models: small class among AGM
  $\rightarrow$ construction of new classical integrable $\sigma$-models
  (coupling together known ones)
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1 Integrable $\sigma$-models

2 Affine Gaudin models

3 Integrable $\sigma$-models from affine Gaudin models

4 Conclusion
Integrable sigma models

Principal chiral model
Principal Chiral Model

- PCM: integrable $\sigma$-model on semi-simple Lie group $G$

- Field

$$g : \Sigma \rightarrow G$$

2d Minkowski $(t, x)$
Principal Chiral Model

- PCM: integrable $\sigma$-model on semi-simple Lie group $G$

Field and currents

$$g : \Sigma \rightarrow G$$
$$j_{\pm} = g^{-1} \partial_{\pm} g : \Sigma \rightarrow g$$

2d Minkowski $(t, x)$
Light-cone coordinates $x_{\pm} = (t \pm x)/2$
Principal Chiral Model

- PCM: integrable $\sigma$-model on semi-simple Lie group $G$

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2d Minkowski $(t, x)$
Light-cone coordinates $x_\pm = (t \pm x)/2$

- Action:

\[ S_{PCM}[g] = K \int_\Sigma dx \, dt \, \kappa (j_+, j_-) \]

Killing form
PCM: integrable $\sigma$-model on semi-simple Lie group $G$

Field and currents

$$g : \Sigma \rightarrow G \quad j_\pm = g^{-1} \partial_\pm g : \Sigma \rightarrow g$$

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Action:

$$S_{PCM}[g] = K \int_{\Sigma} dx \, dt \, \kappa(j_+, j_-)$$

Equation of motion: $\partial_+ j_- + \partial_- j_+ = 0$
Integrability of the PCM

- Equation of motion of the PCM $\Leftrightarrow$ zero curvature equation

\[ \partial_+ \mathcal{L}_-(z) - \partial_- \mathcal{L}_+(z) + [\mathcal{L}_+(z), \mathcal{L}_-(z)] = 0, \quad \forall \ z \in \mathbb{C} \]

of Zakharov-Mikhailov Lax connection $\mathcal{L}_\pm(z) = \frac{j_\pm}{1 \mp z}$

- Spectral parameter $z$: auxiliary complex parameter
Integrability of the PCM

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- Spectral parameter $z$: auxiliary complex parameter

- Monodromy of the Lax matrix $\mathcal{L}(z) = \frac{1}{2}(\mathcal{L}_+(z) - \mathcal{L}_-(z))$
  \[
  T(z) = \text{Pexp} \left( -\int \text{d}x \ \mathcal{L}(z) \right)
  \]

- Infinite number of conserved charges: $Q_n(z) = \text{Tr}(T(z)^n)$
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- Poisson brackets of $\mathcal{L}(z)$ of Maillet form $\rightarrow \{Q_n(z), Q_m(w)\} = 0$

$\rightarrow$ integrability
Adding a Wess-Zumino term

- Continuous integrable deformations of the PCM ($\eta$-deformations, $\lambda$-deformations, ...)

- Adding a Wess-Zumino term

\[
S[g] = K \int_\Sigma d\chi \, dt \, \kappa (j^+, j^-) + k \, l_{WZ}[g]
\]

Remark: $k = K \rightarrow$ conformal Wess-Zumino-Witten point

- Still possesses a Lax connection $L_{\pm}(z)$

- Lax matrix $L(z) = \frac{1}{2} (L_+(z) - L_-(z))$ still satisfies Maillet bracket

\[\rightarrow\] modifies the form of the bracket
Affine Gaudin models

[Vicedo ’17] and [Delduc SL Magro Vicedo ’18]
Affine Gaudin models

- Algebraic origin related to affine Kac-Moody algebras
  → see Benoit Vicedo’s talk
Affine Gaudin models

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- Defining data:
  - sites: points $z_1, \ldots, z_N$ in $\mathbb{C}$
  - levels: numbers $\ell^{(1)}, \ldots, \ell^{(N)}$

- Defined at the Hamiltonian level
- Phase space: Kac-Moody currents $J^{(r)}(x)$ ($g$-valued fields)
- Poisson bracket: $J^{(r)}(x) = J^{(r)}_{a}(x) l^{a}$

\[
\{ J^{(r)}_{a}(x), J^{(s)}_{b}(y) \} = \delta_{rs} \left( f_{ab}^{~~c} J^{(r)}_{c}(x) \delta(x - y) - \ell^{(r)} \kappa_{ab} \partial_{x} \delta(x - y) \right)
\]

\[
[l_{a}, l_{b}] = f_{ab}^{~~c} l_{c} \quad \text{and} \quad \kappa_{ab} = \kappa(l_{a}, l_{b})
\]
Gaudin Lax matrix, twist function and Hamiltonian

- **Gaudin Lax matrix and twist function**

  \[ \Gamma(z, x) = \sum_{r=1}^{N} \frac{J^{(r)}(x)}{z - z_r}, \quad \varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} - \ell_{\infty} \]
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- Zeroes \( \zeta_1, \ldots, \zeta_N \) of the twist function
Gaudin Lax matrix, twist function and Hamiltonian

- Gaudin Lax matrix and twist function
  \[ \Gamma(z, x) = \sum_{r=1}^{N} \frac{J^{(r)}(x)}{z - z_r}, \quad \varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} = -\ell_{\infty} \prod_{i=1}^{N} \frac{z - \zeta_i}{z - z_r} \]

- Zeroes \( \zeta_1, \cdots, \zeta_N \) of the twist function

- Quadratic charges associated with zeroes:
  \[ Q_i = \text{res}_{z = \zeta_i} Q(z) \, dz, \quad Q(z) = -\frac{1}{2\varphi(z)} \int dx \, \kappa(\Gamma(z, x), \Gamma(z, x)) \]

- Involution: \( \{Q_i, Q_j\} = 0 \)
Gaudin Lax matrix and twist function

\[ \Gamma(z, x) = \sum_{r=1}^{N} \frac{J^{(r)}(x)}{z - z_r}, \quad \varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} - \ell^{\infty} = -\ell^{\infty} \frac{\prod_{i=1}^{N} (z - \zeta_i)}{\prod_{r=1}^{N} (z - z_r)} \]

Zeroes \( \zeta_1, \cdots, \zeta_N \) of the twist function

Quadratic charges associated with zeroes:

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Involution: \( \{ Q_i, Q_j \} = 0 \)

Hamiltonian: \( \mathcal{H} = \sum_{i=1}^{N} \epsilon_i \, Q_i \) (\( Q_i \) conserved and in involution)
Integrability

- **Lax matrix:** $\mathcal{L}(z, x) = \frac{\Gamma(z, x)}{\varphi(z)}$

- **Zero curvature equation:** there exists $\mathcal{M}(z, x)$ such that

$$\{\mathcal{H}, \mathcal{L}(z, x)\} - \partial_x \mathcal{M}(z, x) + [\mathcal{M}(z, x), \mathcal{L}(z, x)] = 0$$

$\rightarrow$ conserved quantities from the monodromy matrix of $\mathcal{L}(z, x)$

- Poisson brackets of the Kac-Moody currents
  $\rightarrow$ Maillet bracket for $\mathcal{L}(z, x)$ (controlled by $\varphi(z)$)
  $\rightarrow$ conserved quantities in involution

- **Integrability automatic!**
A few more results and concepts

Affine Gaudin models with multiplicities:

- Gaudin Lax matrix and twist function
  \[
  \Gamma(z,x) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{J_{[p]}^{(r)}(x)}{(z - z_r)^{p+1}} \quad \text{and} \quad \varphi(z) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{\ell_{[p]}^{(r)}}{(z - z_r)^{p+1}} - \ell_{\infty}
  \]

- Takiff currents \( J_{[0]}^{(r)}(x), \ldots, J_{[m_r-1]}^{(r)}(x) \)

- Quadratic charges \( Q_i \) associated with zeroes \( \zeta_1, \ldots, \zeta_M \) of \( \varphi(z) \)

- Hamiltonian \( \mathcal{H} = \sum_{i=1}^{M} \epsilon_i Q_i \rightarrow \text{integrability} \)
A few more results and concepts

Affine Gaudin models with multiplicities:

- Gaudin Lax matrix and twist function

\[ \Gamma(z, x) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{J_{[p]}^{(r)}(x)}{(z-z_r)^{p+1}} \quad \text{and} \quad \varphi(z) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{\ell_{p}^{(r)}}{(z-z_r)^{p+1}} - \ell_\infty \]

- Takiff currents \( J_{[0]}^{(r)}(x), \cdots, J_{[m_r-1]}^{(r)}(x) \)

- Quadratic charges \( Q_i \) associated with zeroes \( \zeta_1, \cdots, \zeta_M \) of \( \varphi(z) \)

- Hamiltonian \( \mathcal{H} = \sum_{i=1}^{M} \epsilon_i Q_i \rightarrow \text{integrability} \)

Relativistic invariance:

- Lorentz invariance \( \Leftrightarrow \epsilon_i = \pm 1 \)

- Relabel the zeroes into two types: \( \zeta_i^+ \) and \( \zeta_i^- \)
Coupling affine Gaudin models

Model (1): sites $z_1^{(1)}, \ldots, z_{N_1}^{(1)}$

Model (2): sites $z_{N_1+1}^{(2)}, \ldots, z_{N_1+N_2}^{(2)}$
Coupling affine Gaudin models

Model (1): sites $z_1^{(1)}, \ldots, z_{N_1}^{(1)}$

Model (2): sites $z_{N_1+1}^{(2)}, \ldots, z_{N_1+N_2}^{(2)}$

Coupled model ($N_1 + N_2$ sites)

\[ z_r = z_r^{(1)} - \frac{1}{2\gamma}, \quad z_r = z_r^{(2)} + \frac{1}{2\gamma} \]

$\mathcal{H} \xrightarrow{\gamma \rightarrow 0} \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$

Preserves relativistic invariance
Integrable $\sigma$-models from affine Gaudin models
PCM as an affine Gaudin model

[Vicedo '17]

- Canonical fields on $T^*G$:
  - Coordinate fields $\phi_i(x)$ and conjugate momenta fields $\pi_i(x)$

- Canonical bracket:
  \[
  \{\pi_i(x), \phi_j(y)\} = \delta_{ij} \delta(x - y), \quad \{\phi_i(x), \phi_j(y)\} = \{\pi_i(x), \pi_j(y)\} = 0
  \]

- Well chosen combinations of $\phi_i(x)$, $\partial_x \phi_i(x)$ and $\pi_i(x)$
  \[\rightarrow\] Takiff currents $J_{[0]}(x)$ and $J_{[1]}(x)$ of multiplicity 2
PCM as an affine Gaudin model

[Vicedo ’17]

- Canonical fields on $T^* G$:
  - coordinate fields $\phi_i(x)$ and conjugate momenta fields $\pi_i(x)$
- Canonical bracket:
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  \]
- Well chosen combinations of $\phi_i(x)$, $\partial_x \phi_i(x)$ and $\pi_i(x)$
  - $\rightarrow$ Takiff currents $J_{[0]}(x)$ and $J_{[1]}(x)$ of multiplicity 2
- Affine Gaudin model with one site, with Takiff currents $J_{[0]}$ and $J_{[1]}$
- Hamiltonian field theory on $T^* G \Leftrightarrow$ Lagrangian field theory on $G$
- Inverse Legendre transform $\rightarrow$ action
  \[
  S[g] = K \int_{\Sigma} dx \, dt \, \kappa (j_+, j_-) + k \, l_{WZ}[g]
  \]
  - $\rightarrow$ principal chiral model with Wess-Zumino term
Integrable coupled $\sigma$-model as affine Gaudin model

[Delduc SL Magro Vicedo 1811.12316 and 1903.00368]

- PCM with Wess-Zumino term $\leftrightarrow$ model with 1 site of multiplicity 2
- Application of the general coupling procedure
  $\rightarrow$ model with $N$ sites of multiplicity 2

- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - levels $\ell_0^{(r)}$ and $\ell_1^{(r)}$

- Twist function: $\varphi(z) = \sum_{r=1}^{N} \left( \frac{\ell_0^{(r)}}{z - z_r} + \frac{\ell_1^{(r)}}{(z - z_r)^2} \right) - 1$
Integrable coupled $\sigma$-model as affine Gaudin model

[Delduc SL Magro Vicedo 1811.12316 and 1903.00368]

- PCM with Wess-Zumino term $\leftrightarrow$ model with 1 site of multiplicity 2
- Application of the general coupling procedure $\rightarrow$ model with $N$ sites of multiplicity 2

- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - zeroes $\zeta_1, \cdots, \zeta_{2N}$

- Twist function: $\varphi(z) = -\frac{\prod_{i=1}^{2N}(z - \zeta_i)}{\prod_{r=1}^{N}(z - z_r)^2}$

Lorentz invariance: two types of zeroes $\zeta^\pm_i$ (parameter $\epsilon_i = \pm 1$ in $H$

Factorisation of the twist function $ϕ(z) = -ϕ^+(z)ϕ^-(z)$, $ϕ^\pm(z) = N\prod_{r=1}^{N}(z - \zeta^\pm_r)$
Integrable coupled $\sigma$-model as affine Gaudin model

[Delduc SL Magro Vicedo 1811.12316 and 1903.00368]

- PCM with Wess-Zumino term $\leftrightarrow$ model with 1 site of multiplicity 2
- Application of the general coupling procedure $\rightarrow$ model with $N$ sites of multiplicity 2

- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - zeroes $\zeta^+_1, \cdots, \zeta^+_N$ and $\zeta^-_1, \cdots, \zeta^-_N$

- Twist function: $\varphi(z) = -\frac{\prod_{i=1}^{2N} (z - \zeta_i)}{\prod_{r=1}^N (z - z_r)^2}$

- Lorentz invariance: two types of zeroes $\zeta^\pm_i$ (parameter $\epsilon_i = \pm 1$ in $\mathcal{H}$)
- Factorisation of the twist function
  $$
  \varphi(z) = -\varphi_+(z)\varphi_-(z),
  \varphi_\pm(z) = \prod_{r=1}^N \frac{z - \zeta_r^\pm}{z - z_r}
  $$
Integrable coupled $\sigma$-model

- Inverse Legendre transform $\rightarrow$ action in terms of $j_{\pm}^{(r)} = g^{(r)-1} \partial_{\pm} g^{(r)}$

$$S = \sum_{r,s=1}^{N} \int dx \, dt \, \rho_{rs} \kappa(j_{\pm}^{(r)}, j_{\pm}^{(s)}) + \sum_{r=1}^{N} k_r \, l_{WZ}[g^{(r)}]$$

$$k_r = \frac{1}{2} \text{res}_{z=z_r} \varphi_+(z) \varphi_-(z) \, dz, \quad \rho_{rs} = \frac{1}{2} \text{res}_{w=z_s} \left(\text{res}_{z-z} \frac{\varphi_+(z) \varphi_-(w)}{z-w} \, dz \, dw\right) - \delta_{rs} \frac{k_r}{2}$$

- Action directly related to the Hamiltonian integrable structure

- $\rho_{rs}$ and $k_r$ invariant under $z_r \mapsto z_r + a$ and $\zeta_r^{\pm} \mapsto \zeta_r^{\pm} + a$

$\rightarrow 3N - 1$ free parameters

- Lax pair:

$$L_{\pm}(z) = \sum_{r=1}^{N} \frac{\varphi_{\pm, r}(z_r)}{\varphi_{\pm, r}(z)} j_{\pm}^{(r)}, \quad \varphi_{\pm, r}(z) = (z - z_r) \varphi_{\pm}(z)$$
Conclusion
Conclusion: new classical integrable $\sigma$-models

- Integrable $\eta$ and $\lambda$ deformation of the PCM as AGM?
  $\rightarrow$ split double pole into two simple poles
  $\rightarrow$ change the Takiff currents

- Integrable deformation of the coupled $\sigma$-model with $N$ copies
  $\rightarrow$ whole panorama of models to explore
  (first results in [Delduc SL Magro Vicedo ’19] and [SL ’19])
  $\rightarrow$ contains and extends the models of [Georgiou Sfetsos ’18]?

- Integrable $\sigma$-model on symmetric space $G/H$ (or $\mathbb{Z}_T$-coset)
  $\rightarrow$ cyclotomic affine Gaudin model with 1 site
  $\rightarrow$ gauge symmetry

- Model with $N$ sites:
  $\rightarrow$ conjecture: integrable $\sigma$-model on $G \times \cdots \times G/H_{\text{diag}}$
Conclusion: quantum level

- Quantisation of integrable coupled \( \sigma \)-models
- Renormalisation group flow
  - \( \rightarrow \) preserves integrability ?
  - \( \rightarrow \) CFT fixed points ? rich structure in [Georgiou Sfetsos ’18]
    for deformed models
- S-matrix ?
- Quantum integrability ?
- Problem of non-ultralocality
- Analogy with the quantisation of finite Gaudin models
  - \( \rightarrow \) see Benoit Vicedo’s talk
Thank you!