Integrable coupled sigma-models from affine Gaudin models

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Collaboration with François Delduc (CNRS, ENS de Lyon), Marc Magro (ENS de Lyon) and Benoit Vicedo (University of York): [1811.12316] and [1903.00368]

Introduction

Introduction: integrable σ-models

- σ-models: 2d fields theories
- Applications to high energy physics, condensed matter, ...
- Integrable σ-models
  - Principal chiral model, symmetric space σ-model, ...
  - Superstring on $AdS_5 \times S^5$ (AdS/CFT)
  - Integrable deformations
- Common algebraic structure behind their integrability
  $\rightarrow$ affine Gaudin models

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- Common algebraic structure behind their integrability
  $\rightarrow$ affine Gaudin models

Introduction: affine Gaudin models

- Gaudin models: integrable systems associated with Lie algebras $\mathfrak{g}$
  - See talk by Benoit Vicedo
  - Finite Gaudin models $\leftrightarrow$ $\mathfrak{g}$ semi-simple finite dimensional
    $\rightarrow$ integrable spin chains
  - Affine Gaudin models (AGM) $\leftrightarrow$ $\mathfrak{g}$ affine Kac-Moody algebra
    $\rightarrow$ integrable 2d field theories
    $\rightarrow$ contain integrable σ-models

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    $\rightarrow$ contain integrable σ-models

- Application of seeing integrable σ-models as AGM ?

- Quantisation: circumvent non-ultralocality issues

- Previously known integrable σ-models: small class among AGM
  $\rightarrow$ construction of new classical integrable σ-models
  (coupling together known ones)

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Contents

Integrable sigma models

Principal chiral model

Principal Chiral Model

- PCM: integrable σ-model on semi-simple Lie group G
- Field

$$g : \Sigma \longrightarrow G$$

Σ: 2d Minkowski (t, x)

Principal Chiral Model

- PCM: integrable σ-model on semi-simple Lie group G
- Field and currents

$$g : \Sigma \longrightarrow G \qquad\qquad j_\pm = g^{-1}\partial_\pm g : \Sigma \longrightarrow \mathfrak{g}$$

2d Minkowski (t, x)
Light-cone coordinates $x_\pm = (t \pm x)/2$

Lie algebra

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Lie algebra

- Action:

$$S_{\text{PCM}}[g] = K \int_\Sigma \text{d}x \, \text{d}t \;\, \kappa\,(j_+, j_-)$$

Killing form

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Killing form

- Equation of motion: $\partial_+ j_- + \partial_- j_+ = 0$

Integrability of the PCM

- Equation of motion of the PCM $\Leftrightarrow$ zero curvature equation

$$\partial_+ \mathcal{L}_-(z) - \partial_- \mathcal{L}_+(z) + [\mathcal{L}_+(z), \mathcal{L}_-(z)] = 0, \qquad \forall\, z \in \mathbb{C}$$

  of Zakharov-Mikhailov Lax connection $\mathcal{L}_\pm(z) = \dfrac{j_\pm}{1 \mp z}$

- Spectral parameter z: auxiliary complex parameter

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- Monodromy of the Lax matrix $\mathcal{L}(z) = \frac{1}{2}\big(\mathcal{L}_+(z) - \mathcal{L}_-(z)\big)$

$$T(z) = \mathrm{P}\overleftarrow{\exp}\left(- \int \mathrm{d}x\; \mathcal{L}(z) \right)$$

- Infinite number of conserved charges: $\mathcal{Q}_n(z) = \mathrm{Tr}\big(T(z)^n\big)$

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- Poisson brackets of $\mathcal{L}(z)$ of Maillet form $\rightarrow$ $\{\mathcal{Q}_n(z), \mathcal{Q}_m(w)\} = 0$
  $\rightarrow$ integrability

Adding a Wess-Zumino term

- Continuous integrable deformations of the PCM (η-deformations, λ-deformations, ...)

- Adding a Wess-Zumino term

$$S[g] = K \int_\Sigma \mathrm{d}x\,\mathrm{d}t\ \kappa\,(j_+, j_-) + k\ I_{\mathrm{WZ}}[g]$$

Remark: $k = K \rightarrow$ conformal Wess-Zumino-Witten point

- Still possesses a Lax connection $\mathcal{L}_\pm(z)$

- Lax matrix $\mathcal{L}(z) = \frac{1}{2}\big(\mathcal{L}_+(z) - \mathcal{L}_-(z)\big)$ still satisfies Maillet bracket
  $\rightarrow$ modifies the form of the bracket

Affine Gaudin models

[Vicedo '17] and [Delduc SL Magro Vicedo '18]

Affine Gaudin models

- Algebraic origin related to affine Kac-Moody algebras
  $\rightarrow$ see Benoit Vicedo's talk

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- Defining data:
  - sites: points $z_1, \cdots, z_N$ in $\mathbb{C}$
  - levels: numbers $\ell^{(1)}, \cdots, \ell^{(N)}$

- Defined at the Hamiltonian level
- Phase space: Kac-Moody currents $J^{(r)}(x)$ ($\mathfrak{g}$-valued fields)
- Poisson bracket: $J^{(r)}(x) = J^{(r)}_a(x)\, I^a$

$$\left\{ J^{(r)}_a(x), J^{(s)}_b(y) \right\} = \delta_{rs} \left(f_{ab}{}^c\, J^{(r)}_c(x)\, \delta(x-y) - \ell^{(r)}\, \kappa_{ab}\, \partial_x \delta(x-y) \right)$$

$$[I_a, I_b] = f_{ab}{}^c\, I_c \qquad \text{and} \qquad \kappa_{ab} = \kappa(I_a, I_b)$$

Gaudin Lax matrix, twist function and Hamiltonian

- Gaudin Lax matrix and twist function

$$\Gamma(z,x) = \sum_{r=1}^{N} \frac{J^{(r)}(x)}{z - z_r}, \qquad \varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} - \ell^{\infty}$$

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- Zeroes $\zeta_1, \cdots, \zeta_N$ of the twist function

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- Zeroes $\zeta_1, \cdots, \zeta_N$ of the twist function

- Quadratic charges associated with zeroes:

$$\mathcal{Q}_i = \operatorname*{res}_{z=\zeta_i} \mathcal{Q}(z)\, \mathrm{d}z, \qquad \mathcal{Q}(z) = -\frac{1}{2\varphi(z)} \int \mathrm{d}x\, \kappa(\Gamma(z,x), \Gamma(z,x))$$

- Involution: $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$

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- Involution: $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$

- Hamiltonian: $\mathcal{H} = \sum_{i=1}^{N} \epsilon_i \, \mathcal{Q}_i$ ($\mathcal{Q}_i$ conserved and in involution)

Integrability

- Lax matrix: $\mathcal{L}(z,x) = \dfrac{\Gamma(z,x)}{\varphi(z)}$
- Zero curvature equation: their exists $\mathcal{M}(z,x)$ such that

$$\{\mathcal{H}, \mathcal{L}(z,x)\} - \partial_x \mathcal{M}(z,x) + \big[\mathcal{M}(z,x), \mathcal{L}(z,x)\big] = 0$$

$\rightarrow$ conserved quantities from the monodromy matrix of $\mathcal{L}(z,x)$

- Poisson brackets of the Kac-Moody currents
  $\rightarrow$ Maillet bracket for $\mathcal{L}(z,x)$ (controlled by $\varphi(z)$)
  $\rightarrow$ conserved quantities in involution
- Integrability automatic !

A few more results and concepts

Affine Gaudin models with multiplicities:

- Gaudin Lax matrix and twist function

$$\Gamma(z,x) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{J^{(r)}_{[p]}(x)}{(z-z_r)^{p+1}} \quad \text{and} \quad \varphi(z) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{\ell^{(r)}_p}{(z-z_r)^{p+1}} - \ell^\infty$$

- Takiff currents $J^{(r)}_{[0]}(x), \cdots, J^{(r)}_{[m_r-1]}(x)$
- Quadratic charges $\mathcal{Q}_i$ associated with zeroes $\zeta_1, \cdots, \zeta_M$ of $\varphi(z)$
- Hamiltonian $\mathcal{H} = \sum_{i=1}^{M} \epsilon_i \, \mathcal{Q}_i$ → integrability

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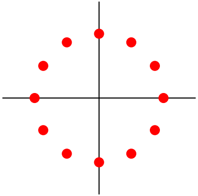
- Hamiltonian $\mathcal{H} = \sum_{i=1}^{M} \epsilon_i \, \mathcal{Q}_i \to$ integrability

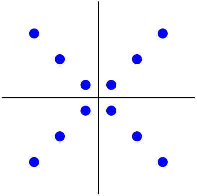
Relativistic invariance:

- Lorentz invariance $\Leftrightarrow \epsilon_i = \pm 1$
- Relabel the zeroes into two types: ζ_i^+ and ζ_i^-

Coupling affine Gaudin models



Model (1): sites $z_1^{(1)}, \cdots, z_{N_1}^{(1)}$



Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

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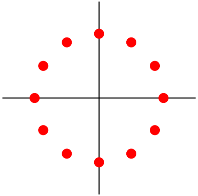

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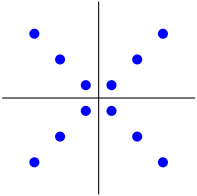


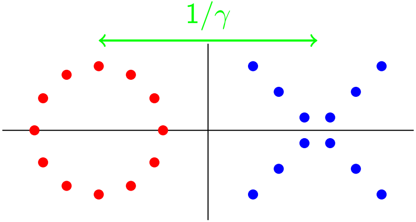
Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

Coupled model ($N_1 + N_2$ sites)

$$z_r = z_r^{(1)} - \frac{1}{2\gamma}, \qquad z_r = z_r^{(2)} + \frac{1}{2\gamma}$$

$$\mathcal{H} \xrightarrow{\gamma \to 0} \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$$

Preserves relativistic invariance



Integrable σ-models from affine Gaudin models

PCM as an affine Gaudin model

[Vicedo '17]

- Canonical fields on T^*G:

  coordinate fields $\phi_i(x)$ and conjugate momenta fields $\pi_i(x)$

- Canonical bracket:

$$\{\pi_i(x), \phi_j(y)\} = \delta_{ij}\,\delta(x-y), \quad \{\phi_i(x), \phi_j(y)\} = \{\pi_i(x), \pi_j(y)\} = 0$$

- Well chosen combinations of $\phi_i(x)$, $\partial_x \phi_i(x)$ and $\pi_i(x)$
  $\rightarrow$ Takiff currents $J_{[0]}(x)$ and $J_{[1]}(x)$ of multiplicity 2

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  $\rightarrow$ Takiff currents $J_{[0]}(x)$ and $J_{[1]}(x)$ of multiplicity 2

- Affine Gaudin model with one site, with Takiff currents $J_{[0]}$ and $J_{[1]}$
- Hamiltonian field theory on T^*G $\Leftrightarrow$ Lagrangian field theory on G
- Inverse Legendre transform $\rightarrow$ action

$$S[g] = K \int_\Sigma \mathrm{d}x\, \mathrm{d}t\ \kappa\,(j_+, j_-) + \ k\, I_{\mathrm{WZ}}[g]$$

$\rightarrow$ principal chiral model with Wess-Zumino term

Integrable coupled σ-model as affine Gaudin model

[Delduc SL Magro Vicedo 1811.12316 and 1903.00368]

- PCM with Wess-Zumino term $\leftrightarrow$ model with 1 site of multiplicity 2
- Application of the general coupling procedure
  $\rightarrow$ model with N sites of multiplicity 2
- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - levels $\ell_0^{(r)}$ and $\ell_1^{(r)}$
- Twist function: $\varphi(z) = \sum_{r=1}^{N} \left(\frac{\ell_0^{(r)}}{z - z_r} + \frac{\ell_1^{(r)}}{(z - z_r)^2} \right) - 1$

Integrable coupled σ-model as affine Gaudin model

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- Application of the general coupling procedure

  $\rightarrow$ model with N sites of multiplicity 2

- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - zeroes $\zeta_1, \cdots, \zeta_{2N}$

- Twist function: $\varphi(z) = -\dfrac{\prod_{i=1}^{2N}(z-\zeta_i)}{\prod_{r=1}^{N}(z-z_r)^2}$

Integrable coupled σ-model as affine Gaudin model

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- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - zeroes $\zeta_1^+, \cdots, \zeta_N^+$ and $\zeta_1^-, \cdots, \zeta_N^-$

- Twist function: $\varphi(z) = -\dfrac{\prod_{i=1}^{2N}(z-\zeta_i)}{\prod_{r=1}^{N}(z-z_r)^2}$

- Lorentz invariance: two types of zeroes $\zeta_i^\pm$ (parameter $\epsilon_i = \pm 1$ in $\mathcal{H}$)
- Factorisation of the twist function

$$\varphi(z) = -\varphi_+(z)\varphi_-(z), \qquad \varphi_\pm(z) = \prod_{r=1}^{N} \frac{z-\zeta_r^\pm}{z-z_r}$$

Integrable coupled σ-model

- Inverse Legendre transform $\rightarrow$ action in terms of $j_\pm^{(r)} = g^{(r)\,-1}\partial_\pm g^{(r)}$

$$S = \sum_{r,s=1}^{N} \int \mathrm{d}x\,\mathrm{d}t\; \rho_{rs}\, \kappa\Big(j_+^{(r)}, j_-^{(s)}\Big) + \sum_{r=1}^{N} k_r\, I_{\mathrm{WZ}}\Big[g^{(r)}\Big]$$

$$k_r = \frac{1}{2} \operatorname*{res}_{z=z_r} \varphi_+(z)\varphi_-(z)\,\mathrm{d}z, \qquad \rho_{rs} = \frac{1}{2} \operatorname*{res}_{z=z_r} \left(\operatorname*{res}_{w=z_s} \frac{\varphi_+(z)\varphi_-(w)}{z-w}\,\mathrm{d}z\,\mathrm{d}w \right) - \delta_{rs}\frac{k_r}{2}$$

- Action directly related to the Hamiltonian integrable structure

- ρ_{rs} and k_r invariant under $z_r \mapsto z_r + a$ and $\zeta_r^\pm \mapsto \zeta_r^\pm + a$
  $\rightarrow 3N - 1$ free parameters

- Lax pair:

$$\mathcal{L}_\pm(z) = \sum_{r=1}^{N} \frac{\varphi_{\pm,r}(z_r)}{\varphi_{\pm,r}(z)} j_\pm^{(r)}, \qquad \varphi_{\pm,r}(z) = (z - z_r)\varphi_\pm(z)$$

Conclusion

Conclusion: new classical integrable σ-models

- Integrable η and λ deformation of the PCM as AGM?
  $\rightarrow$ split double pole into two simple poles
  $\rightarrow$ change the Takiff currents
- Integrable deformation of the coupled σ-model with N copies
  $\rightarrow$ whole panorama of models to explore
  (first results in [Delduc SL Magro Vicedo '19] and [SL '19])
  $\rightarrow$ contains and extends the models of [Georgiou Sfetsos '18]?
- Integrable σ-model on symmetric space G/H (or $\mathbb{Z}_T$-coset)
  $\rightarrow$ cyclotomic affine Gaudin model with 1 site
  $\rightarrow$ gauge symmetry
- Model with N sites:
  $\rightarrow$ conjecture: integrable σ-model on $G \times \cdots \times G/H_{\text{diag}}$

Conclusion: quantum level

- Quantisation of integrable coupled σ-models
- Renormalisation group flow
  $\rightarrow$ preserves integrability ?
  $\rightarrow$ CFT fixed points ? rich structure in [Georgiou Sfetsos '18] for deformed models
- S-matrix ?
- Quantum integrability ?
- Problem of non-ultralocality
- Analogy with the quantisation of finite Gaudin models
  $\rightarrow$ see Benoit Vicedo's talk

Thank you !