Multi-component Derivative NLS Equations Related to Symmetric Spaces

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Based on a joint work with Vladimir Gerdjikov (Sofia) and Rossen Ivanov (Dublin):

- VSG, GGG, RII - *On integrable wave interactions and Lax pairs on symmetric spaces*, Wave Motion **71** (2017), 53–70 (E-print: `arXiv:1607.06940`).

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Motivation
Integrable Systems

The generalized Zakharov-Shabat systems:

$$L\psi(x,t,\lambda) \equiv i\frac{\partial \psi}{\partial x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0,$$

is a starting point for the integrability via ISM for a wide class of NLEEs. This includes: NLS, sine-Gordon, N-wave equation, Toda chain and many others.

NLS on symmetric spaces

Fordy A.P. and Kulish P.P., Comm. Math. Phys. **89** (1983), 427–443.

DNLS on symmetric spaces

Fordy A.P., J. Phys. A: Math. Gen. **17** (1984), 1235–1245.

Derivative NLS equations (1 of 2)

Derivative NLS equation (Kaup-Newell form)

$$\mathrm{i}q_t + q_{xx} + \epsilon \mathrm{i}(|q|^2 q)_x = 0.$$

D. J. Kaup and A. C. Newell, J. Math. Phys. **19** (1978), 798–801.

Gerdjikov-Ivanov form (gauge-equivalent to KN equation):

$$\mathrm{i}q_t + q_{xx} + \epsilon \mathrm{i} q^2 q_x^* + \frac{1}{2}|q|^4 q(x,t) = 0.$$

V. S. Gerdjikov, and M. I. Ivanov, Bulgarian J. Phys. **10** (1983) No.1, 13–26; No.2, 130–143.

Chen-Lee-Liu form:

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}|q|^2 q_x = 0.$$

Y. C. Lee, H. H. Chen and C. S. Liu, Phys. Scr. **20** (1979) 490–492.

Derivative NLS equations (2 of 2)

Lax pair (quadratic bundle):

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2\sigma_3.$$

Applications – plasma physics:

- small-amplitude nonlinear Alfvén waves in a low-β plasma,
- large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma.

DNLS equations are related also to the 2D massive Thirring model

$$\left| \begin{array}{l} -iu_t - iu_{xx} + 2v + 2|v|^2 u = 0 \\ -iv_t + iv_{xx} + 2u + 2|u|^2 v = 0 \end{array} \right. .$$

E. A. Kuznetsov, A. V. Mikhailov, Theor. Math. Phys. **30** (1977) 193–200.

The aim of this talk is to present multi-component generalisations of DNLS equations to **A.III** symmetric spaces.

$\mathbb{Z}_2$-graded Lie algebras

NLEE related to $\mathbb{Z}_2$-graded Lie algebras:

$$\mathfrak{g} \simeq \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, \qquad \mathfrak{g}^{(0)} \equiv \{X \in \mathfrak{g},\ [J, X] = 0\},$$
$$\mathfrak{g}^{(1)} \equiv \{Y \in \mathfrak{g},\ JY + YJ = 0\}, \quad J = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}.$$

Example ($\mathfrak{g} \simeq sl(n, \mathbb{C})$)

Take the typical representation of $\mathbf{A}_{n-1} \simeq sl(n, \mathbb{C})$, $n = p + q$. Then

- the subalgebra $\mathfrak{g}^{(0)}$ consists of block-diagonal matrices with two nontrivial blocks $p \times p$ and $q \times q$;
- the linear subspace $\mathfrak{g}^{(1)}$ consists of block-off-diagonal matrices with two nontrivial blocks $p \times q$ and $q \times p$.

Symmetric spaces

A homogeneous space of a Lie group $\mathfrak{G}$ is any differentiable manifold $\mathcal{M}$ on which $\mathfrak{G}$ acts transitively.
The subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which leaves a given point $p_0 \in \mathcal{M}$ fixed, is called the isotropy group at p_0.
The coset space $\mathfrak{G}/\mathfrak{K}$ is called symmetric space.
To every symmetric space, there corresponds an involution of $\mathfrak{G}$.

The only Hermitian symmetric spaces, compatible with the dispersion law of DNLS equation are

A.III	$\frac{SU(p+q)}{S(U(p)\times U(q))}$	**C.I**	$\frac{Sp(n)}{U(n)}$
BD.I	$\frac{SO(p+q)}{SO(p)\times SU(q)}, \quad p=2$	**D.III**	$\frac{Sp(n)}{U(n)}$

$\mathbb{Z}_2$-graded Lie algebras and symmetric spaces

The subspace $\mathfrak{g}^{(1)}$ is the co-adjoint orbit passing through J which will play role of phase space for our NLEE.

A. G. Reiman, J. Soviet Math. **19** (1982) No.5, 1507–1545.

A. N. Leznov and M. V. Saveliev, *Group-Theoretical Methods for Integration of Nonlinear Dynamical Systems*, Progress in Mathematical Physics, Birkhäuser, Basel (1992).

Remark

If we introduce in addition a complex structure on $\mathfrak{g}$ the grading from the previous example can be related to the Hermitian symmetric space $\mathbf{A.III} \simeq SU(p+q)/(S(U(p) \otimes U(q))$.

Helgason S., *Differential geometry, Lie groups and Symmetric Spaces*, Graduate Studies in Mathematics **34**, AMS, Providence, Rhode Island (2001).

Example (DNLS, **A.III**, Gerdjikov-Ivanov equation)

Take

$$L \equiv i\frac{\partial}{\partial x} + (U_2(x,t) + \lambda Q(x,t) - \lambda^2 J), \qquad Q(x,t) = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{p} & 0 \end{pmatrix},$$

$$M \equiv i\frac{\partial}{\partial t} + (V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J),$$

where $Q(x,t)$, $V_3(x,t) \in \mathfrak{g}^{(1)}$ and $U_2(x,t)$, $V_2(x,t)$ and $V_4(x,t) \in \mathfrak{g}^{(0)}$. The resulting system of NLEEs is:

$$i\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathbf{q}}{\partial x^2} - \frac{i}{2}\mathbf{q}\frac{\partial \mathbf{p}}{\partial x}\mathbf{q} + \frac{1}{4}\mathbf{qpqpq} = 0,$$

$$-i\frac{\partial \mathbf{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathbf{p}}{\partial x^2} + \frac{i}{2}\mathbf{p}\frac{\partial \mathbf{q}}{\partial x}\mathbf{p} + \frac{1}{4}\mathbf{pqpqp} = 0.$$

V. S. Gerdjikov and M. I. Ivanov, Bulgarian J. Phys. **10** (1983) No.1, 13–26; No.2, 130–143.

Example (DNLS, **A.III**, Kaup-Newell equation)

Taking the gauge-equivalent Lax pair (with $\tilde{Q}(x,t)$, $\tilde{V}_3(x,t) \in \mathfrak{g}^{(1)}$ and $\tilde{U}_2(x,t)$ and $\tilde{V}_2(x,t) \in \mathfrak{g}^{(0)}$):

$$\tilde{L}\tilde{\psi} \equiv i\frac{\partial\tilde{\psi}}{\partial x} + (\lambda\tilde{Q}(x,t) - \lambda^2 J)\tilde{\psi}(x,t,\lambda) = 0, \qquad \tilde{Q}(x,t) = \begin{pmatrix} 0 & \tilde{\mathbf{q}} \\ \tilde{\mathbf{p}} & 0 \end{pmatrix},$$

$$\tilde{M}\tilde{\psi} \equiv i\frac{\partial\tilde{\psi}}{\partial t} + (\lambda\tilde{V}_3(x,t) + \lambda^2\tilde{V}_2(x,t) + \lambda^3\tilde{Q}(x,t) - \lambda^4 J)\tilde{\psi}(x,t,\lambda) = 0.$$

will produce the system:

$$i\frac{\partial\tilde{\mathbf{q}}}{\partial t} + \frac{\partial^2\tilde{\mathbf{q}}}{\partial x^2} + i\frac{\partial(\tilde{\mathbf{q}}\tilde{\mathbf{p}}\tilde{\mathbf{q}})}{\partial x} = 0,$$
$$-i\frac{\partial\tilde{\mathbf{p}}}{\partial t} + \frac{\partial^2\tilde{\mathbf{p}}}{\partial x^2} - i\frac{\partial(\tilde{\mathbf{p}}\tilde{\mathbf{q}}\tilde{\mathbf{p}})}{\partial x} = 0.$$

D. J. Kaup and A. C. Newell, J. Math. Phys. **19** (1978), 798–801.

The class of admissible potentials

Conditions on the class of potentials

- C1 $Q(x,t)$ is smooth enough and falls off to zero fast enough for $x \to \pm\infty$ for all t.
- C2 $Q(x,t)$ is such that L has at most finite number of simple discrete eigenvalues.

Jost solutions:

$$\lim_{x\to\infty} \psi(x,\lambda) e^{i\lambda^2 Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda) e^{i\lambda^2 Jx} = \mathbb{1}.$$

Normalised Jost solutions:

$$X_+(x,\lambda) = \psi(x,\lambda) e^{i\lambda^2 Jx}, \qquad X_-(x,\lambda) = \phi(x,\lambda) e^{i\lambda^2 Jx}.$$

Integral representations (Voltera type):

$$X_\pm(x,\lambda) = \mathbb{1} + i\int_{\pm\infty}^{x} dy e^{-i\lambda^2 J(x-y)} Q(y) X_\pm(y,\lambda) e^{i\lambda^2 J(x-y)}.$$

Scattering matrix and the spectrum of L

Block structure of Jost solutions:

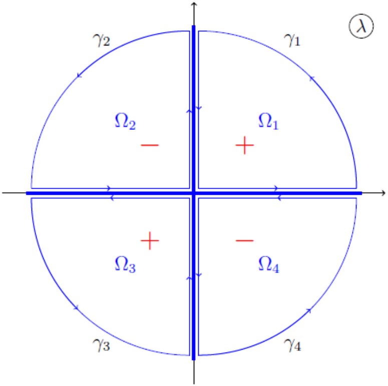
$$\psi(x,\lambda) = \left(|\psi^-(x,\lambda)\rangle, |\psi^+(x,\lambda)\rangle\right),$$
$$\phi(x,\lambda) = \left(|\phi^+(x,\lambda)\rangle, |\phi^-(x,\lambda)\rangle\right).$$

Scattering matrix:

$$\phi(x,\lambda) = \psi(x,\lambda)T(\lambda),$$
$$T(\lambda) = \begin{pmatrix} \mathbf{a}^+(\lambda) & -\mathbf{b}^-(\lambda) \\ \mathbf{b}^+(\lambda) & \mathbf{a}^-(\lambda) \end{pmatrix},$$

and its inverse:

$$\hat{T}(\lambda) \equiv \begin{pmatrix} \mathbf{c}^-(\lambda) & \mathbf{d}^-(\lambda) \\ -\mathbf{d}^+(\lambda) & \mathbf{c}^+(\lambda) \end{pmatrix},$$



The continuous spectrum of a $L(\lambda)$

Reduction conditions

Consider the following types of reductions:

$$
\begin{aligned}
&1) \qquad A_1 J A_1^{-1} = J, \qquad && \kappa_1 A_1 Q^\dagger A_1^{-1} = Q(x,t), \\
&3) \qquad A_3 J A_3^{-1} = -J, \qquad && \kappa_3 A_3 Q^* A_3^{-1} = -Q(x,t),
\end{aligned}
$$

where $\kappa_1^2 = \kappa_3^2 = 1$ and $A_1^2 = A_3^2 = \mathbb{1}$.
From $A_1 J A_1^{-1} = J$ (resp. $A_3 J A_3^{-1} = -J$) we find that A_1 is block-diagonal (resp. A_3 is block-off-diagonal) matrix.
If we introduce

$$
A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix},
$$

we obtain

$$
\begin{aligned}
&1) \qquad \kappa_1 a_1 \mathbf{p}^\dagger \hat{a}_2 = \mathbf{q}, \qquad && \kappa_1 a_2 \mathbf{q}^\dagger \hat{a}_1 = \mathbf{q}, \qquad && A_1 U_2^\dagger A_1^{-1} = U_2; \\
&3) \qquad \kappa_3 b_1 \mathbf{p}^* \hat{b}_2 = -\mathbf{q}, \qquad && \kappa_3 b_2 \mathbf{q}^* \hat{b}_1 = -\mathbf{p}, \qquad && A_3 U_2^* A_3^{-1} = -U_2.
\end{aligned}
$$

Reductions and multi-component DNLS equations

Example

As a result of reductions of type 1) the multicomponent GI equation reduces to:

$$i\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathbf{q}}{\partial x^2} - \frac{i\kappa_1}{2}\mathbf{q}a_2\frac{\partial \mathbf{q}^\dagger}{\partial x}\hat{a}_1\mathbf{q} + \frac{1}{4}\mathbf{q}a_2\mathbf{q}^\dagger\hat{a}_1\mathbf{q}a_2\mathbf{q}^\dagger\hat{a}_1\mathbf{q} = 0.$$

while the multicomponent KN equation reduces to:

$$i\frac{\partial \tilde{\mathbf{q}}}{\partial t} + \frac{\partial^2 \tilde{\mathbf{q}}}{\partial x^2} + i\kappa_1\frac{\partial}{\partial x}\left(\tilde{\mathbf{q}}a_2\tilde{\mathbf{q}}^\dagger\hat{a}_1\tilde{\mathbf{q}}\right) = 0.$$

Fundamental analytic solutions

Introduce Fundamental analytic solutions (FAS) by

$$\chi^+(x,\lambda) \equiv \left(|\phi^+\rangle, |\psi^+\rangle\right)(x,\lambda) = \phi(x,\lambda)\mathbf{S}^+(\lambda) = \psi(x,\lambda)\mathbf{T}^-(\lambda),$$
$$\chi^-(x,\lambda) \equiv \left(|\psi^-\rangle, |\phi^-\rangle\right)(x,\lambda) = \phi(x,\lambda)\mathbf{S}^-(\lambda) = \psi(x,\lambda)\mathbf{T}^+(\lambda),$$

where the block-triangular functions $\mathbf{S}^\pm(\lambda)$ and $\mathbf{T}^\pm(\lambda)$ are given by:

$$\mathbf{S}^+(\lambda) = \begin{pmatrix} \mathbb{1} & \mathbf{d}^-(\lambda) \\ 0 & \mathbf{c}^+(\lambda) \end{pmatrix}, \qquad \mathbf{T}^-(\lambda) = \begin{pmatrix} \mathbf{a}^+(\lambda) & 0 \\ \mathbf{b}^+(\lambda) & \mathbb{1} \end{pmatrix},$$
$$\mathbf{S}^-(\lambda) = \begin{pmatrix} \mathbf{c}^-(\lambda) & 0 \\ -\mathbf{d}^+(\lambda) & \mathbb{1} \end{pmatrix}, \qquad \mathbf{T}^+(\lambda) = \begin{pmatrix} \mathbb{1} & -\mathbf{b}^-(\lambda) \\ 0 & \mathbf{a}^-(\lambda) \end{pmatrix},$$

Generalized Gauss decompositions of $T(\lambda)$ and its inverse:

$$T(\lambda) = \mathbf{T}^-(\lambda)\hat{\mathbf{S}}^+(\lambda) = \mathbf{T}^+(\lambda)\hat{\mathbf{S}}^-(\lambda), \qquad \hat{T}(\lambda) = \mathbf{S}^+(\lambda)\hat{\mathbf{T}}^-(\lambda).$$

Minimal set of scattering data and RHP

On the real and imaginary axes $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$ are related by

$$\begin{aligned} \xi^+(x,\lambda) &= \xi^-(x,\lambda) G(x,\lambda), \qquad G(x,\lambda) = e^{-i\lambda^2 Jx} G_0(\lambda) e^{i\lambda^2 Jx}, \\ G_0(\lambda) &= S^+(\lambda)\hat{S}^-(\lambda). \end{aligned}$$

The function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of $L(\lambda)$.

RHP problem for the scattering data

Thus, the equation above can be understood as a (multiplicative) Riemann-Hilbert problem with canonical normalization ($\lim_{\lambda\to\infty} \xi^{\pm}(x,\lambda) = \mathbb{1}$): given the sewing function $G_0(x,\lambda)$, one can construct $\xi^{\pm}(x,\lambda)$.

Local coordinates on the co-adjoint orbit

Introduce local coordinate $Q_1(x,t)$ on the co-adjoint orbit $\mathfrak{g}^{(1)}$:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{q} \\ -\mathbf{p} & 0 \end{pmatrix},$$

where $\mathbf{q}$ and $\mathbf{p}$ are generic $p \times q$ and $q \times p$ matrices.
Introduce also the solution $\xi(x,t,\lambda)$ of a RHP with canonical normalization

$$\xi(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)) \in \mathfrak{G}, \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t) \in \mathfrak{g},$$

where $\mathcal{Q}(x,t,\lambda)$ is a formal series over the negative powers of λ whose coefficients Q_s take values in $\mathfrak{g}^{(0)}$ if s is even and in $\mathfrak{g}^{(1)}$ if s is odd.

Local coordinates and the RHP

The first few of these coefficients take the form:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{q} \\ -\mathbf{p} & 0 \end{pmatrix}, \quad Q_2(x,t) = \frac{1}{2}\begin{pmatrix} \mathbf{r} & 0 \\ 0 & \mathbf{s} \end{pmatrix}, \quad Q_3(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{v} \\ -\mathbf{w} & 0 \end{pmatrix}.$$

Besides we have requested that $\mathcal{Q}(x,t,\lambda)$ takes values in the Kac-Moody algebra determined by this grading: in other words $\mathcal{Q}(x,t,\lambda)$ satisfies

$$\mathcal{Q}(x,t,\lambda) = C_0\mathcal{Q}(x,t,-\lambda)C_0^{-1}, \qquad C_0 = \exp(\pi i J).$$

Then we can introduce $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as the non-negative parts of:

$$U(x,t,\lambda) = -\left(\lambda^a \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+ ,$$
$$V(x,t,\lambda) = -\left(\lambda^b \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+ ,$$

where a and b can be any integers.

Local coordinates and the multi-component DNLS equation

For DNLS equations we will fix up $a = 2$ and $b = 4$.

$U(x,t,\lambda)$ and $V(x,t,\lambda)$ can be calculated explicitely in terms of $Q_s(x,t)$ using:

$$\xi(x,t,\lambda)J\xi^{-1}(x,t,\lambda) = J + \sum_{s=1}^{\infty} \frac{1}{s!}\operatorname{ad}_{\mathcal{Q}}^{s} J,$$

$$\operatorname{ad}_{\mathcal{Q}} J = [\mathcal{Q}, J], \qquad \operatorname{ad}_{\mathcal{Q}}^{2} J = [\mathcal{Q}, [\mathcal{Q}, J]], \quad \dots$$

In particular for $a = 2$ and $b = 4$ we have:

$$U(x,t,\lambda) = -\left(\lambda^2 \xi J \hat{\xi}\right)_+ = -\lambda^2 J + \lambda Q(x,t) + U_2(x,t),$$

$$Q(x,t) = -[Q_1, J] = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{p} & 0 \end{pmatrix},$$

$$U_2(x,t) = -\frac{1}{2}[Q_1,[Q_1,J]] - [Q_2(x,t),J] = \frac{1}{2}\begin{pmatrix} \mathbf{qp} & 0 \\ 0 & -\mathbf{pq} \end{pmatrix}.$$

Local coordinates and the multi-component DNLS equation

Note that since $Q_2(x,t) \in \mathfrak{g}^{(0)}$ then $[Q_2(x,t), J] = 0$.
Similarly

$$
\begin{aligned}
V(x,t,\lambda) &= -\left(\lambda^4 \xi^{\pm} J \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ \\
&= V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J, \\
V_2(x,t) &= U_2(x,t), \qquad V_3(x,t) = -\frac{1}{2}\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} J - \frac{1}{6}\mathrm{ad}_{Q_1}^3 J, \\
V_4(x,t) &= -\frac{1}{2}\left(\mathrm{ad}_{Q_3}\mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_3} J\right) \\
&\quad - \frac{1}{6}\left(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1}^2 J\right) - \frac{1}{24}\mathrm{ad}_{Q_1}^4 J.
\end{aligned}
$$

Here we used again $[Q_2(x,t), J] = 0$ and $[Q_4(x,t), J] = 0$.

GI equation from RHP (1 of 3)

The special case $p = 1$ corresponds to the vector GI equation.
Here we assume that the FAS satisfy a canonical RHP with a reduction:

$$\xi^{\pm}(x,t,-\lambda) = \xi^{\pm,-1}(x,t,\lambda),$$

i.e., $\mathcal{Q}(x,t,\lambda) = -\mathcal{Q}(x,t,-\lambda)$ and therefore $Q_{2s}(x,t) = 0$.
As a result the expression for the Lax pair simplifies to

$$\begin{aligned}
U(x,t,\lambda) &= U_2(x,t) + \lambda Q(x,t) - \lambda^2 J, \quad Q(x,t) = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{p} & 0 \end{pmatrix}, \\
U_2(x,t) &= \frac{1}{2}\begin{pmatrix} \mathbf{qp} & 0 \\ 0 & -\mathbf{pq} \end{pmatrix}, \quad V_2(x,t) = U_2(x,t), \\
V_3(x,t) &= \begin{pmatrix} 0 & \mathbf{v} - \frac{1}{6}\mathbf{qpq} \\ \mathbf{w} - \frac{1}{6}\mathbf{pqp} & \end{pmatrix}, \\
V_4(x,t) &= \frac{1}{2}\begin{pmatrix} \mathbf{qw} + \mathbf{vp} - \frac{1}{12}\mathbf{qpqp} & 0 \\ 0 & -\mathbf{wq} - \mathbf{pv} + \frac{1}{12}\mathbf{pqpq} \end{pmatrix}.
\end{aligned}$$

GI equation from RHP (2 of 3)

The commutation $[L, M]$ must vanish identically with respect to λ. It is polynomial in λ with the following coefficients:

$$\begin{aligned}
\lambda^5: &\quad -[J, V_1] - [Q, J] = 0, && \Rightarrow \quad V_1 = Q, \\
\lambda^4: &\quad -[J, V_2] + [Q, V_1] - [U_2, J] = 0, && \Rightarrow \quad \text{identity} \\
\lambda^3: &\quad i\frac{\partial V_1}{\partial x} + [U_2, V_1] + [Q, V_2] = [J, V_3], &&
\end{aligned}$$

The last of these equations is fulfilled iff

$$\mathbf{v} = \frac{i}{2}\frac{\partial \mathbf{q}}{\partial x} + \frac{1}{6}\mathbf{qpq}, \qquad \mathbf{w} = -\frac{i}{2}\frac{\partial \mathbf{p}}{\partial x} + \frac{1}{6}\mathbf{pqp}.$$

GI equation from RHP (3 of 3)

The next equations are:

$$\begin{aligned}
\lambda^2 : \quad & i\frac{\partial V_2}{\partial x} + [U_2, V_2] + [Q, V_3] = [J, V_4] \equiv 0, \\
\lambda^1 : \quad & i\frac{\partial V_3}{\partial x} - i\frac{\partial Q}{\partial t} + [U_2, V_3] + [Q, V_4] = 0, \\
\lambda^0 : \quad & i\frac{\partial V_4}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_4] = 0.
\end{aligned}$$

Multi-component GI equations on **A.III** symmetric space

$$\begin{aligned}
i\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathbf{q}}{\partial x^2} - \frac{i}{2}\mathbf{q}\frac{\partial \mathbf{p}}{\partial x}\mathbf{q} + \frac{1}{4}\mathbf{qpqpq} = 0, \\
-i\frac{\partial \mathbf{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathbf{p}}{\partial x^2} + \frac{i}{2}\mathbf{p}\frac{\partial \mathbf{q}}{\partial x}\mathbf{p} + \frac{1}{4}\mathbf{pqpqp} = 0.
\end{aligned}$$

KN equation from RHP (1 of 3)

Lax pair (via gauge transformation):

$$\tilde{L} \equiv i\frac{\partial}{\partial x} + \lambda \tilde{Q}(x,t) - \lambda^2 J,$$
$$\tilde{M} \equiv i\frac{\partial}{\partial t} + \lambda \tilde{V}_3(x,t) + \lambda^2 \tilde{V}_2(x,t) + \lambda^3 \tilde{Q}(x,t) - \lambda^4 J,$$

where

$$\tilde{V}_3(x,t) = g_0^{-1} V_3(x,t) g_0(x,t), \qquad \tilde{V}_2(x,t) = g_0^{-1} V_2(x,t) g_0(x,t),$$
$$\tilde{Q}(x,t) = g_0^{-1} Q(x,t) g_0(x,t)$$

and the gauge $g_0(x,t)$ is defined uniquely by the equations:

$$i\frac{\partial g_0}{\partial x} + U_2(x,t) g_0(x,t) = 0, \qquad i\frac{\partial g_0}{\partial t} + V_4(x,t) g_0(x,t) = 0.$$

KN equation from RHP (2 of 3)

Note that $g_0(x,t)$ must be block-diagonal, so similarity transformations with it preserve the grading (the block-matrix structure).
Introduce

$$\tilde{Q} = \begin{pmatrix} 0 & \tilde{\mathbf{q}} \\ \tilde{\mathbf{p}} & 0 \end{pmatrix}$$

Applying the gauge transformation to $V_2(x,t)$:

$$\tilde{V}_2(x,t) = -\frac{1}{2} g_0^{-1} [Q_1, [Q_1(x,t), J]] = \frac{1}{2} \begin{pmatrix} \tilde{\mathbf{q}}\tilde{\mathbf{p}} & 0 \\ 0 & -\tilde{\mathbf{p}}\tilde{\mathbf{q}} \end{pmatrix}.$$

The compatibility condition of $\tilde{L}$ and $\tilde{M}$ implies

$$V_3 = \mathrm{ad}_J^{-1} \left(i \frac{\partial \tilde{Q}}{\partial x} + [\tilde{Q}, V_2] \right) = \frac{1}{2} \begin{pmatrix} 0 & i\tilde{\mathbf{q}}_x - \tilde{\mathbf{q}}\tilde{\mathbf{p}}\tilde{\mathbf{q}} \\ -i\tilde{\mathbf{p}}_x + \tilde{\mathbf{p}}\tilde{\mathbf{q}}\tilde{\mathbf{p}} & \end{pmatrix}.$$

KN equation from RHP (3 of 3)

Multi-component Kaup-Newell equations on A.III symmetric space

$$i\frac{\partial \tilde{\mathbf{q}}}{\partial t} + \frac{\partial^2 \tilde{\mathbf{q}}}{\partial x^2} + i\frac{\partial (\tilde{\mathbf{q}}\tilde{\mathbf{p}}\tilde{\mathbf{q}})}{\partial x} = 0,$$
$$-i\frac{\partial \tilde{\mathbf{p}}}{\partial t} + \frac{\partial^2 \tilde{\mathbf{p}}}{\partial x^2} - i\frac{\partial (\tilde{\mathbf{p}}\tilde{\mathbf{q}}\tilde{\mathbf{p}})}{\partial x} = 0.$$

Dressing method - main idea

Main idea: construction of a nontrivial (dressed) FAS $\chi^{\pm}(x,t,\lambda)$ from the known (bare) FAS $\chi_0^{\pm}(x,t,\lambda)$:

$$\chi^{\pm}(x,t,\lambda) = u(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda).$$

The dressing factor is analytic in the entire complex λ-plane, with the exception of the newly added simple pole singularities at $\lambda = \lambda_k^{\pm}$:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_k(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_k(x,t) \right).$$

The dressing factor must be a solution of the (dressing chain) equation

$$iu_x + U_2 u - uU_2^{(0)} + \lambda(Qu - uQ^{(0)}) + \lambda^2[u,J] = 0.$$

As a result, the residues $B_k(x,t)$ must satisfy:

$$i\partial_x B_k + (U_2 + \lambda_k^+ Q)B_k - B_k(U_2^{(0)} + \lambda_k^+ Q^{(0)}) + (\lambda_k^+)^2[B_k, J] = 0.$$

Dressing method - rank one projectors

In the simplest nontrivial case, B_k are rank 1 matrices of the form

$$B_k = |n_k\rangle\langle m_k|$$

($|n\rangle$ is a vector-column, $\langle m|$ is a vector-row as usual).
B_k will satisfy the dressing chain equation, if and only if

$$i\partial_x|n_k\rangle + \left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J\right)|n_k\rangle = 0,$$
$$i\partial_x\langle m_k| - \langle m_k|\left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J\right) = 0,$$

i.e.

$$|n_k\rangle = \chi^+(x,t,\lambda_k^+)|n_{k,0}\rangle, \qquad \langle m_k| = \langle m_{k,0}|\hat{\chi}_0^+(x,t,\lambda_k^+),$$

where $|n_{k,0}\rangle$ and $\langle m_{k,0}|$ are some constant vectors.

One can start with the trivial bare solutions $Q^{(0)} = 0$, $U_2^{(0)} = 0$, so that $\chi_0^+(x,t,\lambda) = \exp i(\lambda^2 Jx + \lambda^4 Jt)$ is known explicitly.

One-soliton solution

Introducing the notations

$$\theta(x,t) = 2\rho^2(\sin 2\varphi)x + 2\rho^4(\sin 4\varphi)t,$$
$$\phi(x,t) = 2\rho^2(\cos 2\varphi)x + 2\rho^4(\cos 4\varphi)t,$$

where $r(1) = i\sin\varphi$, and $r(-1) = \cos\varphi$ and when $A_1 = \mathbb{1}$,

$$e^{-2\xi_0} \equiv \frac{\sum_{j=2}^{n} |m_{0j}|^2}{|m_{01}|^2} \in \mathbb{R}_+.$$

One can write the one-soliton solution

$$\mathbf{q}_{j-1}(x,t) = Q_{1j} = 4\rho r(\kappa_1)\frac{m_{0j}e^{\xi_0}e^{-i\phi(x,t)}}{m_{01}\cosh(\theta(x,t)-\xi_0)}, \qquad j = 2,\ldots,n.$$

Conclusions

- We gave a brief overview of the ISM for multi-component integrable equations over symmetric spaces
- We have presented quadratic bundle Lax pairs on $\mathbb{Z}_2$-graded Lie algebras and on **A.III** symmetric spaces.
- We have also constructed the FAS and discussed briefly the spectral properties of the associated Lax operators.
- We have also formulated the Riemann-Hilbert problem for the KN and GI equations on **A.III**-symmetric spaces and derived explicit parametrization of the associated Lax operators.
- we have presented a modification of the dressing method and obtained 1-soliton solution for the multi-component GI equation.

Outlook

The results obtained here can be developed in several directions:

- To construct gauge covariant formulation of the multi-component KN and GI hierarchies on symmetric spaces, including the generating (recursion) operator and it spectral decomposition, the description of the infinite set of integrals of motion, the hierarchy of Hamiltonian structures.
- To study the gauge equivalent systems to the multi-component KN and GI equations on symmetric spaces.
- To study the associated Darboux transformations and their generalizations for DNLS equations over Hermitian symmetric spaces.
- To extend our results for the case of non-vanishing boundary conditions (a non-trivial background).
- To study quadratic bundles associated with other types of Hermitian symmetric spaces both for KN and for GI equations.

Thank you!

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