The stability of elliptic solutions of the focusing NLS equation

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  - Defocusing nonlinear Schrödinger (NLS) equation (Bottman, Deconinck, & Nivala, 2011)
  - Defocusing modified KdV equation (Deconinck & Nivala, 2010)

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  - Defocusing nonlinear Schrödinger (NLS) equation (Bottman, Deconinck, & Nivala, 2011)
  - Defocusing modified KdV equation (Deconinck & Nivala, 2010)
- Analytical description of some spectra of non-self-adjoint problems

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- **For all solutions, we obtain a fully analytical description of the stability spectrum**
- **For almost all spectrally stable solutions, we establish their orbital stability**

Stationary travelling wave solutions

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

- Ansatz

$$\psi = e^{-i\omega t}\phi(x) = e^{-i\omega t}R(x)e^{i\theta(x)}$$

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- Solutions

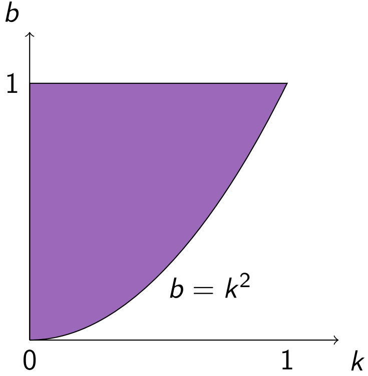
$$\begin{aligned} R^2(x) &= b - k^2\,\mathrm{sn}^2(x,k) \\ \omega &= \frac{1}{2}(1+k^2) - \frac{3}{2}b \\ \theta(x) &= \int_0^x \frac{c}{R^2(y)}dy \\ c^2 &= b(1-b)(b-k^2) \end{aligned}$$

Parameter space

$$R^2(x) = b - k^2 \operatorname{sn}^2(x, k)$$
$$c^2 = b(1-b)(b-k^2)$$

We require $R, c \in \mathbb{R}$ so

- $0 \leq k < 1$
- $k^2 \leq b \leq 1$



Parameter space

$$\phi(x) = R(x)e^{i\theta(x)}$$

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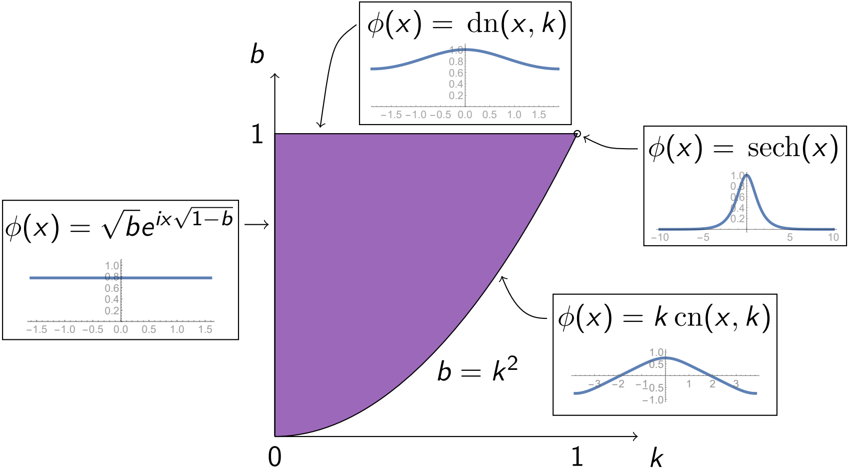

Stability of stationary solutions: Orbital, Spectral

Introducing

$$\psi(x,t) = e^{-i\omega t}\Psi(x,t),$$

we have that stationary solutions $\Psi(x,t) = \phi(x)$ are fixed points of

$$i\Psi_t + \omega\Psi + \frac{1}{2}\Psi_{xx} + |\Psi|^2\Psi = 0. \qquad (*)$$

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A fixed-point solution $\Psi(x,t) = \phi(x)$ of (*) is **orbitally stable** if

$$\forall\epsilon > 0, \exists\delta > 0 : \text{if } ||\Psi(x,0) - \phi(x)|| < \delta$$
$$\Rightarrow \inf_{\gamma,x_0} ||\Psi(x,t) - e^{i\gamma}\phi(x+x_0)|| < \epsilon.$$

Stability of solutions: Orbital, Spectral

To start, we consider infinitesimal perturbations: let

$$\Psi(x,t) = e^{i\theta(x)}\left(R(x) + \varepsilon(u(x,t) + iv(x,t)) + \mathcal{O}(\varepsilon^2)\right).$$

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$$\Psi(x,t) = e^{i\theta(x)}\left(R(x) + \varepsilon(u(x,t) + iv(x,t)) + \mathcal{O}(\varepsilon^2)\right).$$

To first order in ε, $u(x,t)$ and $v(x,t)$ satisfy

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} L_+ & S \\ -S & L_- \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

with

$$L_- = -\frac{1}{2}\partial_x^2 - R^2(x) - \omega + \frac{c^2}{2R^4(x)},$$
$$L_+ = -\frac{1}{2}\partial_x^2 - 3R^2(x) - \omega + \frac{c^2}{2R^4(x)},$$
$$S = \frac{c}{R^2(x)}\partial_x - \frac{cR'(x)}{R^3(x)}.$$

Stability of solutions: Orbital, Spectral

Since L_+, L_- and S do not depend on t, let

$$u(x,t) = e^{\lambda t} U(x;\lambda), \quad v(x,t) = e^{\lambda t} V(x;\lambda).$$

This gives the spectral problem

$$\begin{pmatrix} -S & L_- \\ -L_+ & -S \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}.$$

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A fixed-point solution $\Psi(x,t) = \phi(x)$ of (*) is **spectrally stable** if

$$\sigma(J\mathcal{L}) \cap \mathrm{RHP} = \emptyset.$$

Stability of solutions

Following earlier work (Bottman, Deconinck, Nivala 2009, 2011):

- Examine spectrum for Lax pair associated with NLS
- Use squared-eigenfunction connection to associate spectrum of Lax pair with spectrum of the linear operator for NLS

Lax pairs and integrability

The Lax pair

$$\chi_x = \begin{pmatrix} -i\xi & \phi \\ -\phi^* & i\xi \end{pmatrix} \chi,$$

$$\chi_t = \begin{pmatrix} -i\xi^2 + \frac{i}{2}|\phi|^2 + \frac{i}{2}\omega & \xi\phi + \frac{i}{2}\phi_x \\ -\xi\phi^* + \frac{i}{2}\phi_x^* & i\xi^2 - \frac{i}{2}|\phi|^2 - \frac{i}{2}\omega \end{pmatrix} \chi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \chi,$$

with the compatibility condition $\chi_{xt} = \chi_{tx}$ gives that $\Psi = e^{-i\omega t}\phi(x)$ satisfies

$$i\Psi_t + \omega\Psi + \frac{1}{2}\Psi_{xx} + |\Psi|^2\Psi = 0.$$

Lax pairs and integrability

Since A and B are independent of t, we let

$$\chi(x,t) = e^{\Omega t}\varphi(x),$$

leading to

$$\Omega^2 = A^2 + BC = -\xi^4 + \omega\xi^2 + c\xi + \frac{1}{16}\left(-4\omega b - 3b^2 - (1-k^2)^2\right),$$

and

$$\varphi(x) = \gamma(x)\begin{pmatrix} -B(x) \\ A(x) - \Omega \end{pmatrix}.$$

Lax pairs and integrability

The scalar function $\gamma(x)$ is determined from

$$\chi_x = \begin{pmatrix} -i\xi & \phi \\ -\phi^* & i\xi \end{pmatrix} \chi,$$

resulting in a linear, first-order, homogeneous ODE for $\gamma(x)$, so that

$$\gamma(x) = \gamma_0 e^{-\int \frac{(A-\Omega)\phi + B_x + i\xi B}{B} dx}.$$

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Since φ and thus γ should be bounded, we need

$$\mathrm{Re}\left\langle \frac{(A(x,\xi) - \Omega(\xi))\phi(x) + B_x(x,\xi) + i\xi B(x,\xi)}{B(\xi,x)} \right\rangle = 0,$$

where $\langle\cdot\rangle$ denotes a spatial average.

The Lax spectrum

The Lax Spectrum σ_L consists of all $\xi \in \mathbb{C}$ for which

$$\mathrm{Re}\left(-2i\xi K(k) \pm 2\left(\zeta(\alpha)K(k) - \zeta(K(k))\alpha\right)\right) = 0,$$

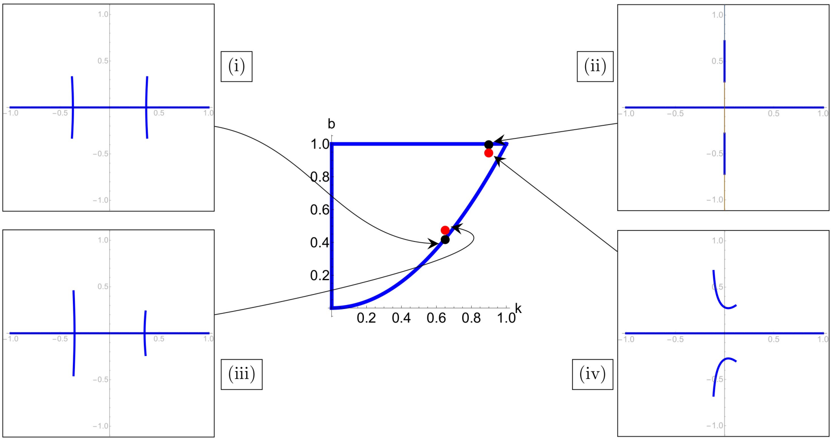
where ζ is the Weierstrass ζ function with lattice invariants

$$g_2 = \frac{4}{3}\left(1 - k^2 + k^4\right), \quad g_3 = \frac{4}{27}\left(2 - 3k^2 - 3k^4 + 2k^6\right),$$

and

$$\alpha = \alpha(\xi) = \wp^{-1}\left(2\Omega(\xi) + 2i\xi^2 + \frac{\omega}{3}, g_2, g_3\right)$$

The Lax spectrum



The stability spectrum

The stability spectrum $\sigma(J\mathcal{L})$ is given by all $\lambda \in \mathbb{C}$ for which

$$\lambda = 2\Omega(\xi),$$

where $\xi \in \sigma_L$, and

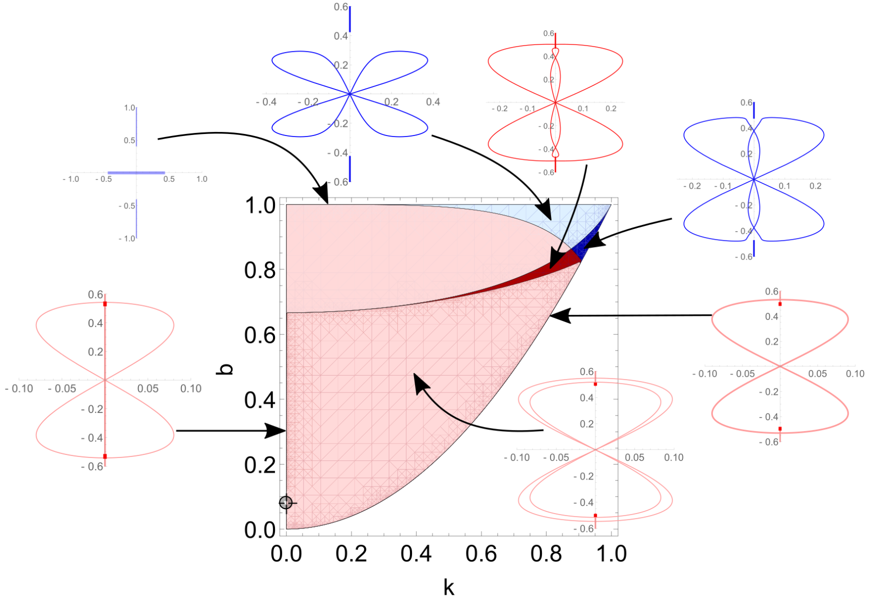
$$\Omega^2 = -\xi^4 + \omega\xi^2 + c\xi + \frac{1}{16}\left(-4\omega b - 3b^2 - (1-k^2)^2\right).$$

Also,

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} e^{-i\theta(x)}\varphi_1^2 - e^{i\theta(x)}\varphi_2^2 \\ -ie^{-i\theta(x)}\varphi_1^2 - ie^{i\theta(x)}\varphi_2^2 \end{pmatrix},$$

where φ_1 and φ_2 are known explicitly.

Parameter space: topology of spectra



Stability with respect to subharmonic perturbations

Consider the class of subharmonic perturbations: $g(x)$ is subharmonic with respect to $f(x) = f(x+T)$, if

$$\exists N \in \mathbb{N}_0 : g(x+NT) = g(x).$$

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Since the spectral problem has periodic coefficients, we know from Floquet's Theorem that all eigenfunctions are of the form

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = e^{i\mu x} \begin{pmatrix} \hat{U}(x) \\ \hat{V}(x) \end{pmatrix},$$

where $\hat{U}(x)$ and $\hat{V}(x)$ are periodic with period $T(k)$. Thus,

$$\mu = \frac{\pi}{PT(k)} \mod \frac{2\pi}{T(k)} \in (-\pi/T(K), \pi/T(K)]$$

corresponds to perturbations of period $PT(k)$.

Stability with respect to subharmonic perturbations

Using the explicit form of the eigenfunction in terms of ξ, and noticing that

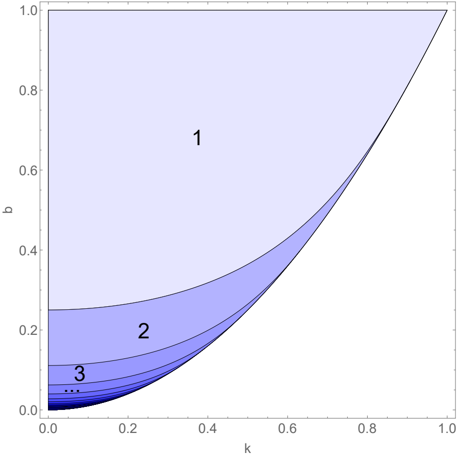
$$e^{i\mu T(k)} = \frac{U(x + T(k))}{U(x)},$$

we find a parametric form of the spectrum as a function of the Floquet parameter μ:

$$\begin{aligned}
\mu &= \mu(\xi), \\
\lambda^2 &= -4\xi^4 + 4\omega\xi^2 + 4c\xi + \frac{1}{4}\left(-4\omega b - 3b^2 - (1-k^2)^2\right).
\end{aligned}$$

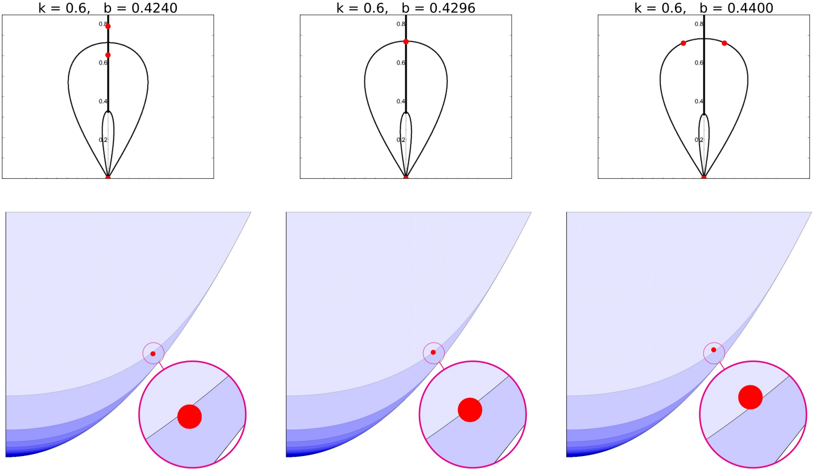
for all ξ in σ_L.

Spectral stability with respect to subharmonic perturbations



Solutions (including those on the border, and all those below) are spectrally stable with respect to perturbations of period $NT(k) = 2K(k)N$.

Spectral stability with respect to subharmonic perturbations



Orbital stability with respect to subharmonic perturbations

We wish to show that the solutions that are spectrally stable with respect to subharmonic perturbations of period $NT(k)$ are orbitally stable with respect to these perturbations.

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- First, establish formal stability: find a Lyapunov functional
- Second, use Grillakis, Shatah and Strauss (87, 90) & Maddocks and Sachs (1993) to establish formal stability

Orbital stability with respect to subharmonic perturbations

- For harmonic perturbations ($N = 1$), Gallay & Haragus showed that the Hamiltonian

$$H_{2,N} = \int_{-NT(k)/2}^{NT(k)/2} \left(\frac{1}{2}|\Psi_x|^2 - \frac{1}{2}|\Psi|^4 - \omega|\Psi|^2 \right) dx$$

  is a Lyapunov functional. In other words, the fixed point solutions minimize $H_{2,1}$.

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- The quadratic part of the Hamiltonian is given by the Krein signature

$$K_{2,N} = \langle \Psi, \mathcal{L}\Psi \rangle_N \,,$$

where the inner product is taken over $N \in \mathbb{N}$ periods.

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- The quadratic part of the Hamiltonian is given by the Krein signature

$$K_{2,N} = \langle \Psi, \mathcal{L}\Psi \rangle_N \,,$$

  where the inner product is taken over $N \in \mathbb{N}$ periods.
- Using the eigenfunctions of the spectral stability problem, $K_{2,N}(\xi)$ is calculated explicitly, with different directions indexed by different ξ. For all N, the set of ξ is countable.

Orbital stability with respect to subharmonic perturbations

- For $N = 1$, no negative directions, and H_2 is a Lyapunov functional for harmonic perturbations, for all fixed point solutions.

Orbital stability with respect to subharmonic perturbations

- For $N = 1$, no negative directions, and H_2 is a Lyapunov functional for harmonic perturbations, for all fixed point solutions.
- For $N > 1$ this is no longer true: this is a larger function space, and negative directions appear.

We need a different Lyapunov functional...

Orbital stability with respect to subharmonic perturbations

- NLS is a member of an infinite hierarchy of integrable equations whose dynamics commute:

$$\Psi_{t_n} = i\frac{\delta H_{n,N}}{\delta \Psi^*}.$$

- Thus $\hat{H}_{n,N} := H_{n,N} + \sum_{j=0}^{n-1} c_{n,j} H_{j,N}$ are conserved quantities, for arbitrary $j \in \mathbb{N}$.

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- Can we find $c_{n,j}$ so that $\hat{H}_{n,N}$ is a Lyapunov functional for the $t = t_2$ dynamics near $\Psi = \phi$?

Orbital stability with respect to subharmonic perturbations

- Impose constraints on $c_{n,j}$ such that

$$\left.\frac{\delta \hat{H}_{n,N}}{\delta \Psi^*}\right|_{\Psi=\phi} = 0,$$

i.e., ϕ is stationary with respect to t_n.

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  i.e., ϕ is stationary with respect to t_n.
- Crucial observation: for $n \geq 2$,

$$K_{n,N}(\xi) := \left\langle \Psi, \hat{\mathcal{L}}_n \Psi \right\rangle = p_n(\xi) K_{2,N}(\xi),$$

  where $p_n(\xi)$ is polynomial in ξ with coefficients determined by $c_{n,j}$, using the same Ψ as before since the different NLS flows commute.

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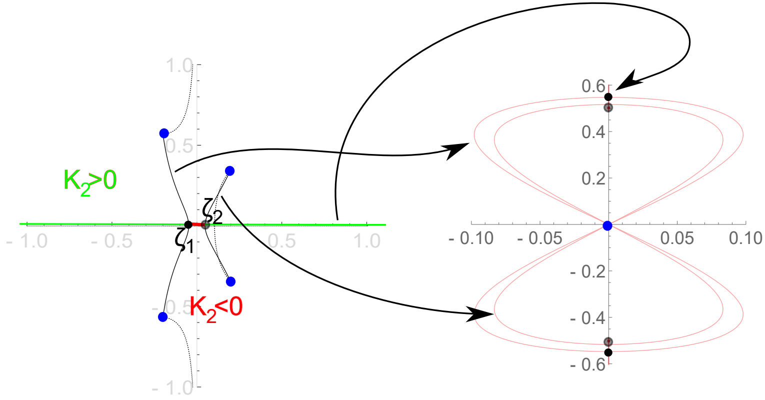
- Impose more constraints on $c_{j,N}$ such that $K_{n,N} \geq 0$, with equality only for $\Psi \in \ker \hat{\mathcal{L}}_n = \ker \mathcal{L}$.

Orbital stability with respect to subharmonic perturbations



- For all N, it suffices to let $n = 4$, thus $p_2(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)$.

Orbital stability with respect to subharmonic perturbations

- The above construction establishes formal stability.
- The conditions of Grillakis, Shatah and Strauss (87, 90) need to be verified. Mainly, we have to check that $\ker\mathcal{L}_4$ is spanned by the generators of the Lie point symmetry group (Maddocks and Sachs, 93).
- For all N, $\ker\mathcal{L}_4$ is identical to $\ker\mathcal{L} = \ker\mathcal{L}_2$, for which the condition is easily verified.

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- For all N, $\ker\mathcal{L}_4$ is identical to $\ker\mathcal{L} = \ker\mathcal{L}_2$, for which the condition is easily verified.
- This establishes orbital stability for all solutions that are spectrally stable, except for the solutions on the boundary curves.

Summary

- Complete understanding of the stability spectra of elliptic solutions of focusing NLS
- Explicit description of the spectra of some non-self-adjoint problems
- Orbital stability with respect to subharmonic perturbations