Field-Theoretical Formulation of the Thermodynamical Bethe Ansatz

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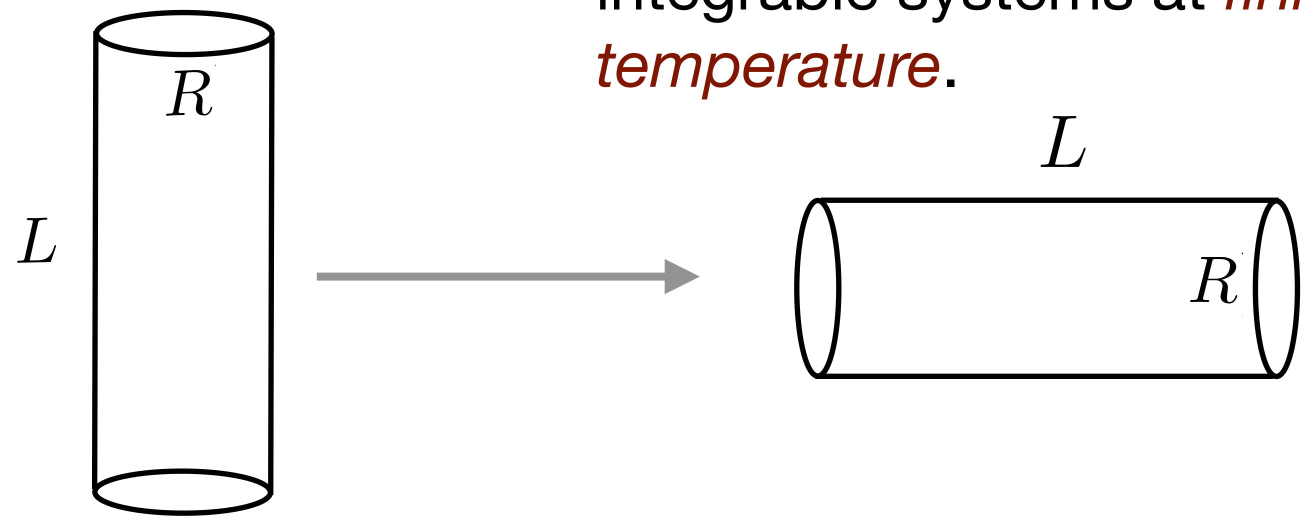
Thermodynamic Bethe Ansatz (TBA) [Yang&Yang, 1966]
— thermodynamics of 1-dim. integrable systems at *finite temperature*.

Since [Al. Zamolodchikov (1990) TBA became the main tool to compute *finite size* effects in 1+1 dim. relativistic field theories



Q. Is there a field/statistical theory behind the Thermodynamic Bethe Ansatz?

Posed 15 years ago (perhaps more, [Saleur 1999])

A. Yes, in principle

Kato&Wadati, 2004: exact cluster expansion.
Woynarovich, 2004: gaussian fluctuations around the saddle point of the Y-Y potential.
Pozsgay, 2010: showed that there is another O(1) contribution from the measure.
I.K., Serban, Vu 2018 graph expansion for the free energy with periodic and open b.c.

Here we give the explicit construction of this effective QFT

1. Massive Quantum Field Theory with factorized scattering

2. Operator formulation of the Asymptotic Bethe Ansatz

3. Partition function at finite temperature as Fock-space expectation value

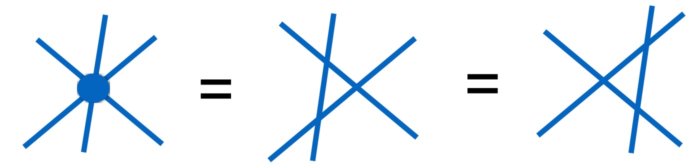
4. Gaussian field representation

5. Path integral formulation and localisation

6. Generalization for open boundary conditions

1. Massive 1+1 dim. QFT with factorized scattering

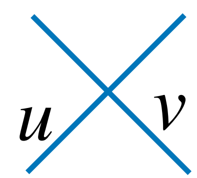
For simplicity assume that there is only one type of particles and no bound states



- **Rapidity** variable: $E^2 = p^2 + m^2, \quad p = p(u),\ E = E(u)\,.$

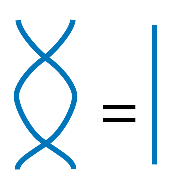
$$E(u) = m\cosh(\pi u),\ p(u) = m\sinh(\pi u)$$

- Two-particle **S-matrix**:



- Analytic in the physical strip |Im(u)|<1

- $S(u,v)\ S(v,u) = 1$



unitarity $\Rightarrow$ $S(u,u) = \pm 1$

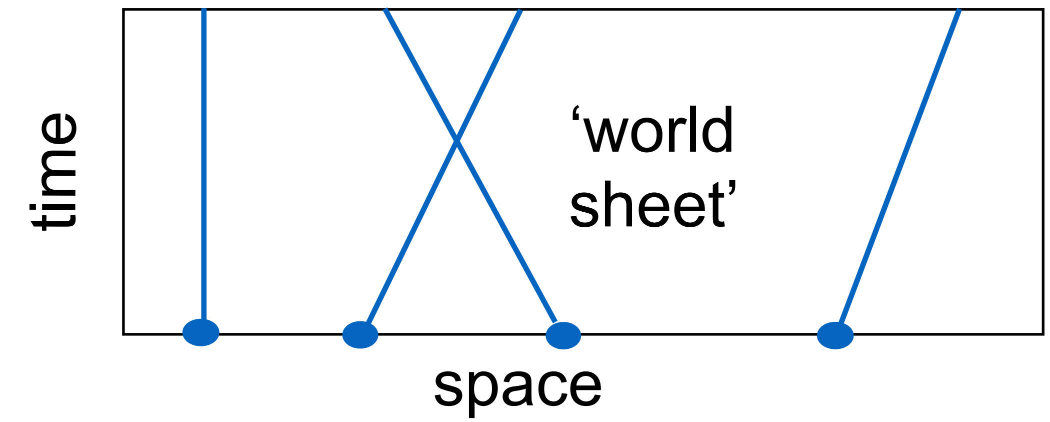
- $S(u, v - i/2) S(u, v + i/2) = 1$

cross unitarity

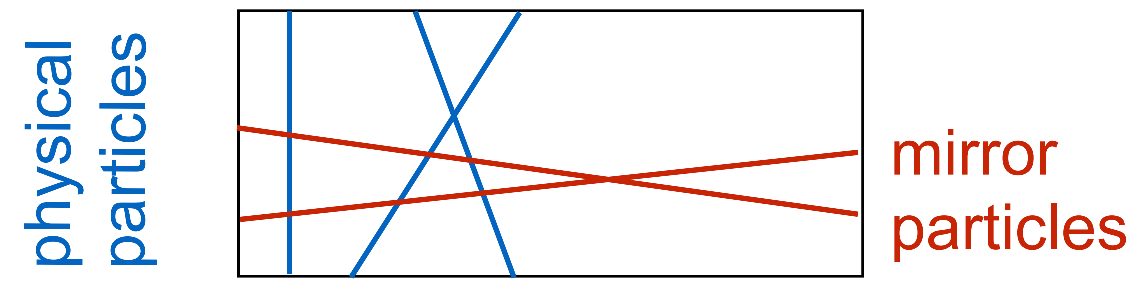
We choose $S(u,u) = -1$

- **Bethe Ansatz**: any physical state is a superposition of one-particle states.

  Works only for asymptotically large volume



- **Mirror transformation:** exchanges space and time $E \to i\tilde{p}, p \to i\tilde{E}$



Corresponds to analytical continuation in the rapidity variable

$$\gamma : \; u \to u^{\gamma} \equiv u + i/2$$

$$\text{Physical channel} \;\to\; \text{Mirror channel}$$

$$S(u,v) \;\to\; \tilde{S}(u,v) \equiv S(u^{\gamma}, v^{\gamma})$$

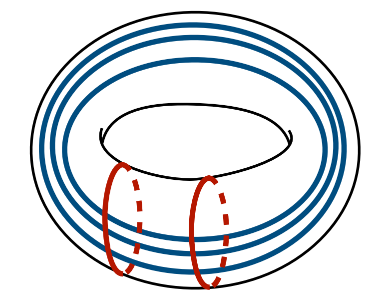
$$p(u) \;\to\; \tilde{p}(u) = -\,iE(u^{\gamma})$$

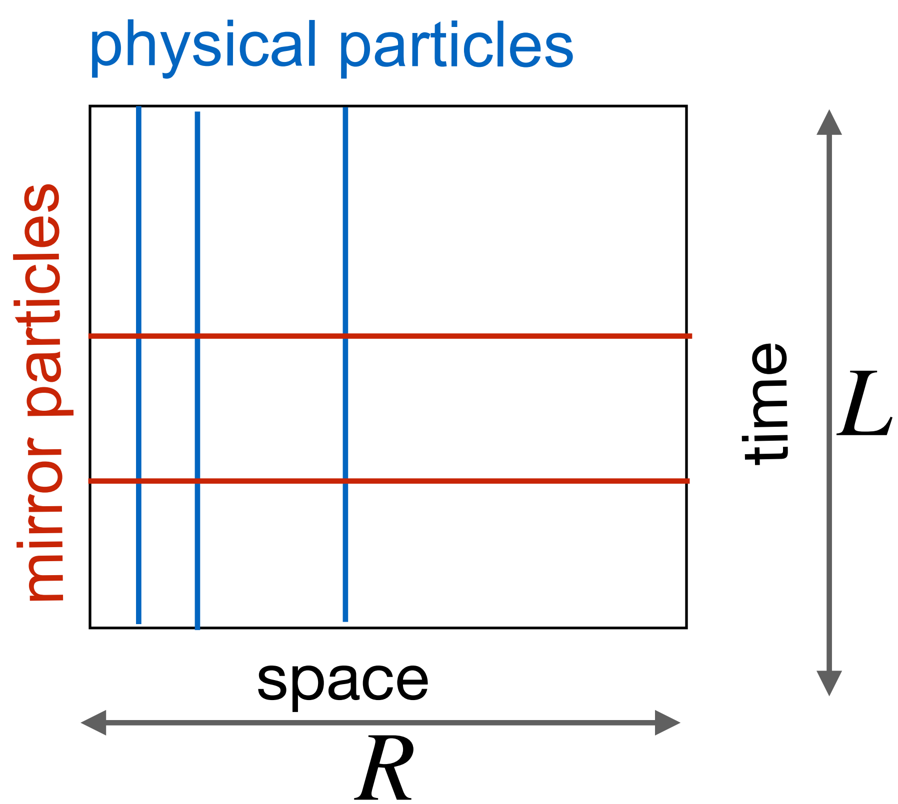
$$E(u) \;\to\; \tilde{E}(u) = -\,ip(u^{\gamma})$$

2. Operator formulation of the Asymptotic Bethe Ansatz

Wrapping operators for periodic b.c.

The elementary excitations on a torus with asymptotically large periods are time-wrapping particles in the physical channel and and space-wrapping particles in the mirror channel

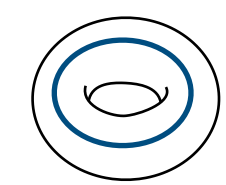




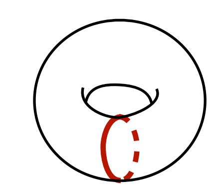
$$mL \gg 1, \quad mR \gg 1$$

Operators creating wrapping particles:

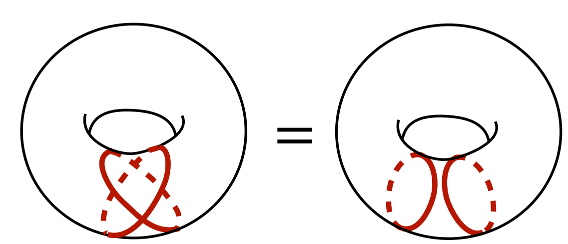
$\mathbf{A}(u)$ = time-wrapping operator

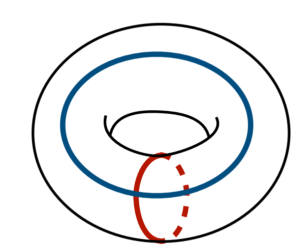


$\mathbf{B}(u)$ = space-wrapping operator



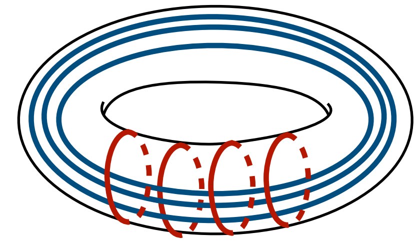
Particles wrapping the same cycle do not scatter but particles wrapping different cycles do:





Algebra of the wrapping operators

Expectation value of N time-wrapping and M space-wrapping operators:



$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle = \prod_{j=1}^{M} \prod_{k=1}^{N} S(v_j^\gamma, w_k) \; \prod_{j=1}^{M} e^{-R\tilde{E}(v_j)} \prod_{k=1}^{N} e^{-LE(w_k)}$$

Fock-space realisation:

$$\mathbf{B}(v)\mathbf{A}(u) = S(v^\gamma, u)\, \mathbf{A}(u)\mathbf{B}(v), \quad [\mathbf{B}(u), \mathbf{B}(v)] = [\mathbf{A}(u), \mathbf{A}(v)] = 0$$

$$\langle L \,|\, \mathbf{A}(u) = e^{-LE(u)} \langle L \,|, \quad \mathbf{B}(u) \,|\, R \rangle = e^{-R\tilde{E}(u)} \,|\, R \rangle \qquad \langle L \,|\, R \rangle = 1$$

Fock-space expectation value:

$$\langle \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \rangle \equiv \langle L \,|\, \prod_{j=1}^{M} \mathbf{B}(v_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \,|\, R \rangle$$

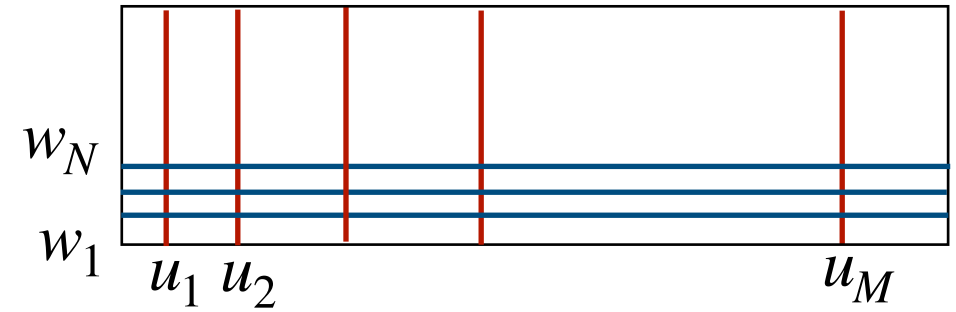
For any operator define

$$\langle \mathcal{O} \rangle = \langle L \,|\!:\! \mathcal{O} :\!|\, R \rangle$$

where $: \;\, :$ is the anti-normal product: all **B**'s are on the left of all **A**'s

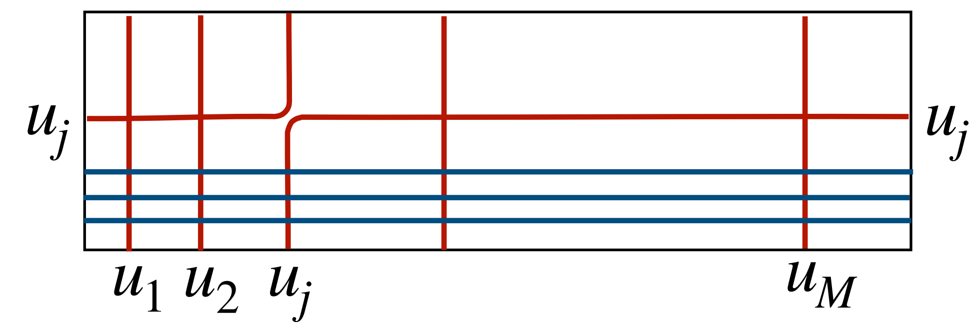
Operator form of Bethe-Yang equations

Consider a state with M space-wrapping and N time-wrapping particles



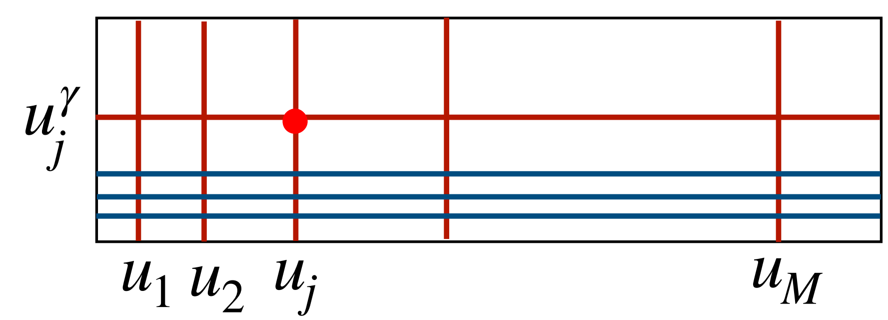
The periodic b.c. impose a quantisation of the momenta in the mirror channel (Bethe-Yang eqs)

$$1 + e^{-iL\tilde{p}(u_j)} \prod_{k=1}^{M} \tilde{S}(u_k, u_j) = 0, \quad j = 1, \ldots, M$$



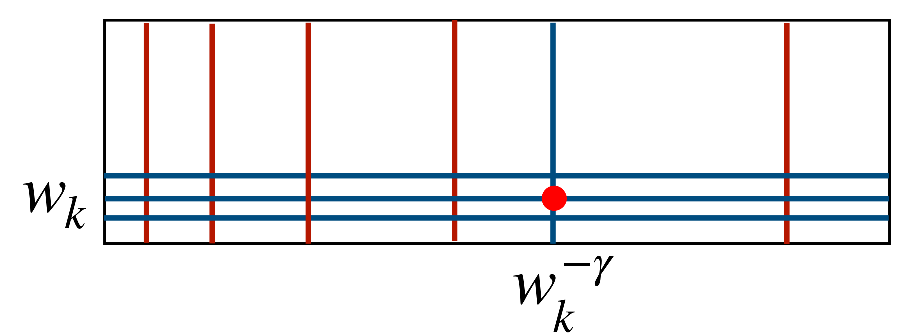
Operator form:

$$\left\langle \prod_{j=1}^{M} \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k)\ (1 + \mathbf{A}(u_j^{\gamma})) \right\rangle = 0, \quad j = 1, \ldots, M$$



Similarly, in the physical channel

$$\left\langle \left(1 + \mathbf{B}(w_k^{-\gamma})\right)\ \prod_{j=1}^{M}\ \mathbf{B}(u_j) \prod_{k=1}^{N} \mathbf{A}(w_k) \right\rangle = 0, \quad k = 1, \ldots, N$$



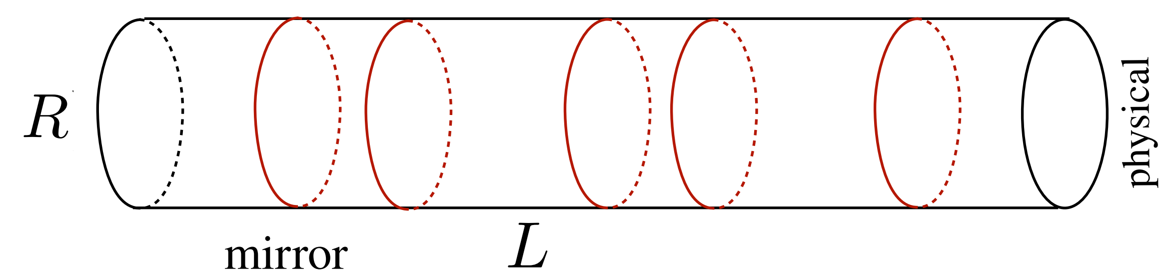
M+N equations for M+N rapidities

3. Partition function at finite volume/finite temperature $mL \gg 1, \quad mR \sim 1$

$$\mathcal{Z}(L,R) = \mathrm{Tr}[e^{-R\tilde{\mathbf{H}}}] = \sum_{\psi} \frac{\langle\psi|e^{-R\tilde{\mathbf{H}}}|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_{M=0}^{\infty} \sum_{u_1,\ldots,u_M} e^{-R(\tilde{E}(u_1)+\ldots+\tilde{E}(u_M))}$$

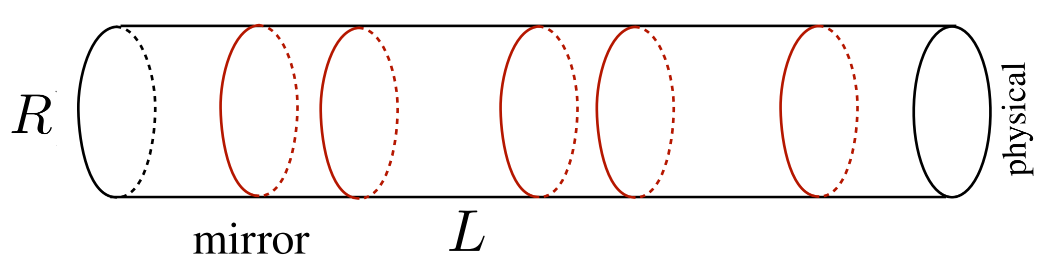
The sum goes over discrete set of rapidities (all solutions of the Bethe-Yang equations)

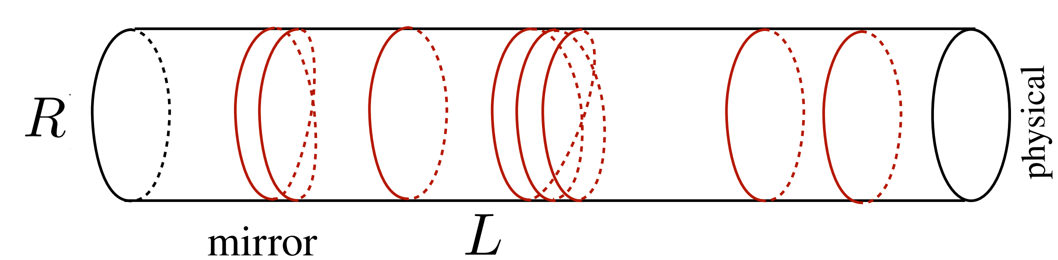
$$L\tilde{p}(v_j) + i\sum_{k=1}^{M} \log \tilde{S}(v_k, v_j) = 2\pi n_j, \quad j = 1,\ldots,M$$



- Relax the constraint $n_1 < n_2 < \ldots < n_M$ by introducing multi-wrapping particles

$$\mathcal{Z}(L,R) = \sum_{M=0}^{\infty} \sum_{n_1<n_2<\ldots<n_M} \prod_{j=1}^{M} e^{-R\tilde{E}(u_j)} \quad \to \quad \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{r_1,\ldots,r_m\geq 1} \sum_{n_1,\ldots,n_m} \prod_{j=1}^{m} \frac{(-1)^{r_j-1}}{r_j} e^{-r_j R\tilde{E}(u_j)}$$





- Impose the operator constraint selecting the solution of the B-Y equations:

$$\mathbf{A}(u_j) + 1 = 0, \quad j = 1,\ldots,M$$

- Express the discrete sum over mode numbers by contour integral

$$\sum_{n_1,\ldots,n_m} \prod_{j=1}^{M} e^{-r_j R \tilde{E}(u_j)} = \left\langle \oint_{\mathscr{C}} \cdots \oint_{\mathscr{C}} \prod_{j=1}^{m} \frac{du_j}{2\pi i} \frac{\partial \log(1 + \ \mathbf{A}(u_j^\gamma)}{\partial u_j} \ \prod_{j=1}^{m} \mathbf{B}(u_j)^{r_j} \right\rangle$$

- perform the sum over all wrapping numbers:

$$\mathscr{Z}(L,R) = \langle \mathbf{\Omega} \rangle$$

$$\mathbf{\Omega} \equiv \exp\left[\oint_{\mathscr{C}} \frac{du}{2\pi i} \log\big(1 + \mathbf{B}(u)\big) \ \frac{\partial}{\partial u} \log\big(1 + \mathbf{A}(u^\gamma)\big) \right]$$

In case of an **excited state** with rapidities $w_1, \ldots, w_N$ in the physical channel

$$\mathscr{Z}(L,R) = \left\langle \prod_{k=1}^{N} \mathbf{A}(w_k) \quad \mathbf{\Omega} \right\rangle$$

The rapidities of the excited state are determined by the exact Bethe equations

$$\left\langle \left(1 + \mathbf{B}(w_j^{-\gamma})\right) \prod_{k=1}^{N} \mathbf{A}(w_k) \quad \mathbf{\Omega} \right\rangle = 0, \quad j = 1,2,\ldots,N$$

The only difference with the Asymptotic Bethe Ansatz is that the bare vacuum is dressed by the operator Omega

4. Free-field representation as vertex operators

$$\mathbf{B}(u) = e^{-\varphi(u)}, \quad \mathbf{A}(u^{\gamma}) = e^{-i\bar{\varphi}(u)}$$

The meaning if the two bosonic fields:
the **scattering phases** acquired after turning around the A- and the B cycle

$$[\varphi(u), \bar{\varphi}(v)] = i \log \tilde{S}(u, v)$$

$$[\bar{\varphi}(u), \bar{\varphi}(v)] = [\varphi(u), \varphi(v)] = 0$$

$$\langle L \,|\, \bar{\varphi}(u) = \bar{\varphi}^{\circ}(u)\langle L \,|\,, \quad \varphi(u)\,|\, R\rangle = \varphi^{\circ}\,|\, R\rangle$$

$$\varphi^{\circ}(u) = R\tilde{E}(u), \qquad \bar{\varphi}^{\circ}(u) = L\tilde{p}(u)$$

Expectation value:

$$\langle \cdots \rangle = \langle L \,|\!: \cdots :\!|\, R\rangle$$

with anti normal product $: \; :$
(all φ's on the left of all φ's)

$$\mathscr{Z}(L, R) = \langle \mathbf{\Omega} \rangle$$

$$\mathbf{\Omega} = \exp\left[\oint_{\mathscr{C}} \frac{du}{2\pi i} \log\langle 1 + e^{-\varphi(u)}\rangle \; \partial_u \log\left(1 + e^{-i\bar{\varphi}(u)}\right)\right]$$

Continuum spectrum approximation

$$\oint_{\mathscr{C}} \frac{d\log\left(1+e^{-i\bar{\varphi}(u)}\right)}{2\pi i}(\cdots) \quad \rightarrow \quad \int_{-\infty}^{\infty} \frac{du}{2\pi}\,\bar{\varphi}'(u) \qquad \text{+ exponentially small}$$

Not so simple! The limit does not commute with the expectation value

$$\sum_{n_1,\ldots,n_m} (\ldots) \quad \rightarrow \quad \int \frac{du_1}{2\pi}\ldots\frac{du_m}{2\pi}\, G(u_1,\ldots,u_m)\,(\ldots)\,.$$

Have to introduce by hand fermions which generate the non-diagonal terms in the Gaudin determinant

$$\boldsymbol{\Omega} \;\rightarrow\; \check{\boldsymbol{\Omega}} = \exp\int_{-\infty}^{\infty}\frac{du}{2\pi}\left[\log\left(1+e^{-\varphi(u)}\right)\bar{\varphi}'(u) + \frac{\bar{\psi}(u)\psi'(u)}{1+e^{\varphi(u)}}\right]$$

5. Path integral and localisation

Impose the correlators by introducing a pair of auxiliary fields $\rho, \bar{\rho}$ $\vartheta, \bar{\vartheta}$

$\varphi, \bar{\varphi}$ $\rho, \bar{\rho}$
commutative

$\bar{\psi}, \psi$ $\vartheta, \bar{\vartheta}$
grassmanian

$$\mathscr{Z}(L,R) = \int \mathscr{D}[\text{fields}]\; e^{-\mathscr{A}[\text{fields}]}$$

$$-\mathscr{A}[\text{fields}] = \int \frac{du}{2\pi}\left(\log(1+e^{-\varphi})\,\partial\bar{\varphi} + \frac{\bar{\psi}\,\partial\psi}{1+e^{\varphi}} + (\bar{\varphi}-\bar{\varphi}^{\circ})\rho + (\varphi-\varphi^{\circ})\bar{\rho} + \bar{\vartheta}\psi + \bar{\psi}\vartheta\right)$$

$$-i\int \frac{du}{2\pi}\frac{dv}{2\pi}\log\tilde{S}(u,v)\big(\bar{\rho}(u)\rho(v) - \vartheta(u)\bar{\vartheta}(v)\big)$$

the dependence on R and L through the classical fields:

$$\varphi^{\circ}(u) = R\tilde{E}(u), \quad \bar{\varphi}^{\circ}(u) = L\tilde{p}(u)$$

Localisation:

$$\mathscr{A} = \mathscr{A}^\circ + \int \frac{du}{2\pi}\, \mathrm{Q}(u)\ \mathscr{B}$$

Q-exact localisation term

$$\mathscr{A}^\circ = -\int \frac{du}{2\pi}\, \bar{\varphi}^\circ \rho$$

$$Q = \bar{\psi}\frac{\delta}{\delta\varphi} + \bar{\varphi}\frac{\delta}{\delta\psi} + \bar{\rho}\frac{\delta}{\delta\vartheta} + \bar{\vartheta}\frac{\delta}{\delta\rho} \qquad Q^2 = 0$$

$$\mathscr{B} = \int \frac{du}{2\pi}\Big(-\log(1+e^{-\varphi})\,\partial\psi + \psi\rho + \theta(\varphi - \varphi^\circ)\Big) - i\int \frac{du}{2\pi}\frac{dv}{2\pi}\,\theta(u)\ \log\tilde{S}(u,v)\,\rho(v)$$

By standard localisation argument integral localises to the critical point:

$$\mathscr{Z} \;\to\; \mathscr{Z}_t = \int e^{-\mathscr{A}_t} \qquad \mathscr{A} \to \mathscr{A}_t = \mathscr{A}^\circ + t\mathrm{Q}\mathscr{B}, \quad \mathscr{A}^\circ \equiv -L\int \frac{du}{2\pi} p'(u)\rho(u)\,.$$

$$\frac{\partial \mathscr{Z}_t}{\partial t} = \int e^{-\mathscr{A}^\circ - t\mathrm{Q}\mathscr{B}}(-\mathrm{Q}\mathscr{B}) = \int \mathrm{Q}\left(e^{-\mathscr{A}^\circ - t\mathrm{Q}\mathscr{B}}\mathscr{B}\right) = 0$$

Take the limit of infinite perturbation: $\quad t \to \infty : \qquad \mathscr{Z} = e^{-\mathscr{A}^\circ}\Big|_{\mathrm{Q}\mathscr{B}=0}$

The critical point:

$$\varphi(u) = R\tilde{E}(u) - i \int \frac{dv}{2\pi} \log \tilde{S}(u,v)\, \rho(v)$$

$$\rho(u) = \partial_u \log(1 + e^{-\varphi(u)})$$

equation for the critical point identical to the TBA integral equation:

$$\epsilon(u) = R\tilde{E}(u) - \int \tilde{K}(v,u) \;\log(1 + e^{-\epsilon(v)})$$

$\epsilon(u) = \varphi^{\text{crit}}(u)$ = pseudo energy

The partition function: $$\mathcal{Z}(L,R) = \exp\left(L \int \frac{d\tilde{p}(u)}{2\pi} \;\log\left[1 + e^{-\epsilon(u)}\right] \right)$$

As a consequence of localisation : the theory is one-loop exact and the gaussian fluctuations of the bosons and the fermions cancel => no quantum corrections to the critical action at all

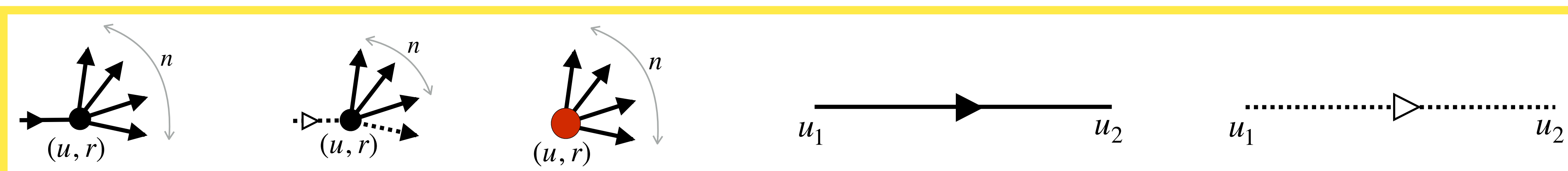
Statistical meaning of the perturbative series - exact cluster expansion

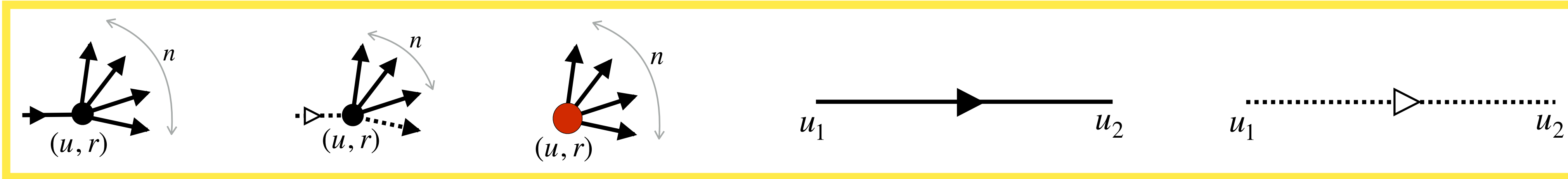
Vertices: $$\bar{\varphi}' \log(1 + e^{-\varphi}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \, \bar{\varphi}' \, \varphi^n$$

Propagators: $$\langle \bar{\varphi}'(u)\varphi(v) \rangle = -\tilde{K}(u,v), \quad \langle \bar{\psi}'(u)\psi(v) \rangle = \tilde{K}(u,v)$$

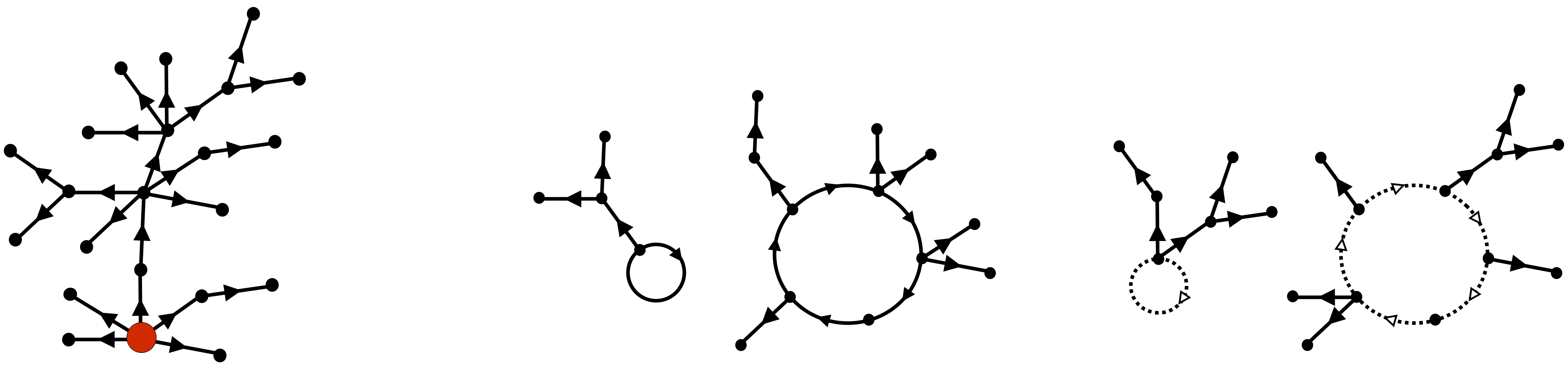
$$\tilde{K}(u,v) = \frac{1}{i}\partial_u \log \tilde{S}(u,v)$$

Feynman rules:





The theory is one-loop exact. Feynman graphs: 0 loops (trees) and 1-loop graphs



trees

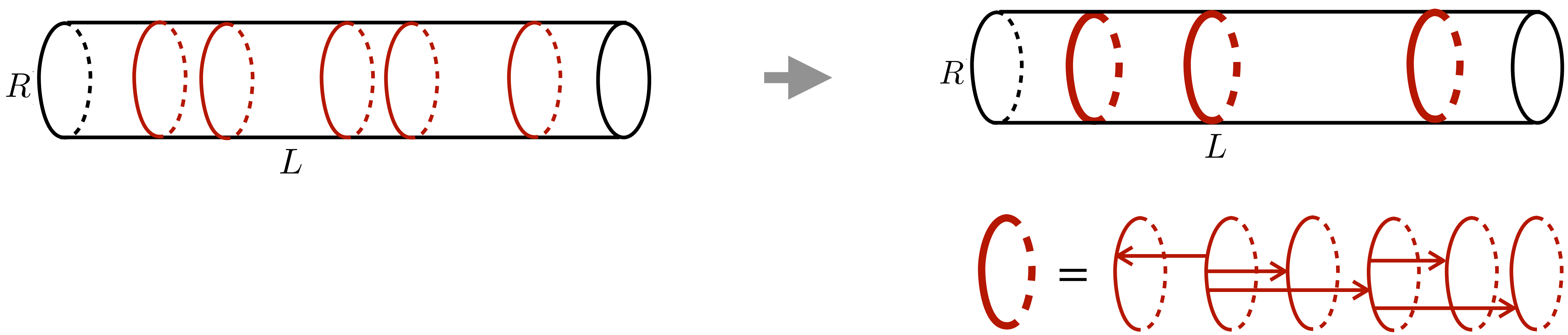
bosonic loops

fermionic loops

The bosonic loops and the fermionic loops cancel and the free energy is given by the sum over tree graphs.

I.K., Didina Serban, D. L. Vu,
arXiv[hep-th]1805.02591,
1809.05705, 1906.01909

Statistical meaning of the perturbative series - exact cluster expansion



Wrapping particles weakly interacting after being put in a large box

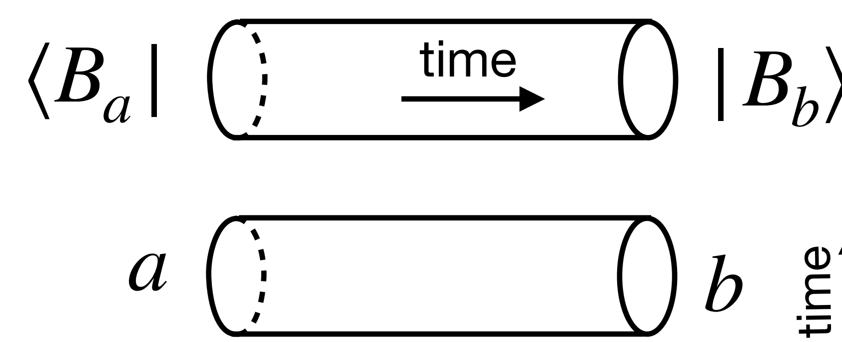
Non-interacting clusters of wrapping particles: behave as free fermions with renormalized energy

6. Generalization for open boundary conditions

Thermal partition function with open b.c. in the mirror channel

$$\mathcal{Z}_{ab}(R,L) = \langle B_a | e^{-H(R)L} | B_b \rangle$$

$$= \mathrm{Tr}[e^{-\tilde{H}_{ab}(L)R}]$$



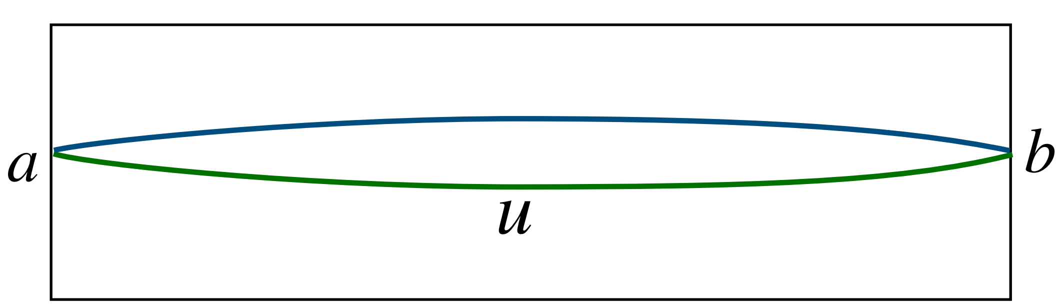
$$S(u,v)S(v,u) = 1$$
$$S(u,-v)S(-u,v) = 1$$

$$\tilde{R}_a(u)\tilde{R}_a(-u) = \tilde{R}_b(u)\tilde{R}_b(-u) = 1$$

boundary reflection matrix
[Ghoshal-Zamolodchikov]

Boundary entropy:

$$\mathcal{F}_{ab}(R,L) \equiv \log \mathcal{Z}_{ab}(R,L) - \log \mathcal{Z}(R,L) = \log g_a(R) g_b(R) + O(e^{-mL})$$

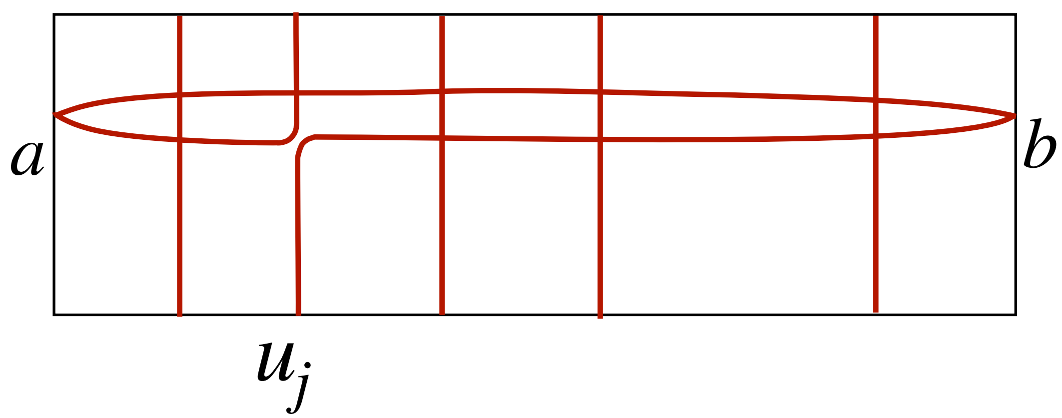


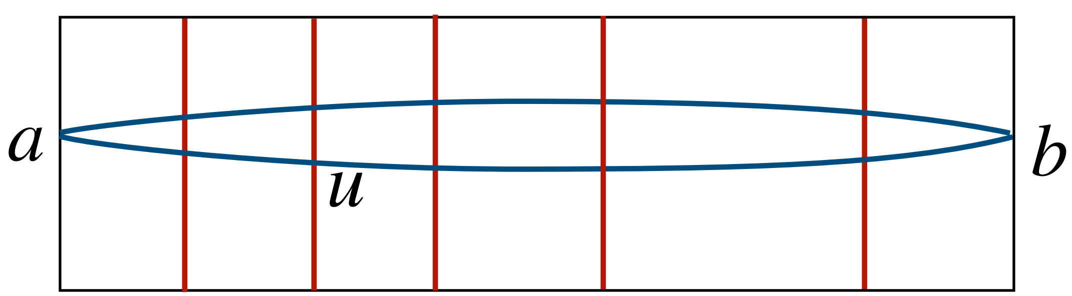
Wrapping operators in the L- direction:

$$e^{-i\bar{\varphi}_{ab}(u)} = e^{-i\bar{\varphi}(u)}\ \tilde{R}_a(u)\ e^{-i\bar{\varphi}(-u)}\ \tilde{R}_b(u)$$

= parity invariant state propagating between the two boundaries

Operator form of the Bethe-Yang equations:





Path integral for open boundary conditions:

$$\mathcal{Z}_{ab}(L,R) = \int \mathcal{D}[\text{fields}]\; e^{-\mathcal{A}[\text{fields}]}$$

$$\mathcal{A}[\text{fields}] = \int \frac{du}{2\pi}\left(\log(1+e^{-\varphi})\,\partial\bar{\varphi}_{ab} - \frac{\bar{\psi}_{ab}\,\partial\psi}{1+e^{\varphi}} + (\bar{\varphi}_{ab}-\bar{\varphi}^{\circ}_{ab})\rho + (\varphi-\varphi^{\circ})\bar{\rho} + \bar{\theta}\psi + \bar{\psi}_{ab}\theta\right)$$
$$+ i\int \frac{du}{2\pi}\frac{dv}{2\pi}\log[\tilde{S}(u,v)\tilde{S}(u,-v)]\big(-\bar{\rho}(u)\rho(v) + \theta(u)\bar{\theta}(v)\big)$$

$$\bar{\varphi}^{\circ}_{ab}(u) = 2Lp(u) + i\log\tilde{R}_a(u) + i\log\tilde{R}_b(u) - 2\pi\,\text{sign}(u), \quad \bar{\varphi}^{\circ}(u) = L\tilde{p}(u)$$

$$\mathcal{Z} = \frac{\text{Det}(1-\hat{K}^{-})}{\text{Det}(1-\hat{K}^{+})}e^{-\mathcal{A}^{\circ}}\bigg|_{\text{Q}\mathcal{B}=0}$$

The theory is one-loop exact

$$\hat{K}^{\pm}(u,v) = \frac{1}{i}\frac{1}{1+e^{\epsilon(u)}}\partial_u\left(\log\tilde{S}(u,v) \pm \log\tilde{S}(u,-v)\right)$$

Same pseudo energy as for the periodic system:

$$\epsilon(u) = R\tilde{E}(u) - \int_0^\infty \frac{dv}{2\pi} K^+(v,u) \ \log(1+e^{-\epsilon(v)}) = R\tilde{E}(u) - \int_{-\infty}^\infty \frac{dv}{2\pi} K(v,u) \ \log(1+e^{-\epsilon(v)})$$

Boundary entropy:

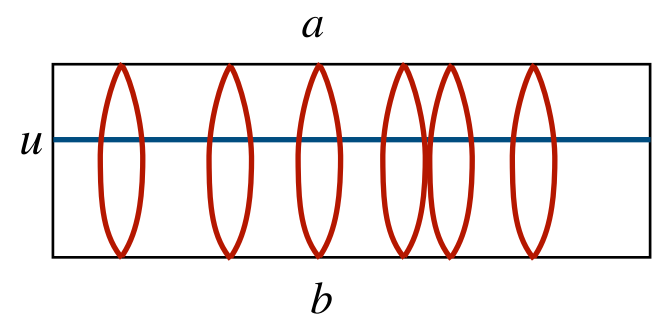
The g-function has universal and non-universal parts

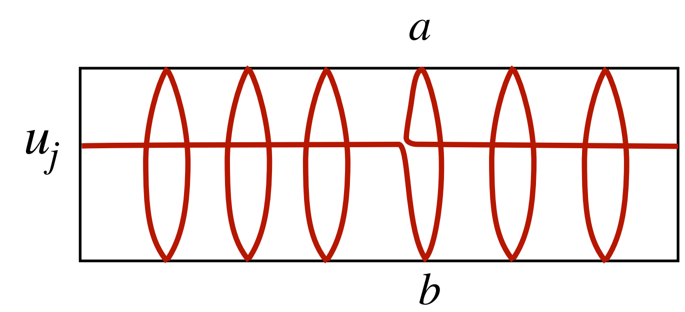
$$\log g_a(R) = \frac{1}{2}\mathrm{Tr}\log(1-\hat{K}^-) - \frac{1}{2}\mathrm{Tr}\log(1-\hat{K}^+) + \int_0^\infty \frac{du}{2\pi}\partial_u[i\log\tilde{R}_a(u) - \pi\,\mathrm{sign}(u)]$$

universal   non-universal

[Dorey, Fioravanti, Rim and Tateo , Pozsgay 2010]

Thermal partition function with open b.c. in the physical channel





Path integral:

$$\mathscr{Z} = \frac{\mathrm{Det}(1-\hat{K}^+)}{\mathrm{Det}(1-\hat{K}^-)} e^{-\mathscr{A}^\circ}\Big|_{\mathrm{Q}\mathscr{B}=0}$$

The theory is one-loop exact

$$\hat{K}^{\pm}(u,v) = \frac{1}{i}\frac{1}{1+e^{\epsilon(u)}}\partial_u \left(\log \tilde{S}(u,v) \pm \log \tilde{S}(u,-v)\right)$$

Example: Sinh-Gordon QFT

$$\mathscr{A} = \int d^2x \left[\frac{1}{4\pi}(\partial_\mu \phi)^2 + \frac{2\mu^2}{\sin \pi b^2} \cosh(b\phi) \right] \qquad \nu = 1 + \frac{1}{b^2}, \quad a = 1 - \frac{2}{\nu} \quad (0 < b \leq 1)$$

One particle, no bound states;
relativistic theory: mirror=physical

$$p(u) = m \sinh \pi u, \quad E(u) = m \cosh \pi u$$

$$S(u, v) = \frac{\tanh\left(\frac{\pi u}{2} - \frac{i\pi}{2\nu}\right)}{\tanh\left(\frac{\pi u}{2} + \frac{i\pi}{2\nu}\right)}$$

$$\log S(u) = (\mathbb{D}^a + \mathbb{D}^{-a}) \log S_0(u)$$

Shift operator: $\mathbb{D} = \exp\left(\frac{i}{2}\frac{\partial}{\partial u}\right)$

$$S_0(u) = \tanh \frac{\pi(u - i/2)}{2}$$

$K_0(u, v) = \frac{1}{i}\partial_u \log S_0(u, v)$ - “universal kernel”

$$(\mathbb{D} + \mathbb{D}^{-1})K_0(u) = 2\pi\delta(u)$$

$K \to \mathbb{K} = \frac{\mathbb{D}^a + \mathbb{D}^{-a}}{\mathbb{D} + \mathbb{D}^{-1}}$ $K(u) = K_0(u + ia/2) + K_0(u - ia/2) = (\mathbb{D}^a + \mathbb{D}^{-a})K_0$ $K_0(u, v) = \frac{\pi}{\cosh \pi (u - v)}$ $K(u, v) = \frac{1}{i}\partial_u \log S(u, v)$ $\mathbb{D} \equiv e^{i\partial_u/2}$

$$\mathscr{A}[\text{fields}] = \int \frac{du}{2\pi} \left(\log(1+e^{-\varphi})\,\bar{\varphi}' - \frac{\bar{\psi}\,\psi'}{1+e^{\varphi}} + (\bar{\varphi}-\bar{\varphi}^{\circ})\rho + (\varphi-\varphi^{\circ})\bar{\rho} + \bar{\theta}\psi + \bar{\psi}\theta \right)$$

$$+ i \int \frac{du}{2\pi}\frac{dv}{2\pi} \big(\bar{\rho}(u)\rho(v) + \theta(u)\bar{\theta}(v)\big)(\mathbb{D}^{a} + \mathbb{D}^{-a}) \log S_0(u,v)$$

$$\begin{aligned} \bar{\varphi} &\to Lp(u) + (\mathbb{D}+\mathbb{D}^{-1})\bar{\varphi} \\ \bar{\psi} &\to (\mathbb{D}+\mathbb{D}^{-1})\bar{\psi} \end{aligned}$$

p(u) is a zero mode
of the shift operator

$$\mathscr{A} = \int \frac{du}{2\pi} \left[\varphi(\mathbb{D}+\mathbb{D}^{-1})\partial\bar{\varphi} - \log(1+e^{-\varphi})(\mathbb{D}^{a}+\mathbb{D}^{-a})\partial\bar{\varphi} \right]$$

$$+ \int \frac{du}{2\pi} \left[\psi(\mathbb{D}+\mathbb{D}^{-1})\bar{\partial}\psi - \frac{1}{1+e^{\varphi}} \bar{\psi}(\mathbb{D}^{a}+\mathbb{D}^{-a})\partial\psi \right]$$

$\varphi \to \varphi + RE(u)$

$$\mathscr{A} = -L\mathscr{E}(R) + \mathrm{Q}\mathscr{B}$$

$$\mathscr{E}(R) = \int \frac{du}{2\pi} m \cosh(u) \ \log(1 + e^{-\varphi(u)})$$

$$\mathrm{Q}(u) = \bar{\varphi}\frac{\delta}{\delta\psi} + \bar{\psi}\frac{\delta}{\delta\varphi}$$

$$\mathscr{B} = \int \frac{du}{2\pi}\left[-\partial\psi\left[\mathbb{D} + \mathbb{D}^{-1}\right]\varphi + \log(1 + e^{-\varphi})(\mathbb{D}^a + \mathbb{D}^{-a})\partial\psi\right]$$

Critical points: $$\left[\mathbb{D} + \mathbb{D}^{-1}\right]\varphi = -(\mathbb{D}^a + \mathbb{D}^{-a})\log(1 + e^{-\varphi})$$

"Discrete Liouville equation"

Y-system, Q-system and quantum spectral curve $Y(u) = Q^{\mathbb{D}^a + \mathbb{D}^{-a}} = e^{-\varphi(u)}$

$$Y(u + i/2)Y(u - i/2) = \left[1 + Y(u + ai/2)\right]\left[1 + Y(u - ai/2)\right]$$ Y-system

$$Q(u + i/2)Q(u - i/2) - Q\,(u + ia/2)\,Q\,(u - ia/2)$$ Q-system

[Zamolodchikov, Lukyanov]

bi-linear equations, typical for integrable systems, known as "quantum spectral curve"

Summary

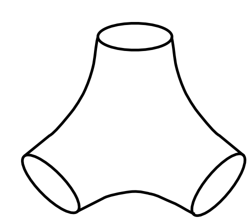
Path integral formulation of the Thermodynamic Bethe Ansatz

The theory is one-loop exact. Explains why there are only exponential corrections to the free energy

Works also for scattering matrices not of difference type, as in AdS/CFT

Can be generalised to the case of non-diagonal scattering (nested Bethe Ansatz) and bound states

Hopefully can be adapted to other geometries with application to AdS/CFT



Thank you!