Integrability of New Class of Hyperbolic Two-Dimensional Linear Equations of Second Order

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The Problem

We consider integrability of Hamiltonian hydrodynamic type systems

\[ u_t = \left( \frac{\partial h}{\partial v} \right)_x, \quad v_t = \left( \frac{\partial h}{\partial u} \right)_x, \]

whose Hamiltonian densities are quadratic with respect to the field variable \( v \), i.e.

\[ h = A(u)v^2 + B(u)v + C(u), \]

where the functions \( A(u), B(u), C(u) \) are arbitrary.
We consider integrability of Hamiltonian hydrodynamic type systems

\[
\begin{align*}
    u_t &= \left( \frac{\partial h}{\partial \nu} \right)_x, \\
    v_t &= \left( \frac{\partial h}{\partial \mu} \right)_x,
\end{align*}
\]

whose Hamiltonian densities are quadratic with respect to the field variable \(\nu\), i.e.

\[
h = A(u)\nu^2 + B(u)\nu + C(u),
\]

where the functions \(A(u), B(u), C(u)\) are arbitrary.

Such hydrodynamic type systems possess infinitely many commuting flows

\[
\begin{align*}
    u_y &= \left( \frac{\partial f}{\partial \nu} \right)_x, \\
    v_y &= \left( \frac{\partial f}{\partial \mu} \right)_x,
\end{align*}
\]

where the Hamiltonian densities satisfy the linear equation

\[
h_{uu} f_{\nu\nu} = h_{\nu\nu} f_{uu}.
\]
In the paper “The motion of a thin liquid layer on the outer surface of a rotating cylinder” the hydrodynamic type system

\[ u_t = (w \ln u)_x, \quad w_t = \left( \frac{w^2}{2u} - \epsilon w \ln u - \frac{1}{2} \epsilon^2 u \right)_x \]

was derived by A.M. Morad and M.Yu. Zhukov.
The Morad–Zhukov System

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Under the linear transformation \( w = v - \frac{\epsilon}{2} u \), this system takes a local Hamiltonian structure

\[ u_t = \left( \frac{\partial h}{\partial v} \right)_x, \quad v_t = \left( \frac{\partial h}{\partial u} \right)_x, \]

where the Hamiltonian density is

\[ h = \frac{1}{2} v^2 \ln u - \frac{\epsilon}{2} vu \ln u + \frac{\epsilon^2}{8} u^2 \ln u - \frac{1}{4} \epsilon^2 u^2. \]
The Camassa–Falqui–Ortenzi system

\[
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    u_t &= \left( \frac{\partial h}{\partial v} \right)_x, \\
    v_t &= \left( \frac{\partial h}{\partial u} \right)_x,
\end{align*}
\]

is determined by the Hamiltonian density quadratic with respect to the field variable \( v \), i.e.

\[
h = \frac{1}{2} \frac{u(1 - u)}{1 - \beta u} v^2 + \frac{1}{2} u^2.
\]
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In the paper “Two-layer interfacial flows beyond the Boussinesq approximation: a Hamiltonian approach” the above authors (R. Camassa, G. Falgui, G. Ortenzi) mentioned: Explicit solutions of

\[ h_{uu} f_{vv} = h_{vv} f_{uu} \]

are in general not available, hence we turn to the perturbative analysis in the small \( \beta \)-limit expansion of the Hamiltonian \( h(u, v) \).
Well-known Results. Pre-History

We construct explicitly infinitely many particular solutions of the linear equation

\[ [A''(u)v^2 + B''(u)v + C''(u)]f_{vv} = 2A(u)f_{uu}, \]

which follows from

\[ h_{uu}f_{vv} = h_{vv}f_{uu} \]

by the substitution

\[ h = A(u)v^2 + B(u)v + C(u). \]
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More than thirty years ago Ya. Nutku and P. Olver introduced the so-called “separable” class of linear equations
\[ Q(v)f_{vv} = G(u)f_{uu}. \]

This linear equation also possesses infinitely many particular solutions written in an explicit form. Both above equations have some obvious intersections (for appropriate choice of the functions \( A(u), B(u), C(u), Q(v), G(u) \)).
Previously, two interesting separable sub-classes selected by the choice \(A(u) = u, B(u) = 0\) and \(A(u) = 1, B(u) = 0\) were considered by Ya. Nutku. Later Ya. Nutku and P. Olver formulated the question “how many of results presented for the separable class can be generalised to the nonseparable classes?” Here we consider precisely such a “non-separable” case. So, we can expect that the linear equation

\[ h_{uu} f_{vv} = h_{vv} f_{uu} \]

also possesses some other interesting non-separable classes, which allow to construct explicitly infinitely many particular solutions.
The Polynomial Ansatz. The Quadratic Case

The second quadratic conservation law density

$$\tilde{h}_2 = \tilde{A}(u)v^2 + \tilde{B}(u)v + \tilde{C}(u)$$

can be found from the linear equation

$$[A''(u)v^2 + B''(u)v + C''(u)]f_{vv} = 2A(u)f_{uu}.$$  

In this case

$$\tilde{A}(u) = A(u) \int \frac{du}{A^2(u)},$$

$$\tilde{B}''(u) = B''(u) \int \frac{du}{A^2(u)},$$

$$\tilde{C}''(u) = C''(u) \int \frac{du}{A^2(u)}.$$
The Polynomial Ansatz. The Cubic Case

The cubic conservation law densities are

\[ h_3 = A_1(u)v^3 + A_2(u)v^2 + A_3(u)v + A_4(u), \]

where the function \( A_1(u) \) can be found from the ordinary differential equation

\[ A(u)A_1''(u) = 3A''(u)A_1(u), \]

while other functions \( A_2(u), A_3(u) \) and \( A_4(u) \) are determined by

\[ A_2(u) = 3A(u) \int \left( \int B''(u)A_1(u)du \right) \frac{du}{A^2(u)}. \]

\[ A_3''(u) = 3 \frac{C''(u)}{A(u)}A_1(u) + 3B''(u) \int \left( \int B''(u)A_1(u)du \right) \frac{du}{A^2(u)}, \]

\[ A_4''(u) = 3C''(u) \int \left( \int B''(u)A_1(u)du \right) \frac{du}{A^2(u)}. \]
The Polynomial Ansatz. The Fourth Order Case

The fourth conservation law densities are

\[ h_3 = A_1(u)\nu^4 + A_2(u)\nu^3 + A_3(u)\nu^2 + A_4(u)\nu + A_5(u), \]

where the function \( A_1(u) \) can be found from the ordinary differential equation

\[ A(u)A''_1(u) = 6A''(u)A_1(u), \]

\[ A(u)A''_2(u) = 3A''(u)A_2(u) + 6B''(u)A_1(u), \]

\[ A_3(u) = 3A(u) \int \left( \int [B''(u)A_2(u) + 2C''(u)A_1(u)] \, du \right) \frac{du}{A^2(u)}, \]

\[ A''_4(u) = 3B''(u) \int \left( \int [B''(u)A_2(u) + 2C''(u)A_1(u)] \, du \right) \frac{du}{A^2(u)} + 3 \frac{C''(u)}{A(u)} A_2(u), \]

\[ A''_5(u) = 3C''(u) \int \left( \int [B''(u)A_2(u) + 2C''(u)A_1(u)] \, du \right) \frac{du}{A^2(u)}. \]
We have the system

\[ u_t = (v \ln u)_x, \quad v_t = \left( \frac{v^2}{2u} - \epsilon v \ln u - \frac{1}{2} \epsilon^2 u \right)_x. \]
The Morad–Zhukov System

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So we are looking for functions \( h(u, v) \) and \( g(u, v) \) such that

\[ h_t = g_x. \]

The compatibility condition \( (g_u)_v = (g_v)_u \) leads to the linear differential equation of second order

\[ \frac{1}{2} \left( \frac{v}{u} + \epsilon \right)^2 h_{vv} + \ln u (h_{uu} - \epsilon h_{uv}) = 0, \]

where

\[ g_u = \frac{v}{u} h_u - \frac{v^2}{2u^2} h_v - \epsilon \frac{v}{u} h_v - \frac{1}{2} \epsilon^2 h_v, \quad g_v = \ln uh_u - \epsilon \ln uh_v + \frac{v}{u} h_v. \]
**The Morad–Zhukov System**

**Theorem:** The Zhukov hydrodynamic type system

\[ u_t = (v \ln u)_x, \quad v_t = \left( \frac{v^2}{2u} - \epsilon v \ln u - \frac{1}{2} \epsilon^2 u \right)_x \]

possesses infinitely many polynomial conservation law densities

\[ h = A_N v^N + A_{N-1} v^{N-1} + \ldots + A_1 v + A_0 \]

with respect to the field variable \( v \), while all functions \( A_k(u) \) can be found in quadratures.

**Sketch of the Proof:** For each \( N \) we have the linear system \((m = 0, \ldots, N - 2)\)

\[ A''_N + \frac{N(N-1)}{2u^2 \ln u} A_N = 0, \]

\[ A''_{N-1} + \frac{(N-1)(N-2)}{2u^2 \ln u} A_{N-1} = \epsilon N A'_N - \frac{\epsilon N(N-1)}{u \ln u} A_N, \]

\[ A''_m + \frac{m(m-1)}{2u^2 \ln u} A_m = \epsilon (m+1) A'_{m+1} - \frac{\epsilon m(m+1)}{u \ln u} A_{m+1} - \frac{\epsilon^2 (m+1)(m+2)}{2 \ln u} A_{m+2}. \]
The Morad–Zhukov System

The second order ordinary differential equation

\[ A''_N + \frac{N(N - 1)}{2u^2 \ln u} A_N = 0, \]

has two independent solutions, and the first solution is given by the polynomial

\[ A^{(1)}_N = R_N(\ln u), \]

where

\[ R_N = zP_K(z) \]

and the degree of the polynomial \( P_K(z) \) is given by

\[ K = \frac{N^2 - N - 2}{2}. \]

So, one can look for a first solution in the polynomial form

\[ P(z) = \sum_{m=0}^{K} B_m z^m. \]
The Morad–Zhukov System. The Generalized Hodograph Method

In a general case the Generalised Hodograph Method yields a general solution presented in implicit form

\[ t = \frac{h_{vv}}{\ln u}, \quad x = h_{uv} - \frac{vh_{vv}}{u \ln u}, \]

where \( h \) is a general solution of the linear equation

\[
\frac{1}{2} \left( \frac{v}{u} + \epsilon \right)^2 h_{vv} + \ln u (h_{uu} - \epsilon h_{uv}) = 0.
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References: