Integrability of New Class of Hyperbolic Two-Dimensional Linear Equations of Second Order

Maxim V. Pavlov

Physical Institute of Russian Academy of Sciences

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The Problem

We consider integrability of Hamiltonian hydrodynamic type systems

$$u_t = \left(\frac{\partial h}{\partial v}\right)_x, \quad v_t = \left(\frac{\partial h}{\partial u}\right)_x,$$

whose Hamiltonian densities are quadratic with respect to the field variable v, i.e.

$$h = A(u)v^2 + B(u)v + C(u),$$

where the functions $A(u), B(u), C(u)$ are arbitrary.

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where the functions $A(u), B(u), C(u)$ are arbitrary.
Such hydrodynamic type systems possess infinitely many commuting flows

$$u_y = \left(\frac{\partial f}{\partial v}\right)_x, \quad v_y = \left(\frac{\partial f}{\partial u}\right)_x,$$

where the Hamiltonian densities satisfy the linear equation

$$h_{uu} f_{vv} = h_{vv} f_{uu}.$$

The Morad–Zhukov System

In the paper "**The motion of a thin liquid layer on the outer surface of a rotating cylinder**" the hydrodynamic type system

$$u_t = (w \ln u)_x, \quad w_t = \left(\frac{w^2}{2u} - \epsilon w \ln u - \frac{1}{2}\epsilon^2 u \right)_x$$

was derived by A.M. Morad and M.Yu. Zhukov.

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Under the linear transformation $w = v - \frac{\epsilon}{2}u$, this system takes a local Hamiltonian structure

$$u_t = \left(\frac{\partial h}{\partial v} \right)_x, \quad v_t = \left(\frac{\partial h}{\partial u} \right)_x,$$

where the Hamiltonian density is

$$h = \frac{1}{2}v^2 \ln u - \frac{\epsilon}{2} vu \ln u + \frac{\epsilon^2}{8} u^2 \ln u - \frac{1}{4}\epsilon^2 u^2.$$

The Camassa–Falqui–Ortenzi System

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$$u_t = \left(\frac{\partial h}{\partial v}\right)_x, \quad v_t = \left(\frac{\partial h}{\partial u}\right)_x,$$

is determined by the Hamiltonian density quadratic with respect to the field variable v, i.e.

$$h = \frac{1}{2}\frac{u(1-u)}{1-\beta u}v^2 + \frac{1}{2}u^2.$$

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In the paper "**Two-layer interfacial flows beyond the Boussinesq approximation: a Hamiltonian approach**" the above authors (R. Camassa, G. Falgui, G. Ortenzi) mentioned: *Explicit solutions of*

$$h_{uu}f_{vv} = h_{vv}f_{uu}$$

are **in general not available**, *hence we turn to the perturbative analysis in the small β-limit expansion of the Hamiltonian $h(u, v)$.*

Well-known Results. Pre-History

We construct explicitly infinitely many particular solutions of the linear equation

$$[A''(u)v^2 + B''(u)v + C''(u)]f_{vv} = 2A(u)f_{uu},$$

which follows from

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More than thirty years ago Ya. Nutku and P. Olver introduced the so called “separable” class of linear equations

$$Q(v)f_{vv} = G(u)f_{uu}.$$

This linear equation also possesses infinitely many particular solutions written in an explicit form. Both above equations have some obvious intersections (for appropriate choice of the functions $A(u), B(u), C(u), Q(v), G(u)$).

Well-known Results. Pre-History

Previously, two interesting separable sub-classes selected by the choice $A(u) = u, B(u) = 0$ and $A(u) = 1, B(u) = 0$ were considered by Ya. Nutku.
Later Ya. Nutku and P. Olver formulated the question "*how many of results presented for the separable class can be generalised to the* **nonseparable** *classes*?". Here we consider precisely such a "non-separable" case. So, we can expect that the linear equation

$$h_{uu} f_{vv} = h_{vv} f_{uu}$$

also possesses some other interesting non-separable classes, which allow to construct explicitly infinitely many particular solutions.

The Polynomial Ansatz. The Quadratic Case

The second quadratic conservation law density

$$\tilde{h}_2 = \tilde{A}(u)v^2 + \tilde{B}(u)v + \tilde{C}(u)$$

can be found from the linear equation

$$[A''(u)v^2 + B''(u)v + C''(u)]f_{vv} = 2A(u)f_{uu}.$$

In this case

$$\tilde{A}(u) = A(u)\int \frac{du}{A^2(u)},$$

$$\tilde{B}''(u) = B''(u)\int \frac{du}{A^2(u)},$$

$$\tilde{C}''(u) = C''(u)\int \frac{du}{A^2(u)}.$$

The Polynomial Ansatz. The Cubic Case

The cubic conservation law densities are

$$h_3 = A_1(u)v^3 + A_2(u)v^2 + A_3(u)v + A_4(u),$$

where the function $A_1(u)$ can be found from the ordinary differential equation

$$A(u)A_1''(u) = 3A''(u)A_1(u),$$

while other functions $A_2(u), A_3(u)$ and $A_4(u)$ are determined by

$$A_2(u) = 3A(u)\int\left(\int B''(u)A_1(u)du\right)\frac{du}{A^2(u)}.$$

$$A_3''(u) = 3\frac{C''(u)}{A(u)}A_1(u) + 3B''(u)\int\left(\int B''(u)A_1(u)du\right)\frac{du}{A^2(u)},$$

$$A_4''(u) = 3C''(u)\int\left(\int B''(u)A_1(u)du\right)\frac{du}{A^2(u)}.$$

The Polynomial Ansatz. The Fourth Order Case

The fourth conservation law densities are

$$h_3 = A_1(u)v^4 + A_2(u)v^3 + A_3(u)v^2 + A_4(u)v + A_5(u),$$

where the function $A_1(u)$ can be found from the ordinary differential equation

$$A(u)A_1''(u) = 6A''(u)A_1(u),$$

$$A(u)A_2''(u) = 3A''(u)A_2(u) + 6B''(u)A_1(u),$$

$$A_3(u) = 3A(u)\int\left(\int[B''(u)A_2(u) + 2C''(u)A_1(u)]du\right)\frac{du}{A^2(u)},$$

$$A_4''(\mathrm{u})=3B''(\mathrm{u})\int\left(\int[B''(\mathrm{u})A_2(\mathrm{u})+2C''(\mathrm{u})A_1(\mathrm{u})]du\right)\frac{du}{A^2(\mathrm{u})}+3\frac{C''(\mathrm{u})}{A(\mathrm{u})}A_2(\mathrm{u}),$$

$$A_5''(u) = 3C''(u)\int\left(\int[B''(u)A_2(u) + 2C''(u)A_1(u)]du\right)\frac{du}{A^2(u)}.$$

The Morad–Zhukov System

We have the system

$$u_t = (v \ln u)_x, \quad v_t = \left(\frac{v^2}{2u} - \epsilon v \ln u - \frac{1}{2}\epsilon^2 u \right)_x .$$

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So we are looking for functions $h(u,v)$ and $g(u,v)$ such that

$$h_t = g_x.$$

The compatibility condition $(g_u)_v = (g_v)_u$ leads to the linear differential equation of second order

$$\frac{1}{2}(\frac{v}{u} + \epsilon)^2 h_{vv} + \ln u(h_{uu} - \epsilon h_{uv}) = 0,$$

where

$$g_u = \frac{v}{u}h_u - \frac{v^2}{2u^2}h_v - \epsilon\frac{v}{u}h_v - \frac{1}{2}\epsilon^2 h_v, \quad g_v = \ln u h_u - \epsilon \ln u h_v + \frac{v}{u}h_v.$$

The Morad–Zhukov System

Theorem: The Zhukov hydrodynamic type system

$$u_t = (v \ln u)_x, \quad v_t = \left(\frac{v^2}{2u} - \epsilon v \ln u - \frac{1}{2}\epsilon^2 u\right)_x$$

possesses infinitely many polynomial conservation law densities

$$h = A_N v^N + A_{N-1} v^{N-1} + \ldots + A_1 v + A_0$$

with respect to the field variable v, while all functions $A_k(u)$ can be found in quadratures.

Sketch of the Proof: For each N we have the linear system ($m = 0, \ldots, N-2$)

$$A_N'' + \frac{N(N-1)}{2u^2 \ln u} A_N = 0,$$

$$A_{N-1}'' + \frac{(N-1)(N-2)}{2u^2 \ln u} A_{N-1} = \epsilon N A_N' - \frac{\epsilon N(N-1)}{u \ln u} A_N,$$

$$A_m'' + \frac{m(m\text{-}1)}{2u^2 \ln u} A_m = \epsilon(m+1)A_{m+1}' - \frac{\epsilon m(m+1)}{u \ln u} A_{m+1} - \frac{\epsilon^2 (m+1)(m+2)}{2 \ln u} A_{m+2}.$$

The Morad–Zhukov System

The second order ordinary differential equation

$$A_N'' + \frac{N(N-1)}{2u^2 \ln u} A_N = 0,$$

has two independent solutions, and the first solution is given by the polynomial

$$A_N^{(1)} = R_N(\ln u),$$

where

$$R_N = zP_K(z)$$

and the degree of the polynomial $P_K(z)$ is given by

$$K = \frac{N^2 - N - 2}{2}.$$

So, one can look for a first solution in the polynomial form

$$P(z) = \sum_{m=0}^{K} B_m z^m.$$

The Morad–Zhukov System. The Generalized Hodograph Method

In a general case the Generalised Hodograph Method yields a general solution presented in implicit form

$$t = \frac{h_{vv}}{\ln u}, \quad x = h_{uv} - \frac{v h_{vv}}{u \ln u},$$

where h is a general solution of the linear equation

$$\frac{1}{2}\left(\frac{v}{u} + \epsilon\right)^2 h_{vv} + \ln u (h_{uu} - \epsilon h_{uv}) = 0.$$

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1. A.M. Morad, M.Yu. Zhukov, The motion of a thin liquid layer on the outer surface of a rotating cylinder, Eur. Phys. J. Plus (2015) 130:8, doi 10.1140/epjp/i2015-15008-6

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