Integrable and near-integrable models for surface and internal waves in stratified shear flows

Karima Khusnutdinova

Department of Mathematical Sciences, Loughborough University, UK

K.Khusnutdinova@lboro.ac.uk

Classical and Quantum Integrability, Dijon, France
2 - 6 September 2019

Overview

- Versions of the Kadomtsev - Petviashvili (KP) equation
- Ostrovsky equation and coupled Ostrovsky equations
- 2+1D cylindrical - Korteweg - de Vries (cKdV) - type equation

Joint work with:

- KK, C. Klein, V.B. Matveev, A.O. Smirnov, On the integrable elliptic cylindrical Kadomtsev-Petviashvili equation, Chaos 23 (2013) 013126.
- A. Alias, R.H.J. Grimshaw, KK, On strongly interacting internal waves in a rotating ocean and coupled Ostrovsky equations, Chaos 23 (2013) 023121.
- A. Alias, R.H.J. Grimshaw, KK, Coupled Ostrovsky equations for internal waves in a shear flow, Phys. Fluids 26 (2014) 126603.
- KK, X. Zhang, Long ring waves in a stratified fluid over a shear flow, J. Fluid Mech. 794 (2016) 17-44.
- KK, X. Zhang, Nonlinear ring waves in a two-layer fluid, Physica D 333 (2016) 208-221.
- KK, M.R. Tranter, D'Alembert-type solution of the Cauchy problem for the Boussinesq-Klein-Gordon equation, Stud. Appl. Math. 142 (2019) 551-585.

1. Introduction

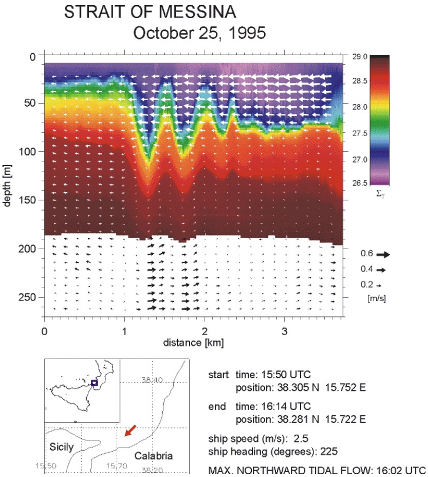


Figure: Internal wave in the Strait of Messina, *from* http://earth.esa.int/ers/instruments/sar/applications/ERS-SARtropical/oceanic/intwaves/intro/

1. Introduction

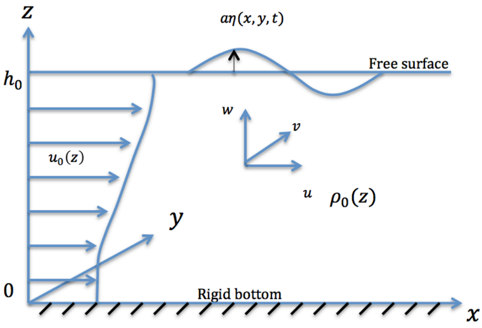


Figure: Schematic of the problem formulation.

1. Introduction

Euler equations for an inviscid, incompressible fluid:

$$\begin{aligned}
\rho(u_t + uu_x + vu_y + wu_z) + p_x &= 0, \\
\rho(v_t + uv_x + vv_y + wv_z) + p_y &= 0, \\
\rho(w_t + uw_x + vw_y + ww_z) + p_z + \rho g &= 0, \\
\rho_t + u\rho_x + v\rho_y + w\rho_z &= 0, \\
u_x + v_y + w_z &= 0.
\end{aligned}$$

Free surface and rigid bottom boundary conditions:

$$\begin{aligned}
&p = p_a \quad \text{at} \quad z = h(x, y, t), \\
&w = h_t + uh_x + vh_y \quad \text{at} \quad z = h(x, y, t), \\
&w = 0 \quad \text{at} \quad z = 0.
\end{aligned}$$

Vertical particle displacement:

$$\zeta_t + u\zeta_x + v\zeta_y + w\zeta_z = w, \quad \zeta|_{z=h} = h - h_0.$$

1. Introduction

These equations can be non-dimensionalised by the transformations:

$$x \to \lambda x,\ y \to \lambda y,\ z \to h_0 z,\ t \to \lambda/\sqrt{gh_0}\, t,$$

$$u \to \sqrt{gh_0}\, u, \quad v \to \sqrt{gh_0}\, v, \quad w \to h_0\sqrt{gh_0}/\lambda w,$$

$$\rho \to \rho_f \rho, \quad \zeta \to h_0 \zeta, \quad h \to h_0(1+\varepsilon\eta),$$

$$p \to p_a + \int_z^{h_0} \rho_0 \rho_f g \,\mathrm{d}z + \rho_f g h_0 p.$$

There are two small parameters in the problem, the amplitude parameter $\varepsilon = a/h_0$ and the wavelength parameter $\delta = h_0/\lambda$.

In the subsequent derivations we impose the condition $\delta^2 = \varepsilon$.

Let $v = 0$ and consider 2D problem formulation (no dependence on y).

In the basic state (in non-dimensional variables), the fluid has the density $\rho_0(z)$, the pressure $p_{0z} = -\rho_0(z)$, $p_0(1) = 0$ and the prescribed current $u_0(z)$ in the x direction. We then consider the equations of motion relative to this basic state.

1. Introduction

Derivation uses asymptotic multiple-scales expansions of the form

$q = q_1 + \varepsilon q_2 + O(\varepsilon^2)$, where $q = \{\zeta,\ \eta,\ u,\ w,\ p,\ \rho\}$.

Known modal decomposition for plane waves (at $O(1)$ leads to Taylor-Goldstein equation, in the long-wave limit):

$$\zeta_1 = A(\xi, T)\phi(z),$$

where $\xi = x - st$, $T = \varepsilon t$ (fast and slow variables) and

$$\begin{aligned} ((s-u_0)^2\rho_0\phi_z)_z - \rho_{0z}\phi &= 0, \\ (s-u_0)^2\phi_z - \phi = 0 \quad \text{at} \quad z &= 1, \\ \phi = 0 \quad \text{at} \quad z &= 0. \end{aligned}$$

(Modal equations: the boundary-value problem defines linear long-wave modes together with their speeds).

1. Introduction

One can systematically derive the amplitude equation in the form of the KdV equation (first versions in Benney 1966, Benjamin 1966; ..., Grimshaw 1981, see review Grimshaw 2015):

$$\mu_1 A_T + \mu_2 A A_\xi + \mu_3 A_{\xi\xi\xi} = 0, \quad \text{where}$$

$$\mu_1 = 2\int_0^1 \rho_0 W \phi_z^2 \, \mathrm{d}z,$$

$$\mu_2 = 3\int_0^1 \rho_0 W^2 \phi_z^3 \, \mathrm{d}z,$$

$$\mu_3 = \int_0^1 \rho_0 W^2 \phi^2 \, \mathrm{d}z,$$

and $W = s - u_0(z)$.

2. Versions of KP equation

The original KP equation (KP, 1970) (Ablowitz and Segur, 1979 for surface waves; Grimshaw, 1985 for internal and surface waves)

$$(U_\tau + 6UU_\xi + U_{\xi\xi\xi})_\xi + 3\alpha^2 U_{YY} = 0 \tag{1}$$

and cylindrical KP (cKP) equation (Johnson, 1980 for surface waves; Lipovskii, 1985 for internal waves)

$$\left(W_\tau + 6WW_\chi + W_{\chi\chi\chi} + \frac{W}{2\tau}\right)_\chi + \frac{3\alpha^2}{\tau^2} W_{VV} = 0 \tag{2}$$

describe the evolution of nearly-plane and nearly-concentric waves, respectively.

2. Versions of KP equation

Transformations between the KP and cKP equations were found by Johnson (1980) and rediscovered by Lipovskii, Matveev, Smirnov (1989). The map

$$W(\tau,\chi,V) \to U(\tau,\xi,Y) := W\left(\tau, \xi + \frac{Y^2}{12\alpha^2\tau}, \frac{Y}{\tau}\right)$$

transforms any solution of the cKP equation (2) into a solution of the KP equation (1). Conversely, the map

$$U(\tau,\xi,Y) \to W(\tau,\chi,V) := U\left(\tau, \chi - \frac{\tau V^2}{12\alpha^2}, \tau V\right)$$

transforms any solution of the KP equation (1) into a solution of the cKP equation (2).

The transformation has been used to construct some special solutions of the cKP equation by Klein, Matveev, Smirnov 2007.

Lax Pair was found by Dryuma, 1983.

2. Versions of KP equation

We aim to consider long waves with the nearly-elliptic front. and we write this set of equations in the elliptic cylindrical coordinate system:

$$x = d\cosh\alpha\cos\beta, \quad y = d\sinh\alpha\sin\beta, \quad = z,$$

where d has the meaning of half a distance between the foci of the coordinate lines.

Equation for the linear waves is obtained as

$$\eta_{tt} = \frac{\eta_{\alpha\alpha} + \eta_{\beta\beta}}{\gamma^2(\sinh^2\alpha + \sin^2\beta)}.$$

The derivation of the cylindrical KP (cKP) equation is based on the existence of the exact reduction of the equation for the linear to the equation which does not depend on the angle variable.

Here, the equation does not have an *exact* reduction to the equation with no dependence on β. However, there is an *asymptotic* reduction, and this is enough to derive a third version of the KP equation related to elliptic-cylindrical geometry (KK, Klein, Matveev, Smirnov 2013).

2. Versions of KP equation

We write the KP equation in the form

$$\left(U_\tau + 6UU_\xi + U_{\xi\xi\xi}\right)_\xi + 3\alpha^2 U_{YY} = 0,$$

the cKP equation in the form

$$\left(W_\tau + 6WW_\chi + W_{\chi\chi\chi} + \frac{1}{2\tau}W\right)_\chi + \frac{3\alpha^2}{\tau^2}W_{VV} = 0,$$

and the ecKP equation as

$$\left(H_\tau + 6HH_\zeta + H_{\zeta\zeta\zeta} + \frac{\tau}{2(\tau^2 - a^2)}H - \frac{a^2\nu^2}{12\sigma^2(\tau^2 - a^2)}H_\zeta\right)_\zeta + \frac{3\sigma^2}{\tau^2 - a^2}H_{\nu\nu} = 0.$$

The map

$$U(\tau, \xi, Y) \to W(\tau, \chi, V) := U\left(\tau, \chi - \frac{\tau V^2}{12\alpha^2}, \tau V\right)$$

transforms any solution of the KP equation into a solution of the cKP equation, and the map

$$U(\tau, \xi, Y) \to H(\tau, \zeta, \nu) := U\left(\tau, \zeta - \frac{\tau\nu^2}{12\alpha^2}, \sqrt{\tau^2 - a^2}\,\nu\right)$$

transforms any solution of the KP eq. into a solution of the ecKP eq.

2. Versions of KP equation

The 1-soliton solution of the ecKP-II equation is explicitly written in the form

$$H(\tau,\zeta,\nu) = \frac{K^2}{2}\,\mathrm{sech}^2\left[\frac{K}{2}\left(\zeta - \frac{\tau\nu^2}{12} + L\sqrt{\tau^2 - a^2}\nu - (K^2+3L^2)\tau + \delta_0\right)\right],$$

where K, L, δ_0 are arbitrary constants. The corresponding surface wave elevation η is plotted below for $\gamma = 1, a = 2, \Delta = 1/2$ and $\delta_0 = 0$.

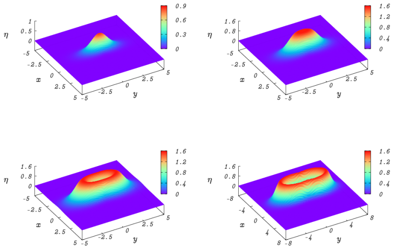


Figure: Surface wave corresponding to the one-soliton solution of the ecKP-II equation with $K = 1, L = 0$ for $t = 0$ (top left), $t = 0.25$ (top right), $t = 0.5$ (bottom left), $t = 1$ (bottom right).

2. Versions of KP equation

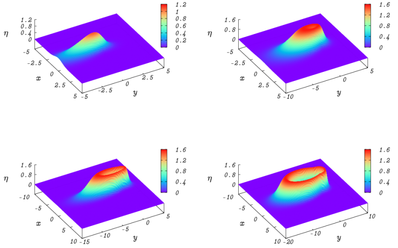


Figure: Surface wave corresponding to the one-soliton solution of the ecKP-II equation with $K = 1, L = 0.1$ for $t = 0$ (top left), $t = 0.25$ (top right), $t = 0.5$ (bottom left), $t = 1$ (bottom right).

2. Versions of KP equation

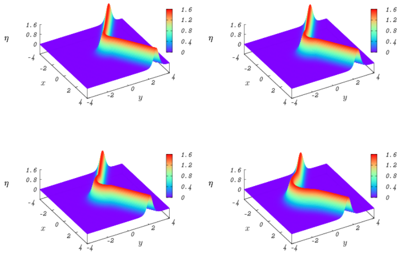


Figure: Surface wave corresponding to the one-soliton solution of the ecKP-II equation with $K = 1, L = -0.5$ for $t = 0$ (top left), $t = 0.25$ (top right), $t = 0.5$ (bottom left), $t = 2$ (bottom right).

2. Versions of KP equation

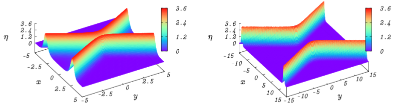


Figure: Surface wave corresponding to the one-soliton solution of the ecKP-II eq. with $K = 1.5, L = 0$ for $t = 0$ (left), $t = 2$ (right).

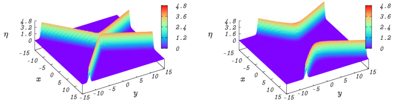


Figure: Surface wave corresponding to the one-soliton solution of the ecKP-II eq. with $K = 1.6, L = 0.1$ for $t = 0$ (left), $t = 2$ (right).

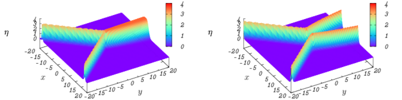


Figure: Surface wave corresponding to the exceptional one-soliton solution of the ecKP-II eq. with $K = 1.5$ and $L \approx 0.1$ for $t = 0$ (left) and $t = 1$ (right).

2 Versions of KP equation

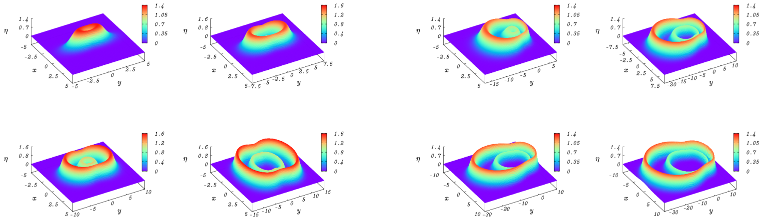


Figure: Surface waves corresponding to two different two-soliton solutions of the ecKP-II equation for $t = 1$ (top left), $t = 2$ (top right), $t = 3$ (bottom left), $t = 4$ (bottom right).

3. Ostrovsky equation and coupled Ostrovsky equations

The evolution of weakly-nonlinear, long internal waves with rotation is described by the Ostrovsky equation (Ostrovsky 1978):

$$\{A_t + \nu A A_x + \lambda A_{xxx}\}_x = \gamma A,$$

where ν and λ are the coefficients of nonlinear and dispersive terms, respectively and $\gamma = f^2/2c$ is the rotation coefficient when there is no shear flow. Here, c is the linear long wave phase speed and f is the local Coriolis parameter.

For oceanic internal waves, in the absence of a shear flow, $\lambda\gamma > 0$ and then here are no steady solitary wave solutions (Galkin and Stepanyants 1991). The long-time effect of rotation in this case is the emergence of a propagating unsteady nonlinear wave packet, associated with the maximum of the group speed (Grimshaw and Helfrich 2008). The same phenomenon was observed independently by Yagi and Kawahara (2001) in a Toda lattice on an elastic substrate.

On the other hand, when $\lambda\gamma < 0$ the Ostrovsky equation can support steady envelope wave packets, associated with the maximum of the phase speed (Galkin and Stepanyants 1991, Obregon and Stepanyants 1998; in the context of a rotating plasma).

3. Ostrovsky equation and coupled Ostrovsky equations

Effects of a shear flow may become important (Alias, Grimshaw, KK, 2013; Grimshaw 2013):

$$(u_t + \nu u u_x + \lambda u_{xxx})_x = \gamma u,$$

where γ is now given by

$$\gamma = \frac{f^2 \int_0^{h_0} \rho_0 \Phi \phi_z dz}{2 \int_0^{h_0} \rho_0 (c - u_0(z)) \phi_z^2 \, \mathrm{d}z},$$

$$\Phi = \frac{\phi_z - (\rho_0 u_0)_z \phi}{\rho_0 (c - u_0(z))}.$$

In the absence of the current, $\gamma = \dfrac{f^2}{2c}$ and $\lambda\gamma > 0$. The underlying current can change this sign (Alias, Grimshaw, KK, 2014). Examples: two-layered fluid, thin top layer, second layer can have either infinite or finite depth, sufficiently strong current in the top layer.

3. Ostrovsky equation and coupled Ostrovsky equations

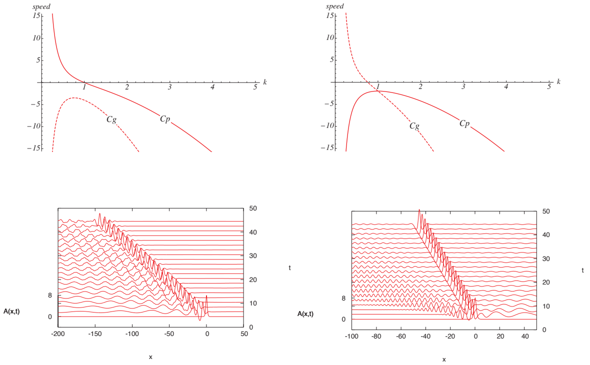


Figure: Dispersion relation and numerical solution of the Ostrovsky equation when $\lambda\gamma > 0$ (left) and $\lambda\gamma < 0$ (right) with $\lambda = \gamma = 1$. The initial condition is given by the KdV solitary wave with the amplitude 8 at $x = 0$: emergence of an unsteady wave packet associated with the maximum of the group speed (left) and a steady wave packet associated with the maximum of the phase speed (right).

3. Ostrovsky equation and coupled Ostrovsky equations

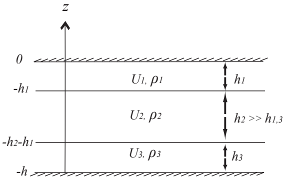
Coupled Ostrovsky equations have been derived for a pair of different long-wave modes with nearly coincident phase speeds c and $c+\Delta$ (Alias, Grimshaw, KK, 2013), generalising the work by Gear and Grimshaw (1984).

$$\begin{aligned} I_1(A_{1\tau} + \mu_1 A_1 A_{1s} + \lambda_1 A_{1sss} - \gamma_1 B_1) + \nu_1 (A_1 A_2)_s \\ + \nu_2 A_2 A_{2s} + \lambda_{12} A_{2sss} = \gamma_{12} B_2 \,, \end{aligned}$$

$$\begin{aligned} I_2(A_{2\tau} + \mu_2 A_2 A_{2s} + \lambda_2 A_{2sss} + \Delta A_{2s} - \gamma_2 B_2) + \nu_2 (A_1 A_2)_s \\ + \nu_1 A_1 A_{1s} + \lambda_{21} A_{1sss} = \gamma_{21} B_1 \,, \end{aligned}$$

where $B_{1s} = A_1, B_{2s} = A_2$, and the coefficients are given in terms of modal functions.

3. Ostrovsky equation and coupled Ostrovsky equations



We assume $U_2 = 0$ without loss of generality. A resonance with two distinct modes (and no implicit critical levels) can take place if

$$h_2 \gg h_1, h_3\,, \quad c = U_1 + \{\frac{gh_1(\rho_2 - \rho_1)}{\rho_1}\}^{1/2} = U_3 + \{\frac{gh_3(\rho_3 - \rho_2)}{\rho_3}\}^{1/2}\,.$$

For given densities $\rho_{1,2,3}$ and layer depths $h_{1,3}$, these determine the allowed shear $U_1 - U_3$. The modal functions and their derivatives, and all coefficients of the scaled cO equations, are then found explicitly:

$$(u_T + uu_X + u_{XXX} + n(uv)_X + mvv_X + \alpha v_{XXX})_X = \beta u + \gamma v,$$
$$(v_T + vv_X + \delta v_{XXX} + \Delta v_X + p(uv)_X + quu_X + \lambda u_{XXX})_X = \mu v + \nu u.$$

3. Ostrovsky equation and coupled Ostrovsky equations

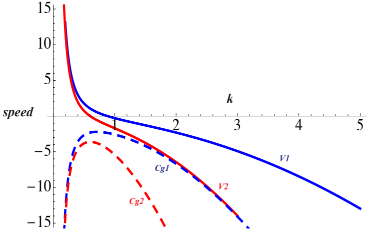


Figure: Dispersion curve in the absence of a shear flow ($\beta = \mu > 0$). The solid curves show the phase speed, and the dashed curves show the group velocity.

In the presence of a shear flow, $\beta \neq \mu$, and each can be either positive or negative.

3. Ostrovsky equation and coupled Ostrovsky equations

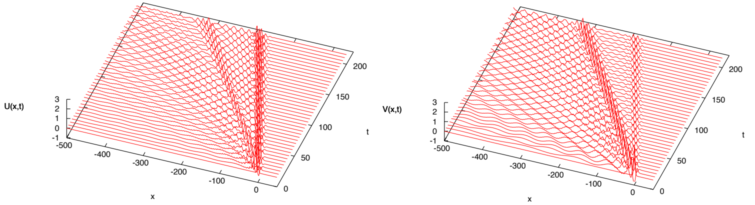


Figure: Typical numerical simulations for the coupled Ostrovsky equations without shear flow.

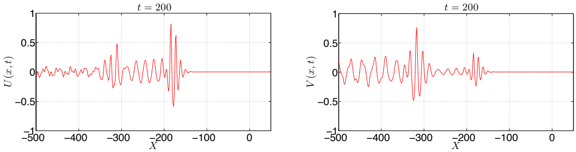


Figure: A cross-section at $t = 200$.

3. Ostrovsky equation and coupled Ostrovsky equations

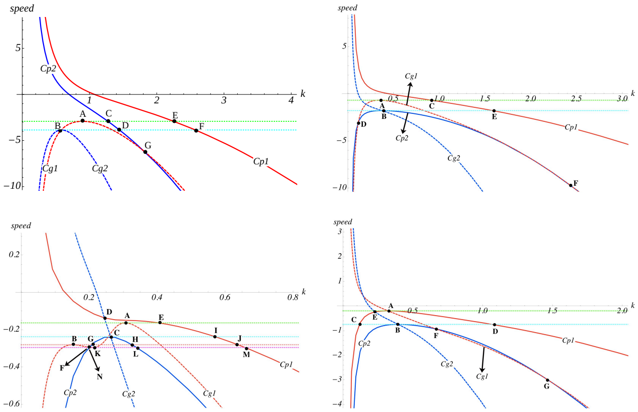


Figure: Typical dispersion curve for Case A ($\beta > 0, \mu > 0$; top left), Case B ($\beta > 0, \mu < 0$; top right), Case C ($\beta > 0, \mu < 0$; bottom left) and Case D ($\beta < 0, \mu > 0$, bottom right)

4. 2+1D cKdV-type equation

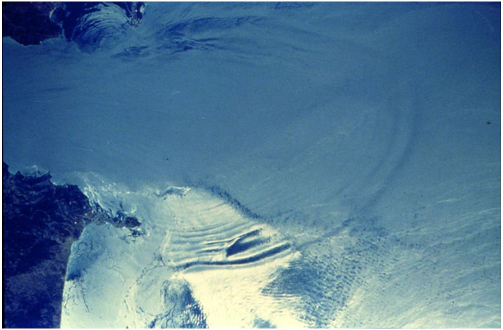


Figure: Ring soliton in the Strait of Gibraltar. NASA image STS17-34-098, Lunar and Planetary Institute.

4. 2+1D cKdV-type equation

- The cylindrical (or concentric) Korteweg - de Vries (cKdV) equation

$$2A_R + 3AA_\xi + \frac{1}{3}A_{\xi\xi\xi} + \frac{A}{R} = 0$$

  is a universal weakly-nonlinear weakly-dispersive wave equation in cylindrical geometry (Maxon & Viecelli 1974, waves in plasma). The equation is integrable (Druma 1976, Calogero & Degasperis 1978).
- In the context of fluids, it was derived for the surface waves problem by Miles in 1978 (from Boussinesq system) and Johnson in 1980 (from Euler equations, homogeneous fluid).
- In 1985, Lipovskii derived the cKdV equation for internal waves (stratified fluid, no shear flow).
- In 1990 Johnson obtained a version of the equation for surface waves in a homogeneous fluid on a shear flow.
- We study the propagation of internal and surface ring waves in a stratified fluid over a prescribed shear flow.

4. 2+1D cKdV-type equation

We use the cylindrical coordinate system moving at a constant speed c, and introduce the variables

$$\xi = rk(\theta) - st, \quad R = \varepsilon rk(\theta), \quad \theta,$$

where s is the wave speed in the absence of a shear flow, and the function $k(\theta)$ is to be determined.
In our problem, to leading order there exists the modal decomposition

$$\zeta_1 = A(\xi, R, \theta)\phi(z, \theta),$$

where the modal functions satisfy the following problem:

$$\left(\frac{\rho_0 F^2}{k^2 + k'^2}\phi_z\right)_z - \rho_{0z}\phi = 0,$$
$$\frac{F^2}{k^2 + k'^2}\phi_z - \phi = 0 \quad \text{at} \quad z = 1,$$
$$\phi = 0 \quad \text{at} \quad z = 0,$$

and $F = -s + (u_0 - c)(k\cos\theta - k'\sin\theta)$. If $c = u_0(0)$, then $F \neq 0$ at $z = 0$ (and then $\phi F = 0 \quad \text{at} \quad z = 0 \quad \text{yields} \quad \phi = 0 \quad \text{at} \quad z = 0$).

Here, the function $k(\theta)$ has to be found as a part of the solution.

4. 2+1D cKdV-type equation

The amplitude equation has the form (KK and Zhang 2016):

$$\mu_1 A_R + \mu_2 A A_\xi + \mu_3 A_{\xi\xi\xi} + \frac{\mu_4}{R} A + \frac{\mu_5}{R} A_\theta = 0, \quad \text{where}$$

$$\mu_1 = 2s \int_0^1 \rho_0 F \phi_z^2 \, \mathrm{d}z, \qquad \mu_2 = -3 \int_0^1 \rho_0 F^2 \phi_z^3 \, \mathrm{d}z,$$

$$\mu_3 = -(k^2 + k'^2) \int_0^1 \rho_0 F^2 \phi^2 \, \mathrm{d}z,$$

$$\mu_4 = -\int_0^1 \left\{ \frac{\rho_0 \phi_z^2 k(k + k'')}{(k^2 + k'^2)^2} \Big((k^2 - 3k'^2) F^2 \right.$$

$$-4k'(k^2 + k'^2)(u_0 - c)\sin\theta F - \sin^2\theta (u_0 - c)^2 (k^2 + k'^2)^2 \Big)$$

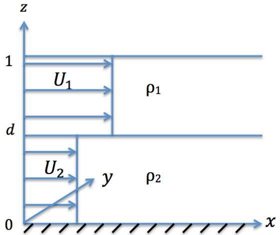
$$\left. + \frac{2\rho_0 k}{k^2 + k'^2} F \phi_z \phi_{z\theta} (k' F + (k^2 + k'^2)(u_0 - c)\sin\theta) \right\} \, \mathrm{d}z,$$

$$\mu_5 = -\frac{2k}{k^2 + k'^2} \int_0^1 \rho_0 F \phi_z^2 [k' F + (u_0 - c)(k^2 + k'^2)\sin\theta] \, \mathrm{d}z.$$

For a homogeneous fluid, the amplitude equation reduces to a 1+1-dimensional cKdV-type equation (i.e. $\mu_5 = 0$) for *any parallel current*, and not just for particular currents, as previously thought (Johnson 1990, Johnson 1997).

4. 2+1D cKdV-type equation

Example:



'Dispersion relation':

$$(\rho_2-\rho_1)d(1-d)(k^2+k'^2)^2-\rho_2[dF_1^2+(1-d)F_2^2](k^2+k'^2)+\rho_2 F_1^2 F_2^2=0,$$

with $F_1=-s+(U_1-U_2)(k\cos\theta-k'\sin\theta)$, $F_2=-s$.

This nonlinear first-order differential equation for the function $k(\theta)$ is further generalisation of both Burns and generalised Burns conditions (Burns 1953, Johnson 1990).

4. 2+1D cKdV-type equation

The singular solution for $k(\theta)$ (two branches) is given by

$$\begin{cases} k(\theta) = a\cos\theta + b(a)\sin\theta, \\ b'(a) = -1/\tan\theta, \\ a^2 + b^2 = \frac{\rho_2(d(-1+a(U_1-U_2))^2+(1-d))\pm\sqrt{\Delta}}{2(\rho_2-\rho_1)d(1-d)}, \end{cases}$$

where

$$\begin{aligned} \Delta \quad &= \rho_2^2\left[d(-1+a(U_1-U_2))^2-(1-d)\right]^2 \\ &\quad +4\rho_1\rho_2 d(1-d)\left[1-a(U_1-U_2)\right]^2 \geq 0. \end{aligned}$$

The upper / lower sign corresponds to interfacial / surface waves.

In the rigid lid approximation, for the interfacial mode,

$$k(\theta) = \sqrt{\frac{\alpha^2 s^2 - \alpha(U_1-U_2)+1}{1+(1-\alpha(U_1-U_2))\tan^2\theta}}\left(\frac{\cos\theta}{1-\alpha(U_1-U_2)}+\frac{\sin^2\theta}{\cos\theta}\right)\ \mathrm{sign}(\cos\theta)$$

$$-\frac{\alpha s\cos\theta}{1-\alpha(U_1-U_2)}, \quad \text{where} \quad \alpha = \frac{\rho_1(U_1-U_2)}{(1-d)(\rho_2-\rho_1)}, \quad s^2 = \frac{(\rho_2-\rho_1)d(1-d)}{\rho_1 d+\rho_2(1-d)}.$$

4. 2+1D cKdV-type equation

Surface (left) and interfacial (right) ring waves

Let $\rho_1 = 1$, $\rho_2 = 1.2$, $d = 0.5$.

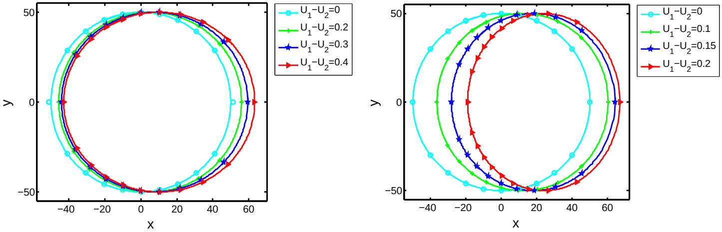


Figure: Wavefronts of surface and interfacial ring waves described by $k(\theta)r = 50$.

4. 2+1D cKdV-type equation

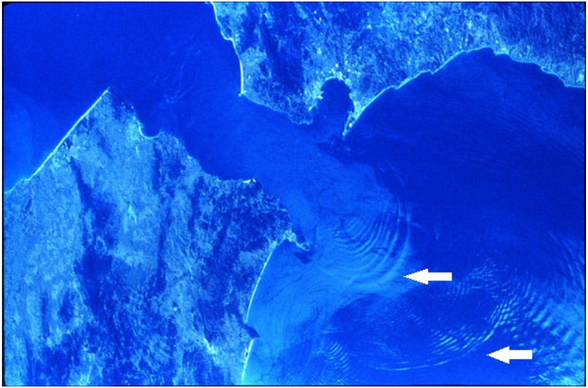


Figure: Internal waves in the Strait of Gibraltar. NASA image STS17-34-081, Lunar and Planetary Institute.

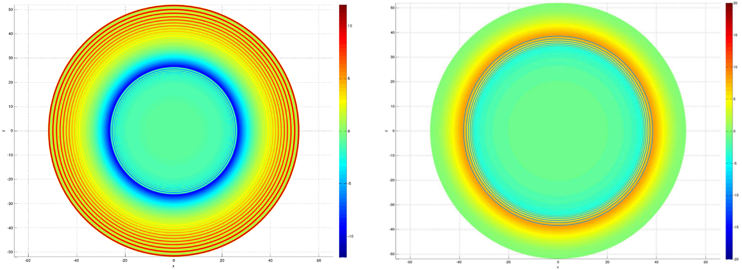
4. 2+1D cKdV-type equation

2D dam-break problem (no current):

To generate initial conditions for the cKdV-type equation we solve:

$$A_{tt} - s_{\pm}^2 (A_{xx} + A_{yy}) = 0,$$
$$A(x, y, 0) = \frac{1}{2} Q \Big[\tanh \big(-0.15(x^2 + y^2 - 64) \big) + 1 \Big], \quad A_t(x, y, 0) = 0,$$

where Q is a scaling factor (cKdV-type equation has a scaling symmetry).



(a) (b)

Figure: Surface (a) and interfacial (b) ring DSWs for $d = 0.6$, $Q = 40$ and $t = 40$.

4. 2+1D cKdV-type equation

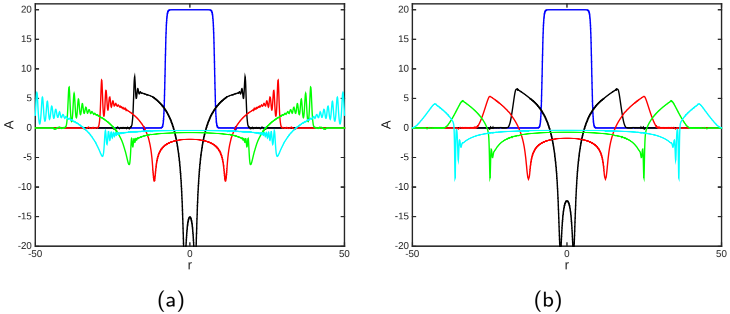


Figure: (a) Surface DSWs in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$, $Q = 20$ at $t = 10$ (black), 20 (red), 30 (green) and 40 (cyan). (b) Interfacial DSWs in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$, $Q = 20$ at $t = 50$ (black), 100 (red), 150 (green) and 200 (cyan).

Conclusions

- Reduced mathematical models have interesting mathematical properties (in this talk, models of the KdV type and their extensions, but there are several other models). Current work: zero-mass contradiction.
- Reduced models are powerful additional tools to direct numerical simulations of the full equations.
- Efficient semi-analytical numerical methods can be developed using our knowledge about these models.
- It is vital to study the analytical properties of the reduced mathematical models.