Short review of results on Isomonodromy Deformations and Applications

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With G. Cotti and B. Dubrovin, Duke Math.J. 168(2019), + 3 papers on arXiv

We consider a linear $n \times n$ differential system:

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, ..., u_n),$$

$A(u)$ *holomorphic* of $u = (u_1, ..., u_n)$ in a polydisc $\mathbb{D}$ (specified later).

Some eigenvalues may coalesce

$$u_j - u_k \to 0 \qquad \text{for some } j \neq k$$

along a coalescence locus in $\mathbb{D}$.

The purpose of the talk is to extend the isomonodromy deformation theory to include the case of coalescing eigenvalues.

This is a non-admissible deformation [Jimbo-Miwa-Ueno, Fokas-Its-Kapaev-Novokshenov].

System

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, ..., u_n),$$

is important in several respects.

- It is related to Fuchsian systems by Laplace transform [Balser-Jurkat-Lutz] Coalescence of eigenvalues $\longleftrightarrow$ coalescence of Fuchsian singularities.
- $n = 3$. Isomonodromy deformation equations for $A(u)$ (if of specific form) are equivalent to the **Painlevé VI equation**.
- **Analytic theory of semisimple Dubrovin-Frobenius Manifolds** [B. Dubrovin]

Dubrovin-Frobenius Manifold: An analytic manifold M of complex dimension n, with a flat metric $< \cdot, \cdot >$. Any tangent space T_pM is a Frobenius algebra (commutative, associative product $\circ$, unity e).

Structure constant of multiplication $\circ$ are the derivatives of "potential" F satisfying the *WDVV* equations of associativity of $2D$-topological field theory:

$$\partial_\alpha \partial_\beta \partial_\gamma F \; \eta^{\gamma\rho} \; \partial_\rho \partial_\mu \partial_\nu F \; = \; \text{the same with exchange } \alpha \leftrightarrow \nu.$$

$$(\eta^{\gamma\rho}) = (< \cdot, \cdot >)^{-1}, \quad \partial_\alpha := \frac{\partial}{\partial t^\alpha}, \quad \textit{flat coordinates } t = (t^1, ..., t^n).$$

Examples

- Singularity theory: M is space of versal deformation of simple singularities.

Example: $M = \{f(x; a) = x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\}$

- Orbit spaces *Example*: $M = \mathbb{C}^n/W$, $W =$ Coxeter group.

- Integrable hierarchies.

- Quantum Cohomology $QH^\bullet(X) :=$ t-deformation of the classical cohomology $H^\bullet(X, \mathbb{C})$ of a smooth projective variety X. $F(t)$ generating function of Gromov-Witten invariants of genus zero.

Important characterisation [Dubrovin]. *A manifold M is Frobenius if and only if there is on M a certain family of flat connections depending on parameter $z \in \mathbb{C}$.*

If $(T_pM, \cdot)$ is semisimple (no nilpotents) on open-dense subset of M, there are local canonical coordinates $(u_1, ..., u_n)$, $u_j \neq u_k$ for all $j \neq k$. Fatness is equivalent to

$$\begin{cases} \frac{\partial Y}{\partial z} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, & A \text{ skew-symmetric,} \\ \text{plus Isomonodromy Deformation Equations.} \end{cases}$$

Important consequence: Monodromy data as local moduli

Locally constant monodromy data parameterise local charts of the Frobenius manifold. From one chart to another through explicit action of braid group on data.

It is important to know monodromy data if we want to study the global structure of the manifold (analytic continuation of Frobenius structure).

They may have also an enumerative-geometric significance.

Back to our problem: compute fundamental solutions and monodormy data of an isomonodromic

$$(DS) \qquad \frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, ..., u_n),$$

$A(u)$ holomorphic in polydisc $\mathbb{D}$

1. $\mathbb{D}(u^0) := \{u \in \mathbb{C}^n \mid \max_j |u_j - u_j^0| < \epsilon\}$; center u^0, $\underline{u_j^0 \neq u_k^0 \ \forall j \neq k}$.

   No coalescence of eigenvalues in the polydisc.

   Admissible deformations [Jimbo-Miwa-Ueno]: just compute the data of

$$\frac{dY}{dz} = \left(\Lambda(u^0) + \frac{A(u^0)}{z}\right) Y.$$

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$A(u)$ holomorphic in polydisc $\mathbb{D}$

2. $\mathbb{D}(u^c)$, center u^c, $\underline{u_j^c = u_k^c \text{ for some } j \neq k}$.

Non-admissible deformations. We have *coalescence of eigenvalues of* Λ at a

$$\text{coalescence locus} \quad \Delta := \mathbb{D}(u^c) \cap \left(\bigcup_{j \neq k} \{u_j - u_k = 0\}\right).$$

Question: Can we define and compute data <u>for the whole $\mathbb{D}(u^c)$</u> starting from

$$(DS|_{u^c}) \qquad \frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y \qquad ??$$

Several problems (Δ branching locus, Asymptotics expansions diverge...)

We need to extend isomonodromy deformation theory at coalescence points despite problems, because there are cases when we expect $A(u)$ to be holomorphically defined at coalescence points:

- **Painlevé transcendents** remaining holomorphic at a fixed singularity

$$\text{Singularities } t = 0, 1, \infty, \qquad t = \frac{u_2 - u_1}{u_3 - u_1} \quad \Longleftrightarrow \quad \begin{array}{c}\text{coalescence of eigenvalues} \\ u_j - u_k \to 0\end{array}$$

Can we compute the monodromy data starting from the system at $u = u^c$?
This is a computational approach of monodromy data different from the one of Jimbo 1982.

- **Frobenius manifolds** remaining semisimple at a coalescence point $u = u^c$ [Cotti-Dubrovin-Guzzetti].

For some important Frobenius manifolds the **manifold structure is explicitly known only at coalescence points**!
Example: Quantum cohomology of *almost all* Grassmannians $G(k, r)$ [Cotti '16].

Can we **compute monodromy data on a chart** from the **only knowledge of $A(u^c)$ at just one point**?

Main motivation !

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Can we **compute monodromy data on a chart** from the **only knowledge of $A(u^c)$ at just one point**?

Main motivation !

• Review of admissible isomonodormy deformations with no coalescence in $\mathbb{D}(u^0)$, $u_j^0 \neq u_k^0$ for all $j \neq k$.

Assume a Frobenius integrable Pfaffian system is given:

$$(PS) \quad dY = \omega Y, \qquad \text{integrability: } d\omega = \omega \wedge \omega$$

where

$$\omega := \Big(\Lambda(u) + \frac{A(u)}{z}\Big)dz + \sum_{j=1}^{n} \omega_j(z,u)du_j, \quad \omega_j(z,u) \text{ holomorphic on } \mathbb{C} \times \mathbb{D}(u^0)$$

Proposition [Yoshida-Takano '76]: *There is a "weakly isomonodromic" fundamental matrix solution in Levelt form at $z = 0$*

$$Y(z,u) = G(u)\Big(I + \sum_{j=1}^{\infty} \Psi_j(u) z^j\Big) z^D z^L \quad \textit{uniformly convergent in } \mathbb{D}(u^0),$$

$$\underline{dL = dD = 0}.$$

- *D diagonal and integer.*

$$Y(ze^{2\pi i}, u) = Y(z,u)M, \quad M = e^{2\pi i L}, \quad \underline{dM = 0}.$$

- *Jordan form of A is $J = D + \lim_{z\to 0} z^D L z^{-D}$, so that $\underline{dJ = 0}$ (isospectrality);*

- $dA = \left[\sum_{j=1}^{n} \omega_j(0,u)du_j, A\right]$, $dG = (\sum_{j=1}^{n} \omega_j(0,u)du_j)\, G$, *and* $G(u)$, $G(u)^{-1}$ *and* $\Psi_j(u)$*'s holomorphic in* $\mathbb{D}(u^0)$

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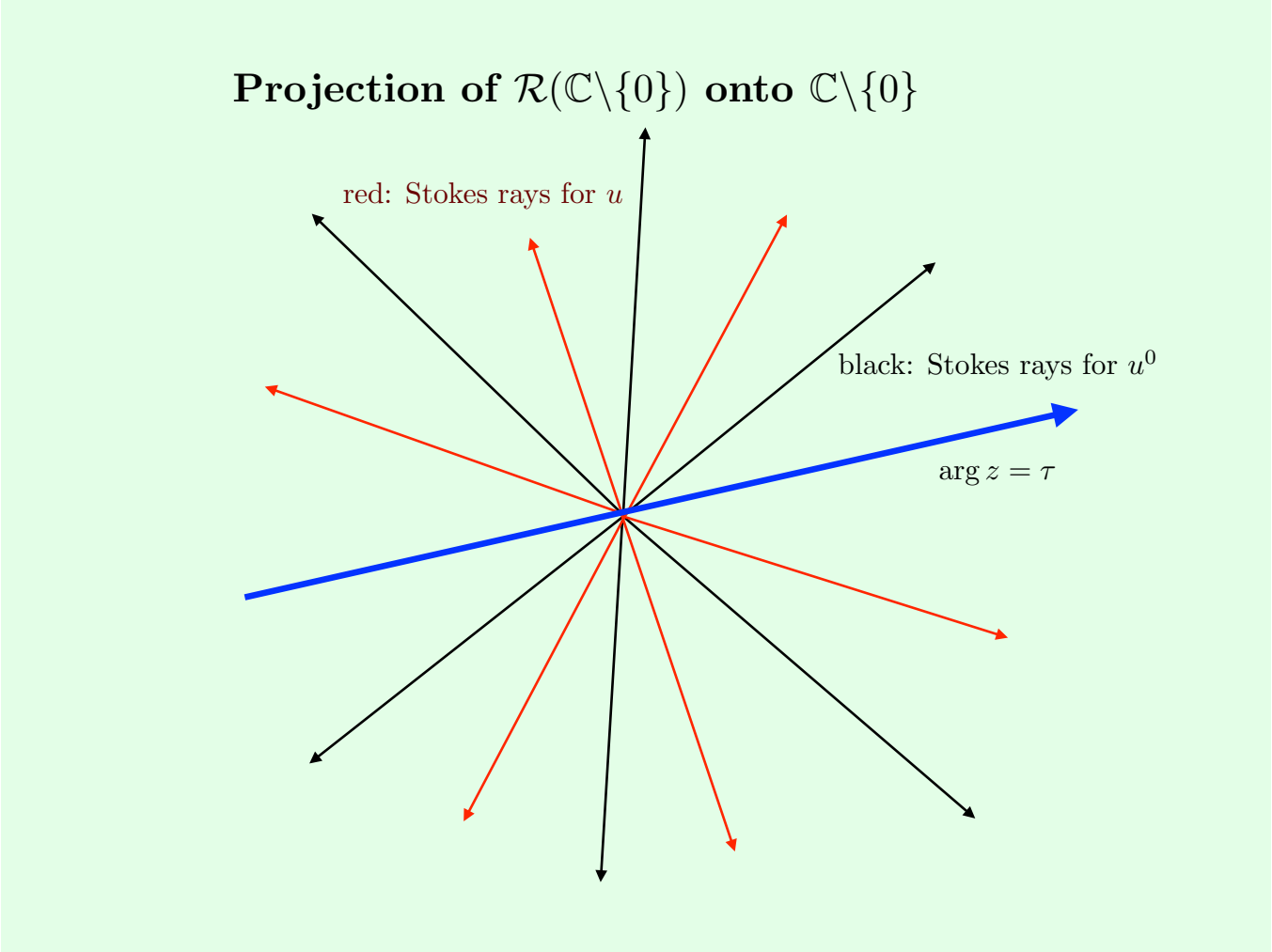
Definition: Stokes rays of $\Lambda(u)$: $z \in \mathcal{R}(\mathbb{C}\backslash\{0\})$ such that:

$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0, \quad 1 \leq j \neq k \leq n.$$

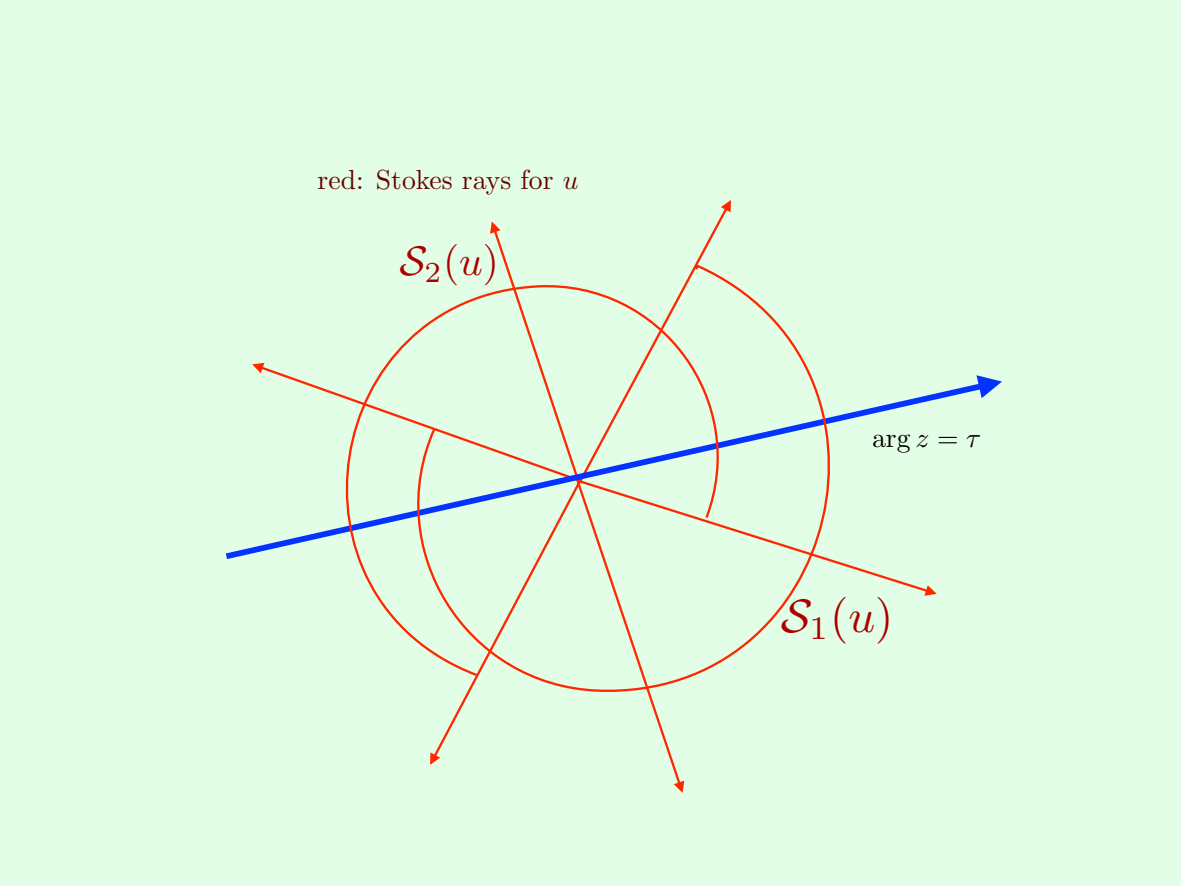
Definition: *Admissible direction τ at u^0*: if $\arg z = \tau$ does not coincide with any of the Stokes rays of $\Lambda(u^0)$.



Sectors $\mathcal{S}_r(u)$ in universal covering of $\mathbb{C}\backslash\{0\}$ extending

$$\tau - (r-2)\pi < \arg z < \tau - (r-1)\pi, \quad r \in \mathbb{Z},$$

to the nearest Stokes rays of $\Lambda(u)$: central angle <u>greater than π</u>.



For u varying in $\mathbb{D}(u^0)$ (sufficiently small), the Stokes rays of $\Lambda(u)$ rotate without crossing $\arg z = \tau \mod \pi$. Then any sector

$$\mathcal{S}_r(\mathbb{D}(u^0)) := \bigcap_{u \in \mathbb{D}(u^0)} \mathcal{S}_r(u), \quad r \in \mathbb{Z}, \qquad \text{has central angle } \underline{\text{greater than } \pi}.$$

Unique formal fundamental matrix solution at $z = \infty$:

$$Y_F(z,u) = \; F(z,u)\; z^{B(u)}\; \exp\{z\Lambda(u)\}; \quad B(u) := \mathrm{diag}(A_{11}(u), ..., A_{nn}(u)).$$

$$F(z,u) = I + \sum_{k=1}^{\infty} F_k(u) z^{-k}, \qquad \text{holomorphic } F_k(u)$$

Proposition: [Sibuya] *There exist unique fundamental matrix solutions*

$$Y_r(z,u) = \widehat{Y}_r(z,u) z^{B(u)} \exp\{z\Lambda(u)\}, \quad r \in \mathbb{Z},$$

with uniform in $u \in \mathbb{D}(u^0)$ *asymptotic expansion*

$$\widehat{Y}_r(z,u) \sim F(z,u), \quad z \to \infty \text{ in } \mathcal{S}_r(\mathbb{D}(u^0)).$$

IMPORTANT: the Proposition allow to well define Stokes matrices in $\mathbb{D}(u^0)$.

$$Y_{r+1}(z,u) = Y_r(z,u)\; S_r(u) \qquad \text{Stokes matrices}$$

$$Y_r(z,u) = Y(z,u)\; C_r(u) \qquad \text{Connection matrix}$$

- The integrability $d\omega = \omega \wedge \omega$ implies $dB = 0$.

Definition: The system (DS) is strongly isomonodromic if all $Y_r(z,u)$ satisfy $dY_r = \omega Y_r$.

Proposition:[D.G. - SIGMA] *Strongly isomonodromic if and only if*

$$\underline{dS_0 = dS_1 = dC_0 = 0}.$$

In this case

$$\omega_j(z,u) = zE_j + [F_1(u), E_j], \qquad j = 1, ..., n.$$

Important Remark: All the other S_r, C_r ($r \in \mathbb{Z}$) and the monodromy matrices of fundamental solutions (and spectrum of A) are completely determined by the

(constant) essential monodromy data $\quad S_0, S_1, B, L, C_0, D$

Conclusion for admissible deformations (no coalescence)

- For strong isomonodromy deformations, the essential mondromy data **are well defined in $\mathbb{D}(u^0)$ and constant** .
- In particular, **Stokes phenomenon and Stokes matrices are well defined** for u varying in $\mathbb{D}(u^0)$.

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• Non-admissible deformations. Coalescence of Eigenvalues

$$u_i^c - u_j^c = 0 \text{ for some } i \neq j, \quad \Delta := \mathbb{D}(u^c) \cap \Big(\bigcup_{k \neq l} \{u_k - u_l = 0\} \Big).$$

Definition: Stokes rays of $\Lambda(u)$: As before, $z \in \mathcal{R}(\mathbb{C} \backslash \{0\})$ such that:

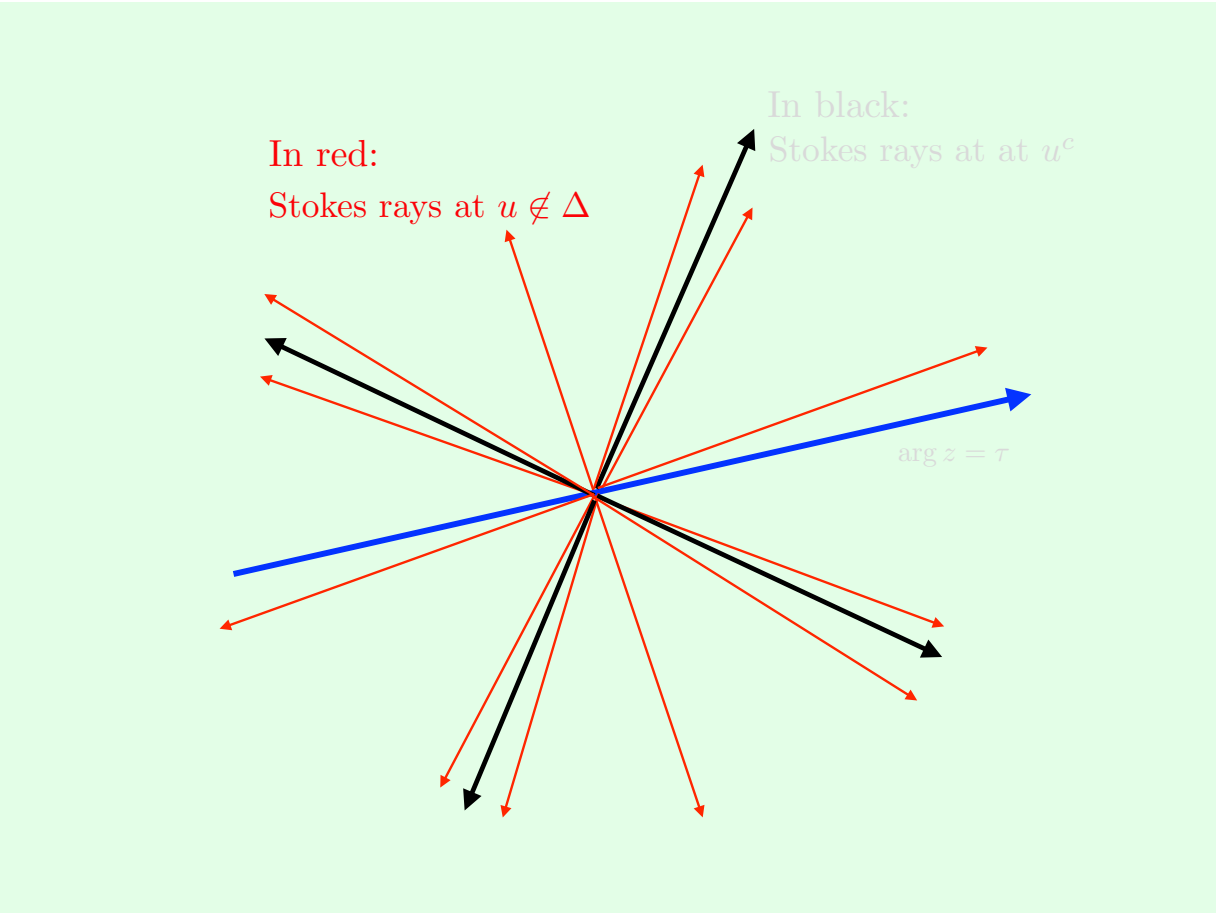
$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0, \quad 1 \leq j \neq k \leq n.$$

Admissible direction τ at u^c if $\arg z = \tau$ does not coincide with any of the Stokes rays of $\Lambda(u^c)$.

This time,

there are more rays

at $u \notin \Delta$ then at $u \in \Delta$.



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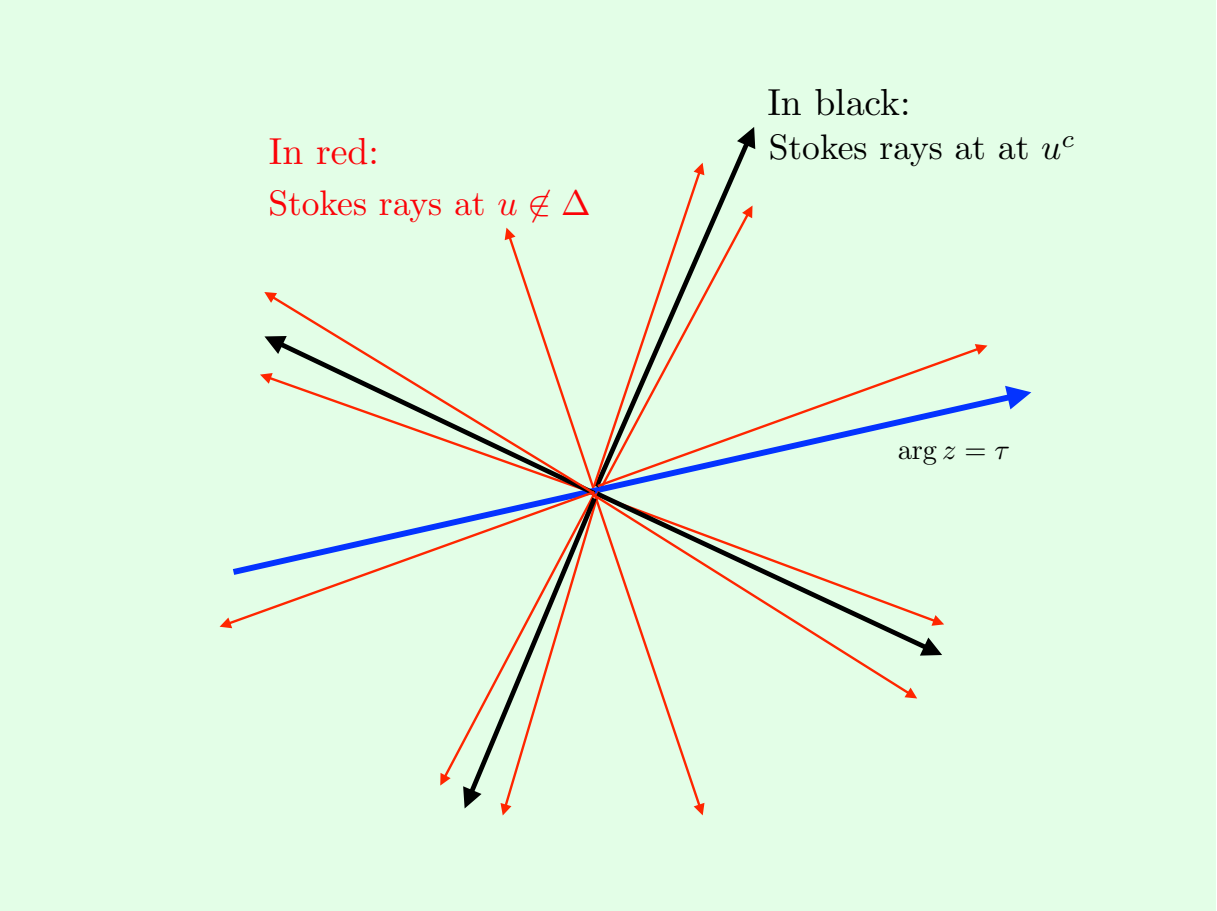
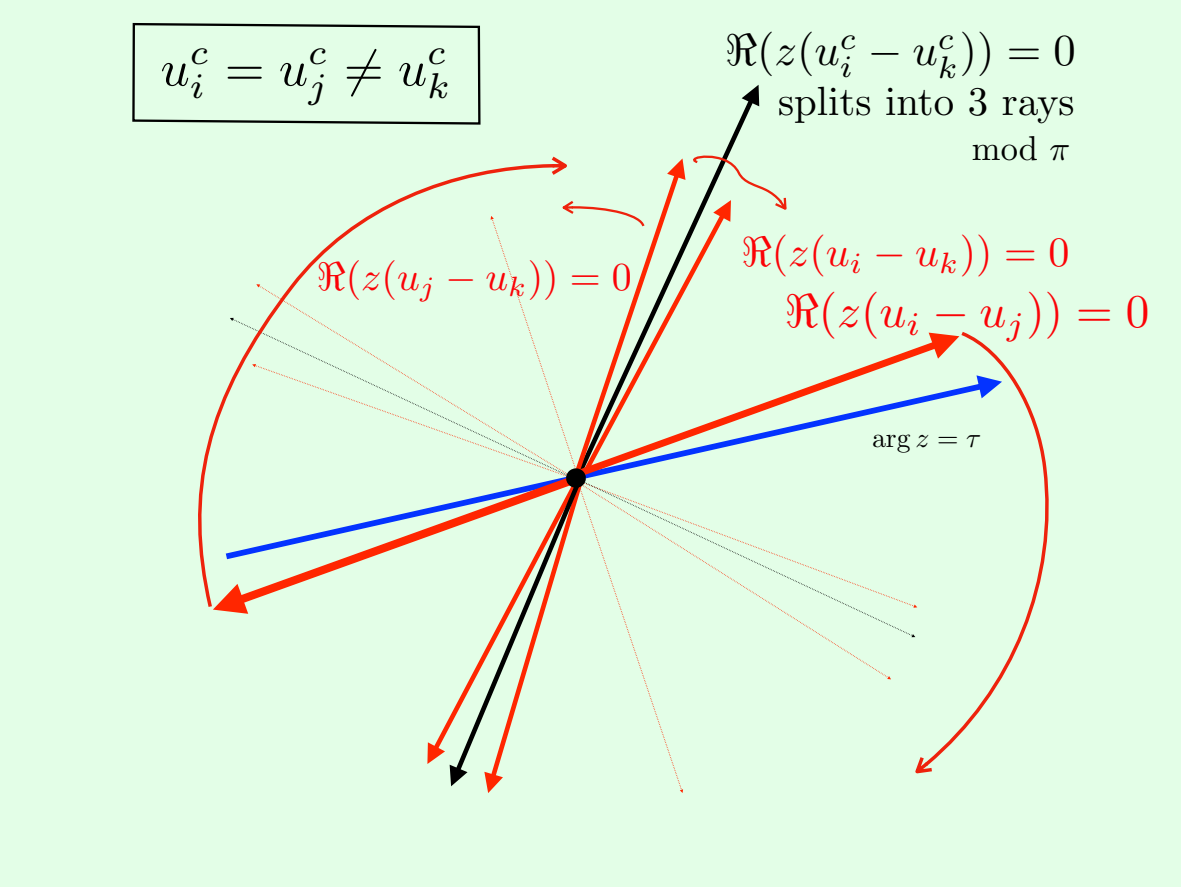


Each ray of $\Lambda(u^c)$ splits into 3 rays when u moves away from Δ.



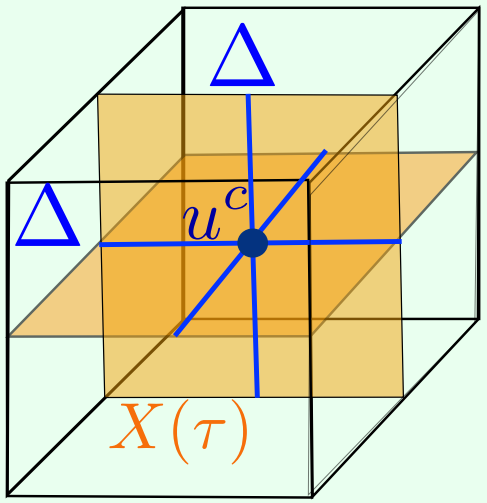
• If $(u_i - u_j) \mapsto (u_i - u_j)e^{2\pi i}$ the ray $\Re(z(u_i - u_j)) = 0$ crosses once the admissible line $\arg z = \tau \mod 2\pi$, and once $\arg z = \tau - \pi \mod 2\pi$, making a complete 2π rotation.

Crossing occurs for $u \in X(\tau)$, **"crossing" locus** of points u in $\mathbb{D}(u^c)$ such that there is a Stokes ray of $\Lambda(u)$ with direction τ (mod π).

Proposition[Cotti-Dubrovin-Guzzetti]. **Cell decomposition of $\mathbb{D}(u^c)$:** *Each connected component of*

$$\mathbb{D}(u^c)\backslash(\Delta \cup X(\tau))$$

is simpliy connected and homeomorphic to a ball, namely it is a cell, called τ-cell.

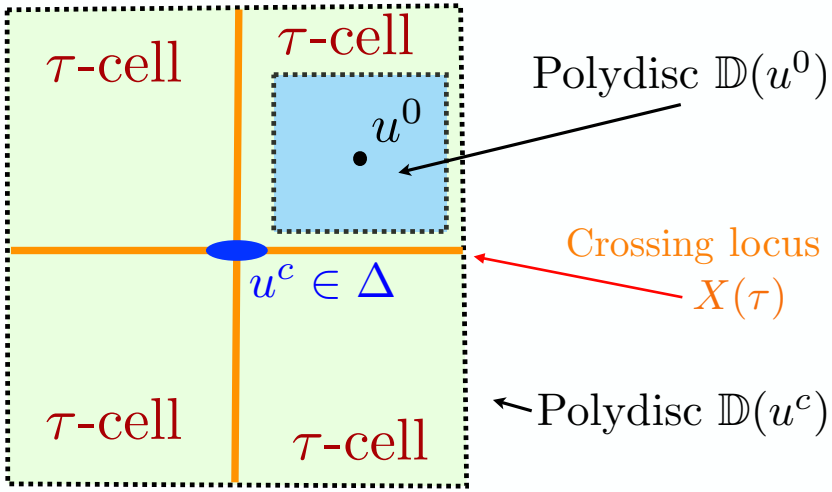


Polidisc $\mathbb{D}(u^c)$

Coalescence locus Δ

Crossing locus $X(\tau)$

We can apply the theory of admissible deformations in $\mathbb{D}(u^0)$ *all contained* **in a** τ**-cell**.



In $\mathbb{D}(u^0)$ are well defined $Y(z,u)$ and $Y_r(z,u)$ and admissible isomonodromy deformations:

$$dY = \omega Y \quad \equiv \quad \left\{ \left(\Lambda + \frac{A}{z} \right) dz + \sum_{k=1}^{n} \omega_k \; du_k \right\} Y, \qquad \omega_k(z,u) = zE_k + [F_1(u), E_k].$$

! Essential monodromy data S_0, S_1, B, C_0, L, D defined and constant on $\mathbb{D}(u^0)$

From formal computation of $F_k(u)$:

$$\omega_k(z,u) = zE_k + \left(\frac{A_{ij}(u)(\delta_{ik}-\delta_{jk})}{u_i - u_j}\right)_{i,j=1}^{n}$$

holomorphic in $\mathbb{D}(u^c)\backslash\Delta$.
Poles at Δ.

Problems arise:

1. Δ may be as branching locus for $Y(z,u)$ and $Y_r(z,u)$'s.
2. The coeff. $F_k(u)$ diverge at Δ:

$$(F_1)_{ij} = \frac{A_{ij}(u)}{u_j - u_i}, \quad i \neq j;$$

$$(F_k)_{ij} = \frac{1}{u_j - u_i}\left\{\Big(A_{ii} - A_{jj} + k - 1\Big)(F_{k-1})_{ij} + \sum_{p\neq i} A_{ip}(F_{k-1})_{pj}\right\}, \quad i \neq j.$$

3. Existence of the crossing locus $X(\tau)$: it is not possible to define Stokes phenomenon and matrices on the whole $\mathbb{D}(u^c)$:

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Sufficient condition $(**)$ to overcome problems and extend isomonodromy deformations to the whole $\mathbb{D}(u^c)$

Lemma: $\omega_k(z,u) = zE_k + \left(\frac{A_{ij}(u)(\delta_{ik}-\delta_{jk})}{u_i-u_j}\right)_{i,j=1}^{n}$ *is holomorphic on the whole* $\mathbb{D}(u^c)$ *if and only if*

$$(A_1)_{ij} = \mathcal{O}(u_i - u_j) \quad \text{as } u_i - u_j \to 0 \text{ at } \Delta. \qquad (**)$$

- **Problem 1** is solved: $Y(z,u)$, $Y_r(z,u)$, $r \in \mathbb{Z}$ extend holomorphically on $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$. .

- **Problem 2** is solved by the following

  Proposition: *If* $(**)$ *holds* $\Longrightarrow$ *all the* $F_k(u)$, $k \geq 1$, *are holomorphic on* $\mathbb{D}(u^c)$.

- **Problem 3** with Stokes phenomenon requires work !! It is solved by the following:

Sufficient condition $(**)$ to overcome problems and extend isomonodromy deformations to the whole $\mathbb{D}(u^c)$

Lemma: $\omega_k(z,u) = zE_k + \left(\frac{A_{ij}(u)(\delta_{ik}-\delta_{jk})}{u_i-u_j}\right)_{i,j=1}^{n}$ *is holomorphic on the whole* $\mathbb{D}(u^c)$ *if and only if*

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- **Problem 2** is solved by the following

  Proposition: *If* $(**)$ *holds* $\Longrightarrow$ *all the* $F_k(u)$, $k \geq 1$, *are holomorphic on* $\mathbb{D}(u^c)$.

- **Problem 3** with Stokes phenomenon requires work !! It is solved by the following:

Fundamental Theorem:[Cotti-Dubrovin-Guzzetti] *Let*

$$(DS) \qquad \frac{dY}{dz} = (\Lambda(u) + \frac{A(u)}{z})Y$$

be strongly isomonodromic with $A(u)$ holomorphic on $\mathbb{D}(u^c)$. Then:

1) *Asymptotics $\widehat{Y}_r(z,u) \sim F(z,u)$ is well defined in $\mathbb{D}(u^c)$ for $z \to \infty$ in wide sectors $\widehat{S}_r$*
This is related to the vanishing

$$(S_r)_{ij} = (S_r)_{ji} = 0 \qquad \text{for } i \neq j \text{ such that } u_i^c = u_j^c.$$

2) *The* **essential monodromy data**, *initially defined on $\mathbb{D}(u^0)$ contained in a τ-cell, are* **well defined and constant on the whole** $\mathbb{D}(u^c)$. *They can be computed as*

$$S_0 = \mathring{S}_0, \quad S_1 = \mathring{S}_1, \quad L = \mathring{L}, \quad C_0 = \mathring{C}_0, \quad D = \mathring{D},$$

where

- $\mathring{S}_0$, $\mathring{S}_1$, $\mathring{L}$, $\mathring{C}_0$, $\mathring{D}$ *are essential monodromy data of*

$$(DS|_{u^c}) \qquad \frac{dY}{dz} = (\Lambda(u^c) + \frac{A(u^c)}{z})Y$$

- *If $A_{ii} - A_{jj} \notin \mathbb{Z}\backslash\{0\}$, there is no ambiguity in choosing the essential data of $(DS|_{u^c})$.*

Important conclusion:

• **This result justifies computation of essential monodromy data on the whole $\mathbb{D}(u^c)$ starting from the system at u^c:**
In order to compute the essential monodromy data of

$$(DS) \qquad \frac{dY}{dz} = (\Lambda(u) + \frac{A(u)}{z})Y$$

it suffices to compute the essential monodromy data of

$$(DS|_{u^c}) \qquad \frac{dY}{dz} = (\Lambda(u^c) + \frac{A(u^c)}{z})Y.$$

• It gives **efficient tool for possibly explicit computations**, because $(DS|_{u^c})$ is **simpler** that (DS). Indeed, $A(u^c)$ has some vanishinbg entries:

$$\left(A(u^c)\right)_{ij} = 0 \text{ whenever} u_i^c = u_j^c$$

• In order to do computations, **we just need to know $A(u^c)$**.
Application to those Frobenius manifolds whose structure is know only at a coalescence point. This occurs for example in Quantum Cohomology.

(Incomplete) list of bibliographic references for *resonant irregular singularities*, their unfolding and confluence of regular singularities.

R. Garnier (1919), J. P. Ramis (1989), A. Duval (1991),

A. A. Bolibruch (1994-8), D. G. Babbitt & V. S. Varadarajan (1985),

R. Schäfke (2001), A. A. Glutsyuk (1999-2004),

C. Lambert & C. Rousseau (2012),

J. Hurtubise & C. Lambert & C. Rousseau (2014), M. Klimes (2013-16),

M. Bertola & M. Y. Mo (2005),

M. V. Fedoryuk (1992), Y. P. Bibilo (2012),

P. P. Boalch (2012), T. Bridgeland & V. Toledano Laredo (2013),

etc...

!! The case we have discussed of $\Lambda(u)$ *remaining diagonal* at coalescing eigenvalues (no change of Jordan type) seems to be missing from the existing literature.

Applications tp PVI.

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right]\left(\frac{dy}{dt}\right)^2 - \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right]\frac{dy}{dt}$$
$$+\frac{1}{2}\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[(2\mu-1)^2 + \frac{t(t-1)}{(y-t)^2}\right], \qquad \text{(PVI) with one parameter}$$

For $n = 3$, for suitable $A(u)$ skew-symmetric and algebraic function of $y(t), dy/dt, \int^t y$, and $t = \frac{u_2-u_1}{u_3-u_1}$, (PVI) is equivalent to the isomonodromy deformation equations

$$dA = \left[\sum_{j=1}^{n} \omega_j(0,u)du_j, A\right]$$

By $z \mapsto z(u_3 - u_1)$ we reduce the system to

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A(t)}{z}\right] Y, \qquad t = \frac{u_2 - u_1}{u_3 - u_1}.$$

For $t \to 0$, **coalescence** of first and second eigenvalue ($u_1(t) - u_2(t) \to 0$).

Consider $\mu = -1/4$ and the A_3 algebraic solution [Dubrovin-Mazzocco (2000)], with **holomorphic branch** (Taylor series) **at** $t = 0$ **(**$\leftrightarrow$ **at** u^c**)**:

$$y(t) = \frac{1}{2}t + \frac{13}{32}t^2 + \frac{13}{64}t^3 + \frac{201}{4096}t^4 - \frac{229}{8192}t^5 + \cdots$$

$$A_{23}(t) = i\sqrt{2}\left(\frac{1}{8} - \frac{1}{256}t + \cdots\right), \quad A_{13}(t) = -i\sqrt{2}\left(\frac{1}{8} + \frac{1}{256}t + \cdots\right),$$

$$A_{12}(t) = -\frac{1}{32}t - \frac{1}{64}t^2 + \ldots \longrightarrow 0, \quad \text{corresponding to coalescence } 1 \leftrightarrow 2$$

- System $(DS|_{u^c})$:

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A(0)}{z}\right] Y, \qquad A(0) = \begin{pmatrix} 0 & 0 & -i\sqrt{2}/8 \\ 0 & 0 & i\sqrt{2}/8 \\ i\sqrt{2}/8 & -i\sqrt{2}/8 & 0 \end{pmatrix},$$

is explicitly solvable (equivalent to a Bessel equation of order 2, plus one quadrature)!

- The system (DS) at $t \neq 0$ is not !!

We can explicitly compute all monodromy data at u^c, i.e. at $t = 0$. For example, relatively to the sectors

$$\mathcal{S}_0 = S\Big(-\frac{3\pi}{2}, \frac{\pi}{2}\Big), \qquad \mathcal{S}_1 = S\Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big), \qquad \mathcal{S}_2 = S\Big(\frac{\pi}{2}, \frac{5\pi}{2}\Big).$$

we find

$$S_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}. \quad \text{Notice } (S_j)_{12} = (S_j)_{21} = 0.$$

Example of Frobenius manifolds

Proposition: [Cotti-Dubrovin-Guzzetti] *If a Frobenius manifold remains semisimple at a coalescence point $u = u^c$ of the canonical coordinates $u = (u_1, ..., u_n)$, then the coefficients of the flat connection ω are such that:*

- *$A(u)$ is holomorphic at u^c*
- *and $A_{ij}(u^c) \to 0$.*

We can apply previous results: we can compute monodromy data on a chart from the only knowledge of $A(u^c)$.

For some important Frobenius manifolds the manifold structure is explicitly known only at coalescence points u^c.

Example: quantum cohomology of Grassmannians $QH^\bullet(G(k,n))$. For almost all Grassmannians [Cotti '16], we explicitly know the linear system only at some coalescence points. We can apply our theorem.

From previous results, the whole manifold structure can be reconstructed (in principle) from the monodromy data *computed at a coalescence point*.

• *Simplest example:*

For $QH^\bullet(G(2,4))$. $n = 6$.

$$\Lambda(u^c) = 4\sqrt{2} \cdot \mathrm{diag}(-1, -i, 0, 0, i, 1) \qquad \longleftarrow \quad \text{coalescence}$$

$A(u^c)$ is explicitly known and $(A(u^c))_{34} = (A(u^c))_{43} = 0$.

Computations of S_0, S_1, C can be *explicitly* done. Indeed, the system at $u = u^c$ reduces to a generalised hypergeometric equation.
Up to some admissible transformations (including action of braid group) we obtain

$$S^{-1} = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_{34} = S_{43} = 0.$$

We have also computed C explicitly (too long to write here, see our papers).

Explicit computation of monodromy data allows to *verify or refine* conjectures [Dubrovin at ICM 1998, Strasburg 2013; Gamma-conjecture Galkin-Golyshev-Iritani] *prescribing an explicit coincidence between the monodromy data of quantum cohomology of smooth projective varieties and suitable quantities associated with objects of exceptional collections in derived categories of coherent sheaves on these varieties.*

Results of this talk allows to *justify* the theory, which is based on only knowledge at colaescence points.

If true, these conjectures would allow to obtain monodromy data in algebraic way (and then analytic continuation of Frobenius manifolds), avoiding problems of analytic computations.

Let X be a Fano manifold. The Frobenius manifold $QH^\bullet(X)$ is semisimple iff there exists a full exceptional collection $(E_1, ..., E_n)$ in $\mathcal{D}^b(X)$. Moreover:
– the (inverse of the) Stokes matrix $\mathbb{S}$ is equal to the inverse of the Gram matrix of the Euler-Poincaré-Grothendieck product $\chi(E_i, E_j)$;
– the columns of the connection matrix C coincide with the components of the forms

$$\frac{i^{\overline{d}}}{2\pi^{d/2}}\Gamma^-(X) \cup e^{-i\pi c_1(X)} \cup Ch(E_j),$$

where

$$\Gamma^-(X) = \prod_\ell \Gamma(1-\alpha_\ell), \quad \alpha_\ell' s \text{ Chern roots of } TX.$$

$d = \dim(X)$, $\overline{d} = \dim(X) \bmod 2$.

For Projective spaces [Guzzetti 1999].
For all Grassmannians [Cotti-Dubrovin-D.G '18], [Galkin-Golyshev-Iritani '16].

Thank you!