Monodromy dependence of Painlevé tau functions

Oleg Lisovyy
Institut Denis-Poisson, Université de Tours, France

Dijon, 02/09/2019

collaborations with
A. Its, A. Prokhorov, M. Cafasso, P. Gavrylenko

Example 1: Sine kernel

Introduce

$$\tau(t) = \det\left(\mathbf{1} - K\big|_{(0,t)}\right), \qquad K(x,y) = \frac{\sin\frac{x-y}{2}}{\pi(x-y)}.$$

- $\tau(t)$ is a Painlevé V tau function: $\zeta(t) = t\frac{d}{dt}\ln\tau(t)$ satisfies

$$\left(t\zeta''\right)^2 + \left(t\zeta' - \zeta\right)\left(t\zeta' - \zeta + 4\zeta'^2\right) = 0. \qquad (\zeta\text{-PV})$$

- Asymptotics:

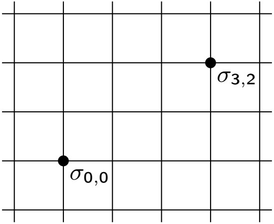
$$\begin{aligned} \tau(t\to 0) &= 1 - \frac{t}{2\pi} + \frac{t^4}{576\pi^2} + O\left(t^6\right), \\ \tau(t\to\infty) &= \tau_{\text{sine}} \cdot t^{-\frac{1}{4}} e^{-\frac{t^2}{32}} \left[1 + \frac{1}{2t^2} + O\left(t^{-4}\right)\right] \end{aligned}$$

Conjecture [Dyson, '76]:

$$\tau_{\rm sine} = 2^{\frac{7}{12}} e^{3\zeta'(-1)} = \sqrt{2}\, G\left(\tfrac{1}{2}\right) G\left(\tfrac{3}{2}\right).$$

- Barnes G-function is essentially defined by the recurrence relation $G(z+1) = \Gamma(z)\, G(z)$; it has integral and product representations, etc
- proved in [Ehrhardt, '04; Krasovsky, '04; Deift, Its, Krasovsky, Zhou, '06]

Example 2: 2D Ising model



- Nearest-neighbor interaction

$$\mathcal{H}[\sigma] = -J \sum_{i,j} \sigma_{i,j} \left(\sigma_{i+1,j} + \sigma_{i,j+1}\right),$$

  of spin variables $\sigma_{i,j} = \pm 1$.

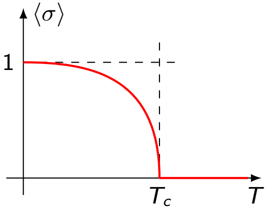
- Spin-spin correlation function:

$$\left\langle \sigma_{0,0}\sigma_{r_x,r_y} \right\rangle = \frac{\sum_{[\sigma]} \sigma_{0,0}\sigma_{r_x,r_y} e^{-\beta H[\sigma]}}{\sum_{[\sigma]} e^{-\beta H[\sigma]}}$$

- Phase transition at $s \equiv \sinh 2\beta J = 1$
- Spontaneous magnetization [Yang, '52]:

$$\langle \sigma \rangle = \begin{cases} \left(1 - s^{-4}\right)^{\frac{1}{8}}, & s > 1, \\ 0, & s < 1, \end{cases}$$

- Correlation length $\Lambda \sim 2^{-\frac{1}{2}} |s-1|^{-1}$



Scaling limit of the 2-point functions is described by

$$T \to T_c, \qquad \Lambda \to \infty, \qquad R = \sqrt{r_x^2 + r_y^2} \to \infty, \qquad \frac{R}{\Lambda} \to t,$$

$$\left\langle \sigma_{0,0} \sigma_{r_x, r_y} \right\rangle \to \Lambda^{-\frac{1}{4}} 2^{\frac{3}{8}} \tau_{\pm}(t), \qquad T \gtrless T_c,$$

Scaled correlations can be written in terms of Fredholm determinants & related to Painlevé functions [McCoy, Tracy, '73; Wu, McCoy, Tracy, Barouch, '76] (PV, PIII(D_6), PIII(D_8)). In particular, both $\zeta_{\pm} = t \frac{d}{dt} \ln \tau_{\pm}(t)$ satisfy

$$\left(t\zeta''\right)^2 = 4\left(\zeta - t\zeta'\right)^2 + 4\left(\zeta'\right)^2\left(\zeta - t\zeta'\right) + \left(\zeta'\right)^2 \qquad (\zeta\text{-PV})$$

- Long distances (form factor expansions):

$$\tau_{+}(t \to \infty) \sim \frac{e^{-t}}{\sqrt{2\pi t}}, \qquad \tau_{-}(t \to \infty) \sim 1.$$

- Short distances (conformal limit):

$$\tau_{\pm}(t \to 0) \sim \tau_{\text{Ising}} \cdot (2t)^{-\frac{1}{4}}.$$

Theorem [Tracy, '91; also, Bothner, '17]:

$$\tau_{\mathrm{Ising}} = 2^{\frac{1}{12}} e^{3\zeta'(-1)} = G\left(\tfrac{1}{2}\right) G\left(\tfrac{3}{2}\right).$$

- constant factors in the asymptotics of tau functions (connection constants) were computed for many other (special!) Fredholm determinant solutions of Painlevé equations:
  - correlator of twist fields in sine-Gordon field theory at the free-fermion point, Airy kernel, Bessel kernel, ...

Summary:

- Ising scaled correlator = specific PV tau function
- it has Fredholm determinant representation
- its asymptotics at one singular point ($t \to \infty$) is "easy"
- the asymptotics at the other singular point ($t \to 0$) is difficult (connection constant)

Questions:

- Can the general solutions of Painlevé equations be written as Fredholm determinants?
- How to compute the relevant connection constants?

In this talk, I will mainly focus on the Painlevé VI case.

Painlevé functions arise in the solution of the inverse monodromy problem for linear ODEs with rational coefficients.

Example: Fuchsian system

$$\partial_z \Phi = \Phi \left(\frac{A_0}{z} + \frac{A_1}{z-1} \right), \qquad A_0, A_1 \in \mathfrak{sl}_2 \left(\mathbb{C} \right).$$

- 3 regular singularities $z = 0, 1, \infty$ on $\mathbb{CP}^1$
- analytic continuation of the fundamental matrix along closed loops, $\gamma : \Phi \mapsto M_{[\gamma]} \Phi$, generates a monodromy representation

$$\rho : \pi_1 \left(\mathbb{CP}^1 \backslash \{0, 1, \infty\} \right) \to SL_2(\mathbb{C})$$

- different points of the space of Fuchsian systems

$$\mathcal{A} = \left\{ (A_0, A_1, A_\infty) \in \mathfrak{sl}_2 \left(\mathbb{C} \right)^3 : A_0 + A_1 + A_\infty = 0 \right\} / \sim$$

  are mapped to the space of monodromy data

$$\mathcal{M} = \left\{ (M_0, M_1, M_\infty) \in SL_2 \left(\mathbb{C} \right)^3 : M_0 M_1 M_\infty = \mathbf{1} \right\} / \sim$$

- denote $\mathrm{Sp}\, A_\nu = \{\theta_\nu, -\theta_\nu\}$, then $\mathrm{Sp}\, M_\nu = \left\{ e^{2\pi i \theta_\nu}, e^{-2\pi i \theta_\nu} \right\}$

Fuchsian system:

$$\partial_z \Phi = \Phi \left(\frac{A_0}{z} + \frac{A_1}{z-1} \right)$$

Problem A: Find (the conjugacy class of) the residues (A_0, A_1, A_∞) in terms of (the conjugacy class of) monodromy (M_0, M_1, M_∞)

- solved in elementary functions:

$$A_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{pmatrix}, \quad A_1 = \frac{1}{2\theta_0} \begin{pmatrix} \theta_\infty^2 - \theta_1^2 - \theta_0^2 & (\theta_1+\theta_0)^2 - \theta_\infty^2 \\ \theta_\infty^2 - (\theta_1 - \theta_0)^2 & \theta_0^2 + \theta_1^2 - \theta_\infty^2 \end{pmatrix},$$

- similar (trigonometric) formulas for (M_0, M_1, M_∞).
- 3-point rank 2 Fuchsian system are rigid: both conjugacy classes are completely determined by local monodromy, $\dim \mathcal{A}_\theta = \dim \mathcal{M}_\theta = 0$.

Problem B: Find $\Phi(z)$ in terms of monodromy (M_0, M_1, M_∞)

- solution involves Gauss hypergeometric functions ${}_2F_1(z)$

Add one more Fuchsian singularity:

$$\partial_z \Phi = \Phi \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right), \qquad A_0, A_t, A_1 \in \mathfrak{sl}_2(\mathbb{C})$$

- this system is no longer rigid, but $\dim \mathcal{A}_{\theta} = \dim \mathcal{M}_{\theta} = 2$
- isomonodromic deformation: given monodromy data, it can be expected that appropriate (A_0, A_t, A_1) exist for any fixed t
- Problem A: find (A_0, A_t, A_1) as functions of t and monodromy
- as functions of t, the residue matrices satisfy Schlesinger equations

$$\frac{dA_0}{dt} = \frac{[A_0, A_t]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_1, A_t]}{t-1}, \qquad A_\infty = \text{const}$$

  where $A_\infty := -A_0 - A_t - A_1$.
- these matrix equations (in rank 2) are equivalent to Painlevé VI.

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = (t-1)\,\mathrm{Tr}\,A_0A_t + t\,\mathrm{Tr}\,A_1A_t = t(t-1)\frac{d}{dt}\ln\tau$
- $\tau(t)$ is the Painlevé VI tau function
- Asymptotics:

$$\tau(t\to 0) \simeq \mathcal{N}_0\, t^{\sigma^2-\theta_0^2-\theta_t^2},$$
$$\tau(t\to 1) \simeq \mathcal{N}_1\,(1-t)^{\overline{\sigma}^2-\theta_1^2-\theta_t^2}$$

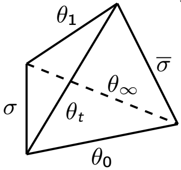
Parameters $\sigma, \overline{\sigma}$ may be thought of as encoding initial conditions. They are explicitly related to monodromy (eigenvalues of M_0M_t and M_1M_t).

Questions:

- express $\tau(t)$ as a Fredholm determinant (constant factor ambiguity!)
- find connection coefficient $\mathcal{N}_1/\mathcal{N}_0$ as function of monodromy.

It turns out $\ln \frac{\mathcal{N}_1}{\mathcal{N}_0}$ coincides (up to an elementary correction) with the complexified volume of the hyperbolic tetrahedron with dihedral angles σ, $\overline{\sigma}$, $\theta_{0,t,1,\infty}$.

$$\ln \frac{\mathcal{N}_1}{\mathcal{N}_0} \sim \mathrm{Vol}\left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right) \sim \ln \prod_{k=1}^{8} \frac{G\left(1+z_k\right)}{G\left(1-z_k\right)}$$

- z_k's are explicit elementary (though complicated) functions of monodromy parameters
- conjecture in [Iorgov, OL, Tykhyy, '13]

There exists a generalization [Jimbo, Miwa, Ueno, '79] of $\tau(t)$ to arbitrary linear systems with rational coefficients (arbitrary number of poles of arbitrary order):

$$d_{\mathcal{T}} \ln \tau = \omega_{JMU}$$

Here ω_{JMU} is a 1-form on the space $\mathcal{T}$ of isomonodromic times explicitly expressed in terms of coefficients of the system.

One of the aims is to extend the Jimbo-Miwa-Ueno differential to the space of monodromy data (idea first appeared in [Bertola, '09]) to be able to capture constant term $\mathcal{N}_1/\mathcal{N}_0$ in the asymptotics killed by the time derivative.

Strategy

1. Introduce a tau function $\tau[J]$ for a Riemann-Hilbert problem (RHP) on a circle with a generic matrix jump

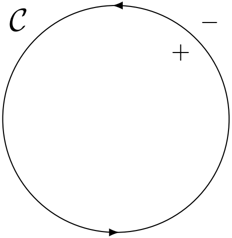
$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}$$

2. Find expressions for derivatives $\partial_s \ln \tau[J]$ with respect to parameters of the jump matrix in terms of $\Psi_\pm(z)$, $\bar{\Psi}_\pm(z)$ [Widom, '76]
3. Reduce 4-point Fuchsian system to solution of a RHP of the above type with explicitly known $\Psi_\pm(z)$ (but unknown $\bar{\Psi}_\pm(z)$ which essentially provide the solution of Problem B/Fuchsian system).
4. Use variational formula with $s = t$ to prove the relation between $\tau[J]$ and Painlevé VI tau function $\tau(t) \Longrightarrow$ explicit solution of Problem A.
5. Use variational formula with $s =$"any monodromy parameter" to study the dependence of $\tau(t)$ on monodromy and find $\mathcal{N}_1/\mathcal{N}_0$.

Riemann-Hilbert setup

- let $\mathcal{C} \subset \mathbb{C}$ be a circle centered at the origin
- pick a loop $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- $J(z)$ continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}} J_k z^k,$$



Two Riemann-Hilbert problems:

$$\text{direct}: \quad J(z) = \Psi_-(z)^{-1} \Psi_+(z)$$
$$\text{dual}: \quad J(z) = \bar{\Psi}_+(z) \bar{\Psi}_-(z)^{-1}$$

Main definition: The tau function of RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

$$\tau[J] = \det_{H_+}\left(\Pi_+ J^{-1} \Pi_+ J \Pi_+\right),$$

where $H = L^2\left(\mathcal{C}, \mathbb{C}^N\right)$ and Π_+ is the orthogonal projection on H_+ along H_-.

Properties:

- dual RHP is solvable iff the operator $P := \Pi_+ J^{-1} \Pi_+$ is invertible on H_+, in which case $P^{-1} = \bar{\Psi}_+ \Pi_+ \bar{\Psi}_-^{-1} \Pi_+$
- likewise, for direct RHP and $Q := \Pi_+ J \Pi_+$, with $Q^{-1} = \Psi_+^{-1} \Pi_+ \Psi_- \Pi_+$

Variational formula

Theorem: Let $(z,t) \mapsto J(z,s)$ be a smooth family of $\mathrm{GL}(N,\mathbb{C})$-loops which depend on an extra parameter s and admit direct & dual factorization. Then

$$\partial_s \ln \tau[J] = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathrm{Tr}\left\{ J^{-1} \partial_s J \left[\partial_z \bar{\Psi}_- \bar{\Psi}_-^{-1} + \Psi_+^{-1} \partial_z \Psi_+ \right] \right\} dz.$$

- due to [Widom, '74]; rediscovered by [Its, Jin, Korepin, '06]
- related results in the study of dependence of isomonodromic tau functions on monodromy [Bertola, '09]

Corollary: in isomonodromic RHPs,

Widom's constant $\tau[J]$ $\simeq$ Jimbo-Miwa-Ueno tau function

Example: 4-point Fuchsian system

4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

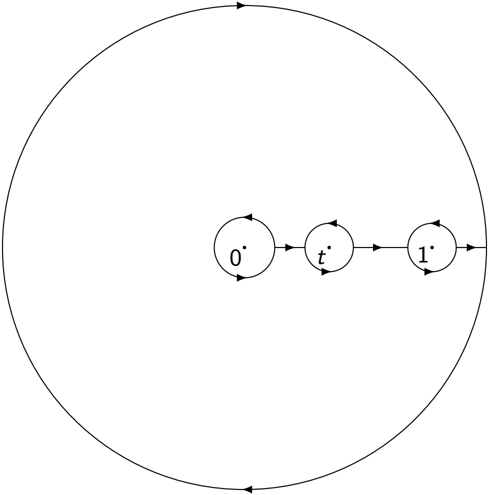
- arbitrary rank: $A_{0,t,1} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$
- generic case: $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ are diagonalizable
- fix the diagonalizations $A_\nu = G_\nu^{-1} \Theta_\nu G_\nu$ with diagonal Θ_ν
- eigenvalues of A_ν are assumed distinct mod $\mathbb{Z}$

There exist unique fundamental solutions $\Phi^{(\nu)}(z)$, holomorphic on the universal covering of $\mathbb{C}\setminus\{0, t, 1\}$ and such that

$$\Phi^{(\nu)}(z) = \begin{cases} (\nu - z)^{\Theta_\nu} G^{(\nu)}(z), & \text{for } \nu = 0, t, 1, \\ (-z)^{-\Theta_\infty} G^{(\infty)}(z), & \text{for } \nu = \infty, \end{cases}$$

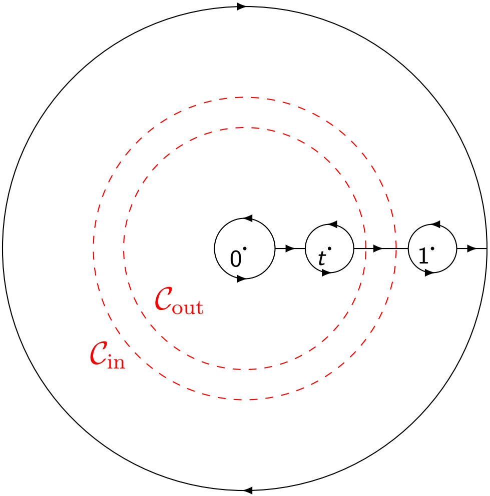
where $G^{(\nu)}(z)$ is holomorphic and invertible in a finite open disk around $z = \nu$ and satisfies $G^{(\nu)}(\nu) = G_\nu$.

Dual RHP$_1$ for $\tilde{\Psi}$



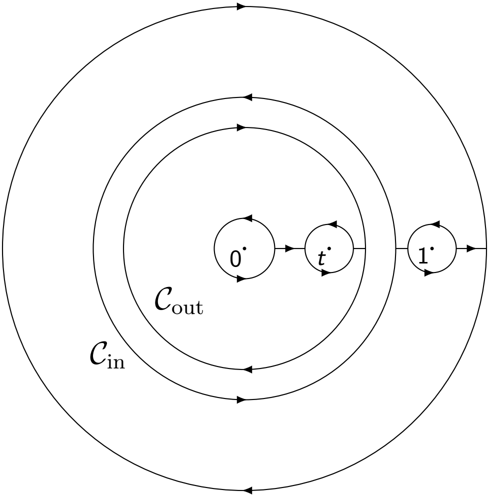
$$\tilde{\Psi}(z) = \begin{cases} G^{(\nu)}(z), & z \in D_\nu, \\ \Phi(z), & z \notin \mathbb{R}_{\geq 0} \cup \bar{D}_0 \cup \bar{D}_t \cup \bar{D}_1 \cup \bar{D}_\infty. \end{cases}$$

Dual RHP_1 for $\tilde{\Psi}$

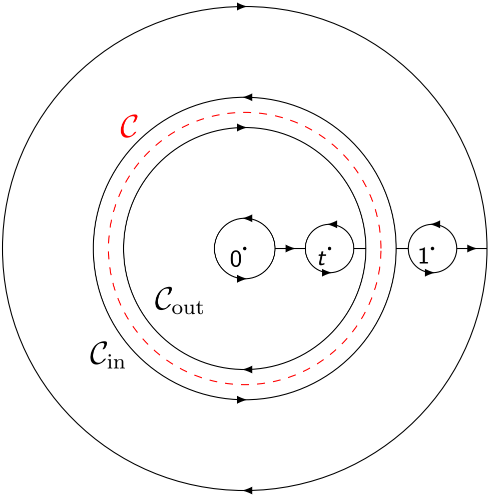


$$\hat{\Psi}(z) = \begin{cases} (-z)^{-\mathfrak{S}}\,\tilde{\Psi}(z)\,, & z \in \mathcal{A}, \\ \tilde{\Psi}(z)\,, & z \notin \bar{\mathcal{A}}. \end{cases}$$

Dual RHP$_2$ for $\hat{\Psi}$

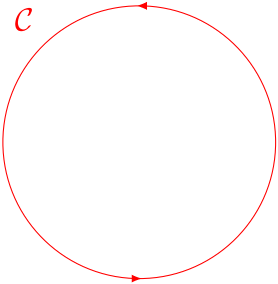


Dual RHP$_2$ for $\hat{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

Dual RHP_3 for $\bar{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\mathcal{C}$ (single circle !), smooth jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ given by

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}$$

Substitute into Widom's differentiation formula

$$\partial_s \ln \tau[J] = \frac{1}{2\pi i} \oint_C \operatorname{Tr} J^{-1}\partial_s J \left(\partial_z \bar{\Psi}_- \bar{\Psi}_-^{-1} + \Psi_+^{-1}\partial_z \Psi_+\right) dz.$$

the expression for the jump $J = \Phi_i^{-1}\Phi_e$ and the dual/direct factorizations,

$$\bar{\Psi}_- = \Phi_e^{-1}\Phi, \quad \bar{\Psi}_+ = \Phi_i^{-1}\Phi, \quad \Psi_- = (-z)^{-\mathfrak{S}}\,\Phi_i, \quad \Psi_+ = (-z)^{-\mathfrak{S}}\,\Phi_e,$$

and use that $\partial_z \Phi = \Phi A(z)$. This gives

$$\partial_s \ln \tau[J] = \frac{1}{2\pi i} \oint_C \operatorname{Tr} \Big\{ A(z)\,\Phi^{-1}\Phi_i \partial_s \left(\Phi_i^{-1}\Phi\right) - A(z)\,\Phi^{-1}\Phi_e \partial_s \left(\Phi_e^{-1}\Phi\right)$$
$$- \frac{\mathfrak{S}}{z}(-z)^{-\mathfrak{S}}\,\Phi_i \partial_s \left(\Phi_i^{-1}(-z)^{\mathfrak{S}}\right) + \frac{\mathfrak{S}}{z}(-z)^{-\mathfrak{S}}\,\Phi_e \partial_s \left(\Phi_e^{-1}(-z)^{\mathfrak{S}}\right) \Big\}\, dz$$

Red terms contribute via the residues at $z = 0, t$, and blue ones via the residues at $z = 1, \infty$.

The log-derivative then reduces to

$$\partial_s \ln \tau\,[J] = \sum_{\nu=0,t,1,\infty} \mathrm{Tr}\,\Theta_\nu \partial_s G_\nu G_\nu^{-1}$$

$$-\sum_{\nu=0,t,\infty} \mathrm{Tr}\,\Theta_{\nu,i} \partial_s G_{\nu,i} G_{\nu,i}^{-1} - \sum_{\nu=0,1,\infty} \mathrm{Tr}\,\Theta_{\nu,e} \partial_s G_{\nu,e} G_{\nu,e}^{-1}$$

where Θ_ν are exponents of the 4-point solution,

$$\Theta_{0,i} = \Theta_0, \quad \Theta_{t,i} = \Theta_t, \quad \Theta_{\infty,i} = \mathfrak{S},$$
$$\Theta_{0,e} = \mathfrak{S}, \quad \Theta_{1,e} = \Theta_1, \quad \Theta_{\infty,e} = \Theta_\infty,$$

and $G_{\nu,i}$, $G_{\nu,e}$ are 3-point counterparts of G_ν.

For $s = t$ (isomonodromic time):

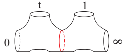
- 1st line is nothing but the Jimbo-Miwa-Ueno definition of τ
- 2nd line corresponds to tau functions of auxiliary 3-point systems

We then obtain

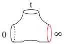
$$\tau_{\mathrm{JMU}}(t) = t^{\frac{1}{2}\,\mathrm{Tr}\left(\mathfrak{S}^2-\Theta_0^2-\Theta_t^2\right)}\tau\,[J]\,.$$

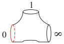
- $\tau_{\mathrm{JMU}}(t)$ for 4-point system written via auxiliary 3-point solutions
- hypergeometric representations for $N = 2 \Longrightarrow$ PVI tau function !

Schematically,

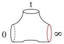
$\tau_{\mathrm{JMU}}\Bigg($ 

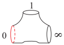
$\Bigg) =$

$\tau_{\mathrm{JMU}}\Bigg($ 

$\Bigg)\,\tau_{\mathrm{JMU}}\Bigg($ 

$\Bigg)\,\mathrm{det}\left(\begin{array}{cc} \mathbf{1} & \mathsf{a}_{+-}\Big(\text{[figure 5]} \Big) \\ \mathsf{a}_{-+}\Big(\text{[figure 4]} \Big) & \mathbf{1} \end{array}\right)$





Connection coefficient

Considering a different pants decomposition which combines t and 1 instead of t and 0, we obtain a different Fredholm determinant representation, which is better adapted for the asymptotic analysis of the regime $t \to 1$.

$$\bar{\tau}_{\mathrm{JMU}}(t) = (1-t)^{\frac{1}{2}\operatorname{Tr}\left(\bar{\mathfrak{S}}^2 - \Theta_1^2 - \Theta_t^2\right)} \tau\left[\bar{J}\right].$$

It is of course proportional to the previous tau function $\tau_{\mathrm{JMU}}(t)$, and their ratio is the connection coefficient that we want to compute.

Corollary: For any monodromy parameter s,

$$\begin{aligned}\partial_s \ln \frac{\mathcal{N}_1}{\mathcal{N}_0} = & \sum_{\nu=0,t,\infty} \operatorname{Tr} \bar{\Theta}_{\nu,i} \partial_s \bar{G}_{\nu,i} \bar{G}_{\nu,i}^{-1} + \sum_{\nu=0,1,\infty} \operatorname{Tr} \bar{\Theta}_{\nu,e} \partial_s \bar{G}_{\nu,e} \bar{G}_{\nu,e}^{-1} \\ & - \sum_{\nu=0,t,\infty} \operatorname{Tr} \Theta_{\nu,i} \partial_s G_{\nu,i} G_{\nu,i}^{-1} - \sum_{\nu=0,1,\infty} \operatorname{Tr} \Theta_{\nu,e} \partial_s G_{\nu,e} G_{\nu,e}^{-1} \\ & + \frac{1}{2} \ln t \, \partial_s \operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right) - \frac{1}{2} \ln(1-t) \, \partial_s \operatorname{Tr}\left(\bar{\mathfrak{S}}^2 - \Theta_1^2 - \Theta_t^2\right)\end{aligned}$$

Theorem [Its, OL, Prokhorov, Duke. Math. J., '19]

For generic monodromy data,

$$\frac{\mathcal{N}_1}{\mathcal{N}_0} = \prod_{\epsilon,\epsilon'=\pm} \frac{G\left(1+\epsilon\overline{\sigma}+\epsilon'\theta_t-\epsilon\epsilon'\theta_1\right) G\left(1+\epsilon\overline{\sigma}+\epsilon'\theta_0-\epsilon\epsilon'\theta_\infty\right)}{G\left(1+\epsilon\sigma+\epsilon'\theta_t+\epsilon\epsilon'\theta_0\right) G\left(1+\epsilon\sigma+\epsilon'\theta_1+\epsilon\epsilon'\theta_\infty\right)} \times$$
$$\times \prod_{\epsilon=\pm} \frac{G(1+2\epsilon\sigma)}{G(1+2\epsilon\overline{\sigma})} \prod_{k=1}^{4} \frac{\hat{G}(\varsigma+\nu_k)}{\hat{G}(\varsigma+\lambda_k)}$$

Here $G(z)$ denotes the Barnes G-function, $\hat{G}(z) = \frac{G(1+z)}{G(1-z)}$, the parameters $\nu_{1\ldots4}$ and $\lambda_{1\ldots4}$ are defined by

$$\begin{aligned} \nu_1 &= \sigma+\theta_0+\theta_t, & \lambda_1 &= \theta_0+\theta_t+\theta_1+\theta_\infty, \\ \nu_2 &= \sigma+\theta_1+\theta_\infty, & \lambda_2 &= \sigma+\overline{\sigma}+\theta_0+\theta_1, \\ \nu_3 &= \overline{\sigma}+\theta_0+\theta_\infty, & \lambda_3 &= \sigma+\overline{\sigma}+\theta_t+\theta_\infty, \\ \nu_4 &= \overline{\sigma}+\theta_t+\theta_1, & \lambda_4 &= 0, \end{aligned}$$

and the quantity ς is determined by

$$e^{2\pi i\varsigma} = \frac{2\cos 2\pi\left(\sigma-\overline{\sigma}\right) - 2\cos 2\pi\left(\theta_0+\theta_1\right) - 2\cos 2\pi\left(\theta_\infty+\theta_t\right) + \operatorname{Tr} M_0 M_1}{\sum_{k=1}^{4}\left(e^{2\pi i(\nu_\Sigma-\nu_k)} - e^{2\pi i(\nu_\Sigma-\lambda_k)}\right)},$$

with $2\nu_\Sigma = \sum_{k=1}^{4}\nu_k = \sum_{k=1}^{4}\lambda_k$.

Some open problems

- 3-point auxiliary solutions are known for 4-point Fuchsian systems of higher rank N whose 2 singularities have special spectral type $(N-1, 1)$. It is then in principle possible to find explicitly the log-differential of the connection coefficient. Is it possible to integrate it? What would be the higher-rank analog of the tetrahedron?
- In the generic case, the 3-point solutions for $N > 2$ are not available. Can we at least find an interpretation of the connection constant in terms of Poisson geometry of the $\mathrm{SL}(N, \mathbb{C})$ character variety of $C_{0,4}$? Affirmative answer [Bertola, Korotkin, '19]
- Connections constants for PI are computed in [OL, Roussillon, '16]. Their evaluation for PII-PV with generic parameters is wide open; for PV conjectural expressions are available [OL, Nagoya, Roussillon, '18].