Hydrodynamics for integrable systems

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What this is about

⋆ This is about **certain differential and integral equations that describe the non-equilibrium dynamics of many body integrable systems**.

⋆ "Hydrodynamics" refers to the **fundamental physical principles of hydrodynamics**. Applied to integrable systems we obtain what we call **generalised hydrodynamics (GHD)**.

⋆ This is **not** about hydrodynamic-type partial differential equations that are integrable (although the GHD equations, it turns out, are integrable...).

⋆ Technically, GHD morphs **the thermodynamic Bethe ansatz (TBA)** into a set of **hydrodynamic-scale equations**.

⋆ Despite the name, this TBA framework, and hence GHD, is **extremely widely applicable**: quantum and classical field theories, chains or gases of all types. It is not really based on the Bethe ansatz, rather on the scattering theory for many-body systems.

Examples of models

Quantum Lieb-Liniger gas: point-like interactions in a Galilean invariant Bose gase

$$H = \int \mathrm{d}x \, \left(\frac{1}{2} \partial_x \Psi^\dagger \partial_x \Psi + \frac{c}{2} \Psi^\dagger \Psi^\dagger \Psi \Psi \right), \quad [\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$$

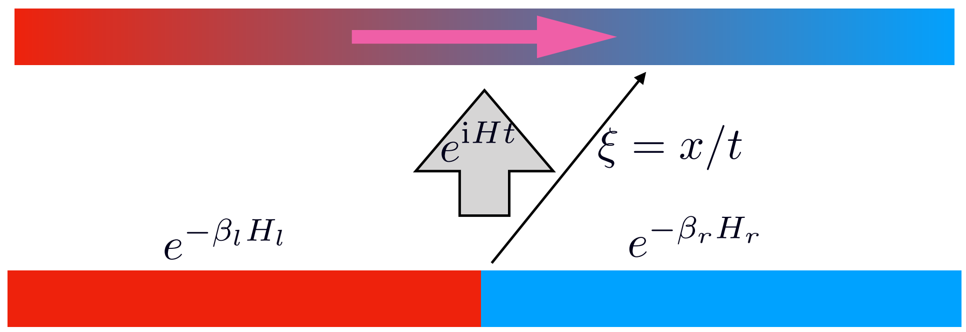
Classical Toda gas: exponential interaction in a classical gas

$$H = \sum_{a\in\mathbb{Z}} \left(\frac{1}{2} p_a^2 + e^{-r_a} + P r_a \right), \quad r_a = x_{a+1} - x_a, \quad \{x_a, p_b\} = \delta_{a,b}$$

and many, many more...

Examples of problems to solve

Non-equilibrium steady states and the Riemann problem (or partitioning protocol)



Hamiltonians H_l and H_r are for the system on half-lines $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively (with some boundary condition at 0), and evolution is with full hamiltonian H on $\mathbb{R}$.

Quantum: $$\langle \mathcal{O} \rangle_\xi = \lim_{t\to\infty} \frac{1}{Z} \mathrm{Tr}\, e^{-\beta_l H_l - \beta_r H_r} \mathcal{O}(\xi t, t)$$

example: $\mathcal{O}(x,t) = \Psi^\dagger(x,t)\Psi(x,t)$

Classical: $$\langle \mathcal{O} \rangle_\xi = \lim_{t\to\infty} \frac{1}{Z} \int \prod_a \mathrm{d}p_a \mathrm{d}x_a \, e^{-\beta_l H_l[p,x] - \beta_r H_r[p,x]} \mathcal{O}(\xi t, t)$$

example: $\mathcal{O}(x,t) = \sum_m \delta(x_a(t) - x) f(x_a(t), p_a(t))$

Examples of problems to solve

Correlation functions in statistical ensembles

Dynamical connected correlation functions at large wavelengths and large times:

$$\mathsf{S}_{\mathcal{O},\mathcal{O}'}(kt) = \lim_{\substack{k\to 0, t\to\infty \\ kt \text{ fixed}}} \int \mathrm{d}x\, e^{\mathrm{i}kx} \langle \mathcal{O}(x,t)\mathcal{O}'(0,0)\rangle^{\mathrm{c}}$$

where, for instance,

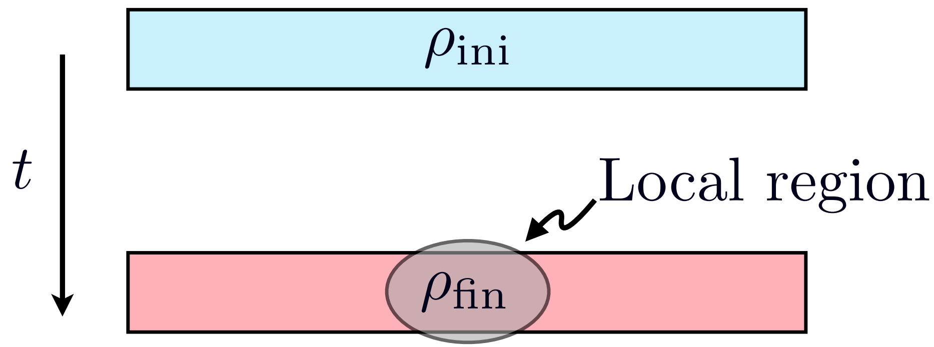
Quantum: $$\langle \mathcal{O}\cdots\rangle_\beta = \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H}\mathcal{O}\cdots$$

and

Classical: $$\langle \mathcal{O}\cdots\rangle_\beta = \lim_{t\to\infty}\frac{1}{Z}\int\prod_a \mathrm{d}p_a\mathrm{d}x_a\, e^{-\beta H[p,x]}\mathcal{O}\cdots$$

Hydrodynamics

1. “Thermalisation” in isolated large systems: the local state obtained after a large time from a homogeneous (but generically not stationary) initial state.

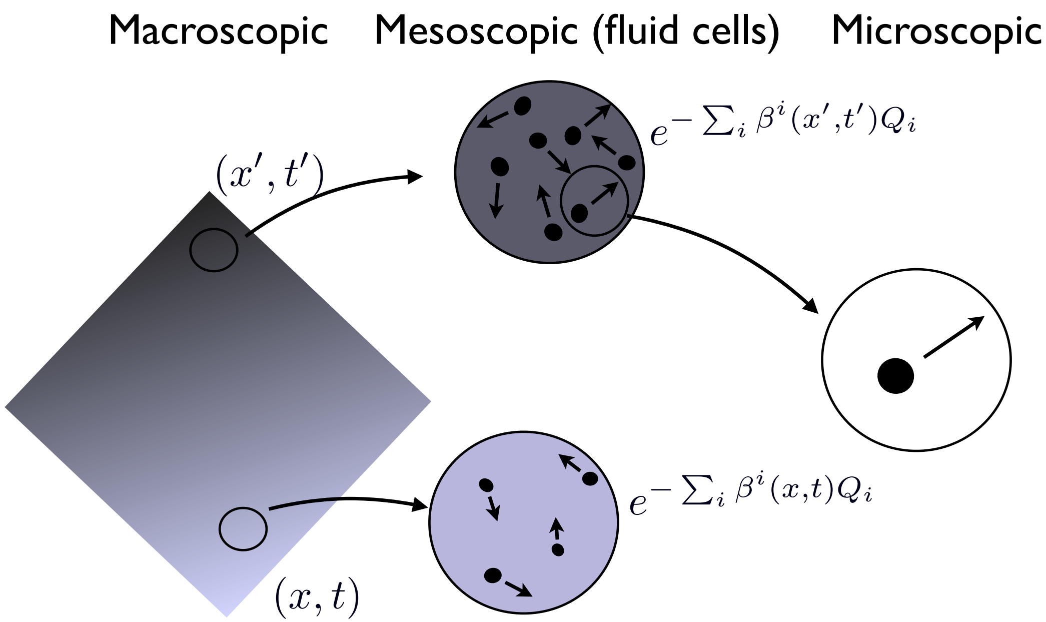


The emerging local state is **homogeneous and stationary**, and has **maximised entropy with respect to all conservation laws available**.

$$\lim_{t\to\infty}\langle \mathcal{O}(x,t)\rangle_{\rm ini} = \langle \mathcal{O}\rangle_{\rm fin},\qquad \rho_{\rm fin} = e^{-\sum_i \beta^i Q_i},\qquad Q_i = \int {\rm d}x\, q_i(x).$$

A maximal entropy state may carry currents if some Q_i are not time-reversible (e.g. momentum). In this case it is not at equilibrium.

2. Euler hydrodynamics. The main idea behind Euler hydrodynamics is **"local thermodynamic equilibrium"**.



It says that **locally** and on **short time scales** (in fluid cells), the many-body system relaxes to **space-time dependent entropy-maximised states**:

$$\langle \mathcal{O}(x,t)\rangle_{\rm ini} \approx \langle \mathcal{O}\rangle_{\boldsymbol{\beta}(x,t)}, \qquad \rho_{\boldsymbol{\beta}(x,t)} = e^{-\sum_i \beta^i(x,t)Q_i}$$

Hydrodynamics

Recall the **local conservation laws**

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0$$

With hydrodynamic approximation, imply hydrodynamic equations for local averages:

$$\boxed{\partial_t \mathsf{q}_i(x,t) + \partial_x \mathtt{j}_i(x,t) = 0}$$

where $\mathsf{q}_i(x,t) = \langle q_i \rangle_{\boldsymbol{\beta}(x,t)}$ and $\mathtt{j}_i(x,t) = \langle j_i \rangle_{\boldsymbol{\beta}(x,t)}$.

As many β_i as q_i: **can invert the map $\boldsymbol{\beta} \to \langle q_i \rangle_{\boldsymbol{\beta}}$, so the local state is fully characterised by the $\mathsf{q}_i(x,t)$'s**. Hence there are **equations of state**:

$$\mathtt{j}_i(x,t) = \mathfrak{J}_i(\mathsf{q}_\bullet(x,t))$$

and therefore one can write the "quasi-linear form" of Euler hydrodynamics:

$$\partial_t \mathsf{q}_i(x,t) + \sum_j \mathsf{A}_i^{\,j}\, \partial_x \mathsf{q}_j(x,t) = 0, \qquad \mathsf{A}_i^{\,j} = \frac{\partial \mathfrak{J}_i}{\partial \mathsf{q}_\mathsf{j}}$$

The matrix $\mathsf{A}_i^{\,j}$ is (often) called the **flux Jacobian**.

Hydrodynamics

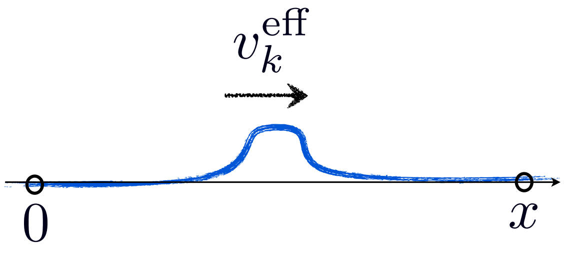
3. Correlation functions. Flux Jacobian also control the **space-time profile of two-point functions in homogeneous, stationary states**:

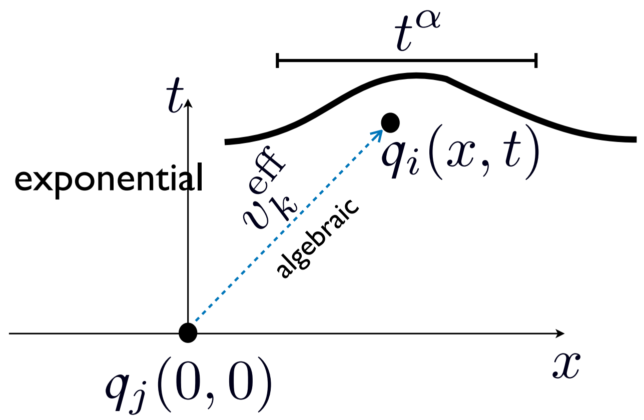
$$\sum_k \left(\delta_i^k \partial_t + \mathsf{A}_i^{\ k} \partial_x\right) \langle q_k(x,t) q_j(0,0)\rangle^{\rm c}_{\boldsymbol{\beta}} = 0$$

The eigenvalues of the flux Jacobian are the **effective velocities**:

$$\sum_j \mathsf{A}_i^{\ j} \mathbf{h}_j = v_i^{\rm eff} \mathbf{h}_i$$

and determine the **directions where correlations don't decay exponentially** (due to ballistic propagation of waves). Around this, there is diffusive or super/subdiffusive spreading.





Hydrodynamics

The Fourier transform has an explicit form:

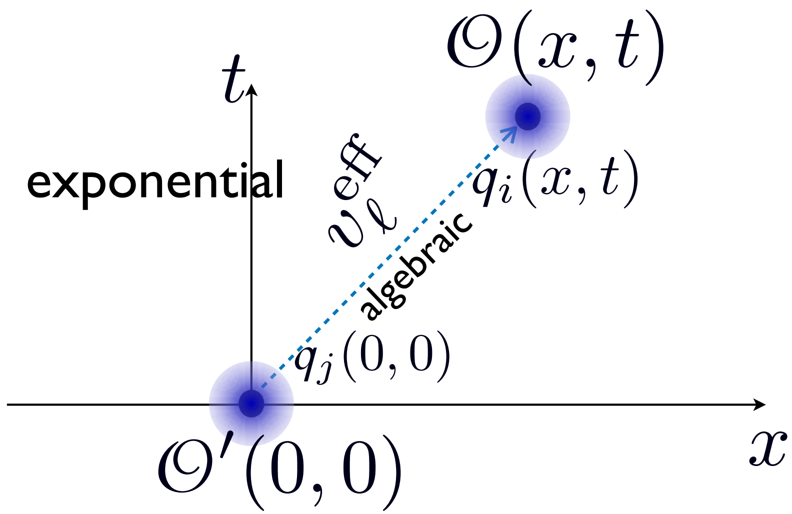
$$\mathsf{S}_{ij}(k,t) = \int \mathrm{d}x \, e^{\mathrm{i}kx} \langle q_i(x,t) q_j(0,0) \rangle^{\mathrm{c}}_{\boldsymbol{\beta}} \sim \Big(\exp\left[\mathrm{i}\mathsf{A}kt\right] \mathsf{C} \Big)_{ij}$$

with the static correlation matrix

$$\mathsf{C}_{ij} = \int \mathrm{d}x \, \langle q_i(x) q_j(0) \rangle^{\mathrm{c}}_{\boldsymbol{\beta}} = -\frac{\partial \mathsf{q}_j}{\partial \beta^i}.$$

4. Hydrodynamic projections. There are very general formulae for **the large-time asymptotics of correlation functions of more generic fields**, again controlled by the flux Jacobian [BD 2017]:

$$\lim_{\substack{k\to 0,t\to\infty \\ kt \text{ fixed}}} \int \mathrm{d}x\, e^{\mathrm{i}kx} \langle \mathcal{O}(x,t)\mathcal{O}'(0,0)\rangle^{\mathrm{c}}_{\boldsymbol{\beta}} = \sum_{i\ell j} \frac{\partial\langle\mathcal{O}\rangle_{\boldsymbol{\beta}}}{\partial\beta^i}\,(C^{-1})^{i\ell}\,\exp[\mathrm{i}\mathsf{A}kt]_\ell^{\;j}\,\frac{\partial\langle\mathcal{O}'\rangle_{\boldsymbol{\beta}}}{\partial\beta^j}$$



Generalised hydrodynamics

In integrable systems: all these structures are expected to be there, but with **infinitely many conserved quantities**

$$Q_i, \qquad i = 1, 2, 3, \ldots, \text{ to infinity}$$

For instance in the LL Model: $Q_0 = \int \mathrm{d}x\, \Psi^\dagger(x)\Psi(x)$, $Q_1 = \int \mathrm{d}x\, \Psi^\dagger(x)\mathrm{i}\partial_x\Psi(x)$, $Q_2 = H, \ldots$. The GGEs are

$$\rho_{\mathrm{GGE}} = e^{-\sum_{i=1}^{\infty} \beta^i Q_i}.$$

We need to evaluate:

- ⋆ Density averages $\langle q_i \rangle_{\boldsymbol{\beta}} = \mathrm{Tr}(\rho_{\mathrm{GGE}}\, q_i)/Z$
- ⋆ Current averages $\langle j_i \rangle_{\boldsymbol{\beta}} = \mathrm{Tr}(\rho_{\mathrm{GGE}}\, j_i)/Z$
- ⋆ Static correlation matrix $\mathsf{C}_{ij} = \int \mathrm{d}x\, \langle q_i(x,t) q_j(0,0) \rangle_{\boldsymbol{\beta}} = -\partial \langle q_i \rangle_{\boldsymbol{\beta}} / \partial \beta^j$
- ⋆ Flux Jacobian $\mathsf{A}_i^{\;j} = \partial \langle j_i \rangle_{\boldsymbol{\beta}} / \partial \langle q_j \rangle_{\boldsymbol{\beta}}$

Generalised hydrodynamics

1. From the Bethe ansatz. Generalized Gibbs ensembles can be described, in Bethe ansatz integrable models, by using the **thermodynamic Bethe ansatz**.

Bethe-ansatz quasi-momenta: **spectral parameter** $\theta \in \mathbb{R}$ characterizing the quasi-particle,

$$|\theta_1, \theta_2, \ldots\rangle.$$

The main property of quasi-momenta is additivity of conserved charge eigenvalues,

$$Q_i|\theta_1, \theta_2, \ldots\rangle = \sum_\ell h_i(\theta_\ell)|\theta_1, \theta_2, \ldots\rangle.$$

For instance $h_0(\theta) = 1$, $h_1(\theta) = \theta$, $h_2(\theta) = \theta^2/2$, $\ldots$ in the LL model.

With finite densities of particles, we introduce the "phase-space" density $\rho_{\rm p}$, and states $|\rho_{\rm p}\rangle$. The equivalence of ensembles allows us to write

$$Z^{-1}{\rm Tr}\rho_{\rm GGE}\,\mathcal{O} = \langle\rho_{\rm p}|\mathcal{O}|\rho_{\rm p}\rangle.$$

Generalised hydrodynamics

Given a set of Lagrange parameters β_i, TBA tells us how to compute the quasi-particle densities $\rho_{\rm p}(\theta)$: minimizing the free energy, taking into account the number of microstates corresponding to a given density. In the LL model:

$$\rho_{\rm p}(\theta) = \frac{\delta}{\delta w(\theta)} \int {\rm d}p(\theta) F(\epsilon(\theta)), \qquad F(\epsilon) = -\log(1+e^{-\epsilon})$$

with $p(\theta) = \theta$ the momentum, and **pseudo-energy** evaluated at $w(\theta) = \sum_i \beta_i h_i(\theta)$

$$\epsilon(\theta) = w(\theta) - \int \frac{{\rm d}\alpha}{2\pi} \varphi(\theta,\alpha) \log(1+e^{-\epsilon(\alpha)}).$$

$\varphi(\theta,\alpha) = 2c/(c^2+(\theta-\alpha)^2)$ is the differential scattering phase from the Bethe ansatz.

Then,

$$\langle q_i \rangle_{\boldsymbol{\beta}} = \int {\rm d}\theta\, h_i(\theta) \rho_{\rm p}(\theta).$$

Generalised hydrodynamics

The average currents do not follow from the free energy. To evaluate them in GGEs, more work needed. We find [Castro-Alvaredo, BD, Yoshimura, 2016; Bertini, Collura, De Nardis, Fagotti, 2016]

$$\mathtt{j}_i = \int \mathrm{d}\theta\, h_i(\theta) v^{\mathrm{eff}}(\theta) \rho_{\mathrm{p}}(\theta)$$

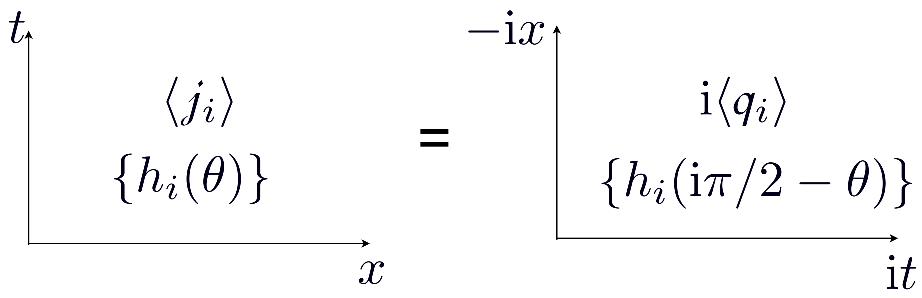
with

$$v^{\mathrm{eff}}(\theta) = \frac{E'(\theta)}{p'(\theta)} + \int \mathrm{d}\alpha\, \frac{\varphi(\theta,\alpha)\,\rho_{\mathrm{p}}(\alpha)}{p'(\theta)}\,(v^{\mathrm{eff}}(\alpha) - v^{\mathrm{eff}}(\theta))$$

where $E(\theta) = \theta^2/2$ is the energy of a quasiparticle.

Shown in relativistic QFT using crossing symmetry, and form factor expansions. Proof in the XXZ chain using form factor expansions.

[Castro-Alvaredo, BD, Yoshimura, 2016; Bertini, Collura, De Nardis, Fagotti, 2016; Vu, Yoshimura 2019]



Generalised hydrodynamics

We now make GGEs space-time dependent. This means we promote

$$\rho_{\rm p}(\theta) \mapsto \rho_{\rm p}(x,t,\theta)$$

The quantity $\rho_{\rm p}(x,t;\theta){\rm d}x{\rm d}\theta$ is the number of quasi-particles in the “phase-space” element $[\theta,\theta+{\rm d}\theta]\times[x,x+{\rm d}x]$.

Using

$$\mathtt{q}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)\rho_{\rm p}(x,t,\theta), \qquad \mathtt{j}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)v^{\rm eff}(x,t,\theta)\rho_{\rm p}(x,t,\theta)$$

and **completeness of** $\{h_i(\theta)\}$, the fundamental GHD equations $\partial_t\mathtt{q}_i + \partial_x\mathtt{j}_i = 0$ become

$$\boxed{\partial_t\rho_{\rm p}(x,t,\theta) + \partial_x\left[v^{\rm eff}(x,t,\theta)\rho_{\rm p}(x,t,\theta)\right] = 0}$$

These are the **GHD hydrodynamic equations in the quasi-particle language**.

Generalised hydrodynamics

Define the **occupation function**:

$$n(\theta) = \frac{\rho_{\rm p}(\theta)}{\rho_{\rm s}(\theta)}, \quad 2\pi\rho_{\rm s}(\theta) = p'(\theta) + \int {\rm d}\alpha\, \varphi(\theta,\alpha)\rho_{\rm p}(\alpha).$$

Here $\rho_{\rm s}$ as the interpretation as a **density of states**: the “availabilities” for quasi-particles.

Then we find the **diagonalised quasi-linear form**

[Castro-Alvaredo, BD, Yoshimura, 2016; Bertini, Collura, De Nardis, Fagotti, 2016]

$$\partial_t n(x,t;\theta) + v^{\rm eff}(x,t;\theta)\partial_x n(x,t;\theta) = 0.$$

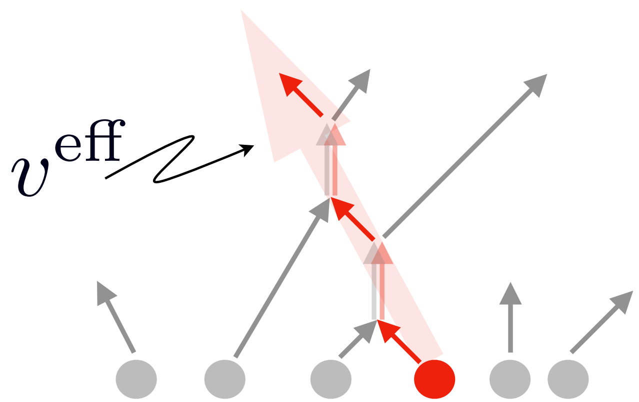
That is, the occupation function is the fluid coordinate that diagonalizes GHD – the normal modes. It is convectively transported by the fluid, with propagation velocities $v^{\rm eff}(x,t;\theta)$. The effective velocities are then the eigenvalues of the flux Jacobian.

Generalised hydrodynamics

2. From classical scattering. Note that there is a clear **soliton scattering picture** underlying the expression of the effective velocity

$$v^{\rm eff}(\theta) = v(\theta) + \int {\rm d}\alpha\, \frac{\varphi(\theta,\alpha)\,\rho_{\rm p}(\alpha)}{p'(\theta)}\,(v^{\rm eff}(\alpha) - v^{\rm eff}(\theta))$$

Just add up the accumulated "soliton time delays" incurred by a particle as it goes through a gas of other particles [Zakharov 1971; El 2003; El, Kamchatnov 2005; El, Kamchatnov, Pavlov, Zykov 2011; BD, Yoshimura, Caux 2017]



One can use such arguments to derive the GHD equations **without the explicit use of the assumption of local entropy maximisation**.

Generalised hydrodynamics

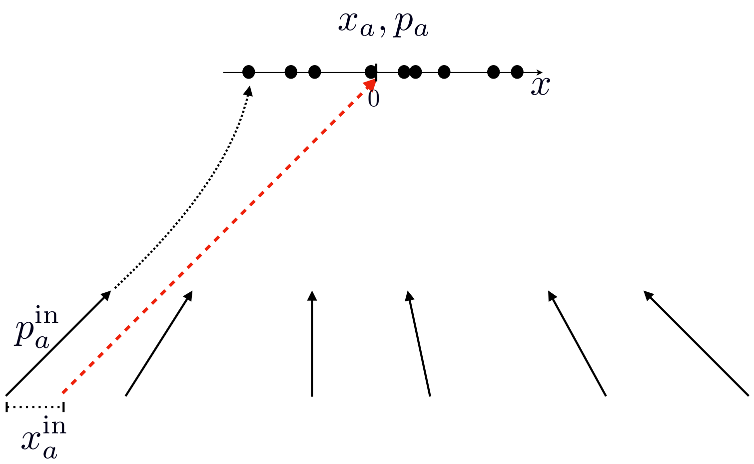
The GHD equations are the scattering transform of the Liouville equations.

Consider e.g. the Toda gas. Transform the phase-space coordinates of the gas's particles x_a, p_a into their **asymptotic coordinates**

$$(x_a, p_a) \mapsto (x_a^{\rm in}, p_a^{\rm in})$$

as

$$x_a(t) = p_a^{\rm in} t + x_a^{\rm in} + O(t^{-\infty}), \quad p_a(t) = p_a^{\rm in} + O(t^{-\infty}) \qquad (t \to -\infty)$$



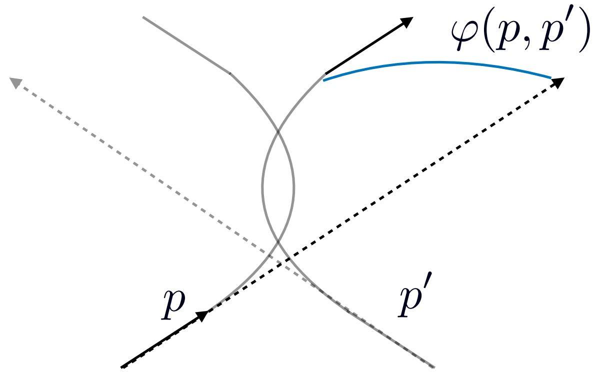
These are **canonical coordinates, which evolve trivially**

$$\{x_a^{\rm in}, p_b^{\rm in}\} = \delta_{a,b}, \qquad H = \sum_a (p_a^{\rm in})^2/2.$$

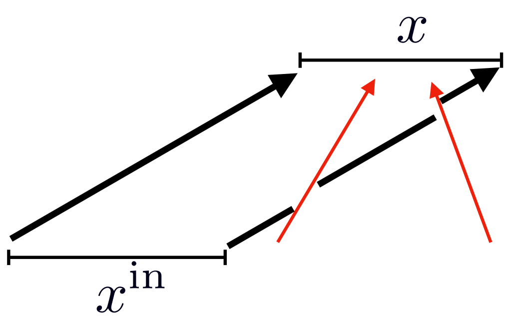
Therefore they have **trivial hydrodynamics** (Liouville's equation)

$$\partial_t \rho_{\rm p}^{\rm in}(x,t,p) + p\,\partial_x \rho_{\rm p}^{\rm in}(x,t,p) = 0.$$

Generalised hydrodynamics



Integrability implies factorised scattering, hence conservation of all momenta. So we can follow the **velocity tracers**: these are the quasi-particles.



Transforming back to the original coordinates is a **momentum-dependent change of metric, due to the space gained / lost by the accumulation of time delays** [BD 2019]:

$$\mathrm{d}x^{\text{in}} = \Big(1 - \int \mathrm{d}q\, \varphi(p,q)\rho_{\text{p}}(x,t,q)\Big)\mathrm{d}x$$

The result of the change of coordinates is the GHD equation [BD, Spohn, Yoshimura 2017]

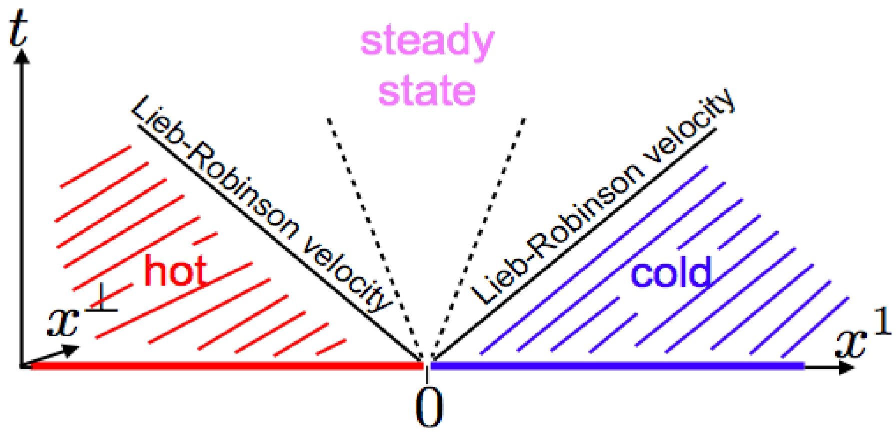
$$\partial_t \rho_{\text{p}} + \partial_x (v^{\text{eff}} \rho_{\text{p}}) = 0.$$

Generalised hydrodynamics

3. General structure. GHD applies to a wide family of integrable models, including **quantum and classical field theories, gases and chains, as well as cellular automata**. The data are

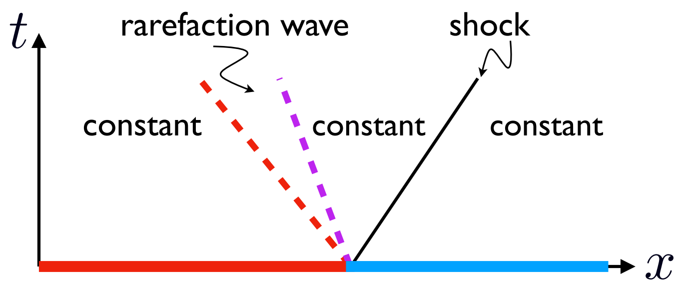
- ⋆ Differential scattering phase $\varphi(\theta,\theta')$
- ⋆ Energy and momentum functions $E(\theta)$ and $p(\theta)$ (as well as functions for other conserved charges $h_i(\theta)$)
- ⋆ Free energy function $F(\epsilon)$ that enters the TBA formulation, encoding the statistics of the quasi-particles: $-\log(1+e^{-\epsilon})$ for fermions, $\log(1-e^{-\epsilon})$ for bosons, $-e^{-\epsilon}$ for classical particles, $1/\epsilon$ for classical radiative modes (also for anyons, etc.). In particular, it is related to the occupation function as $n(\theta) = F'(\epsilon(\theta))$.

GHD solution to Riemann problem



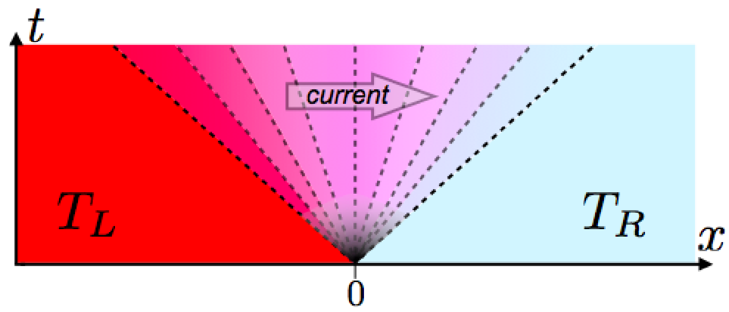
Currents emerge in steady-state region if there is ballistic transport.

Phenomenology is well known in conventional hydrodynamics, with **shocks and rarefaction waves**.



GHD solution to Riemann problem

In integrable systems, the phenomenology is different: **generically smooth profiles**. Technically, this is because of the **continuum of normal mode's velocities**.



The problem is invariant under $x, t \mapsto \lambda x, \lambda t$: functions of the ray $\xi = x/t$:

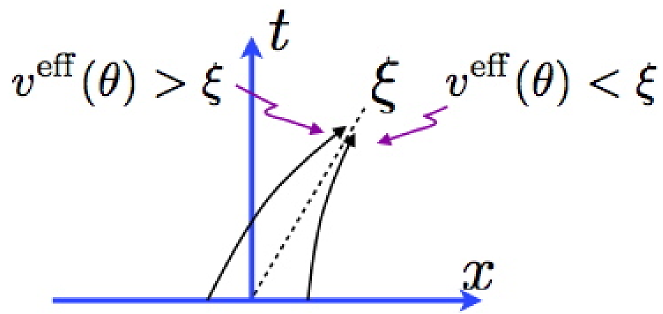
$$(v^{\rm eff} - \xi)\partial_\xi n(\xi;\theta) = 0.$$

In fluid dynamic terms, solution is a family (parametrised by θ) of **contact singularities**:

[Castro Alvaredo, BD, Yohimura 2016; Bertini, Collura, De Nardis, Fagotti 2016]

$$n(\xi;\theta) = n^L(\theta)\Theta(\theta - \theta_\star(\xi)) + n^R(\theta)\Theta(\theta_\star - \theta(\xi)), \qquad v^{\rm eff}(\xi;\theta_\star(\xi)) = \xi.$$

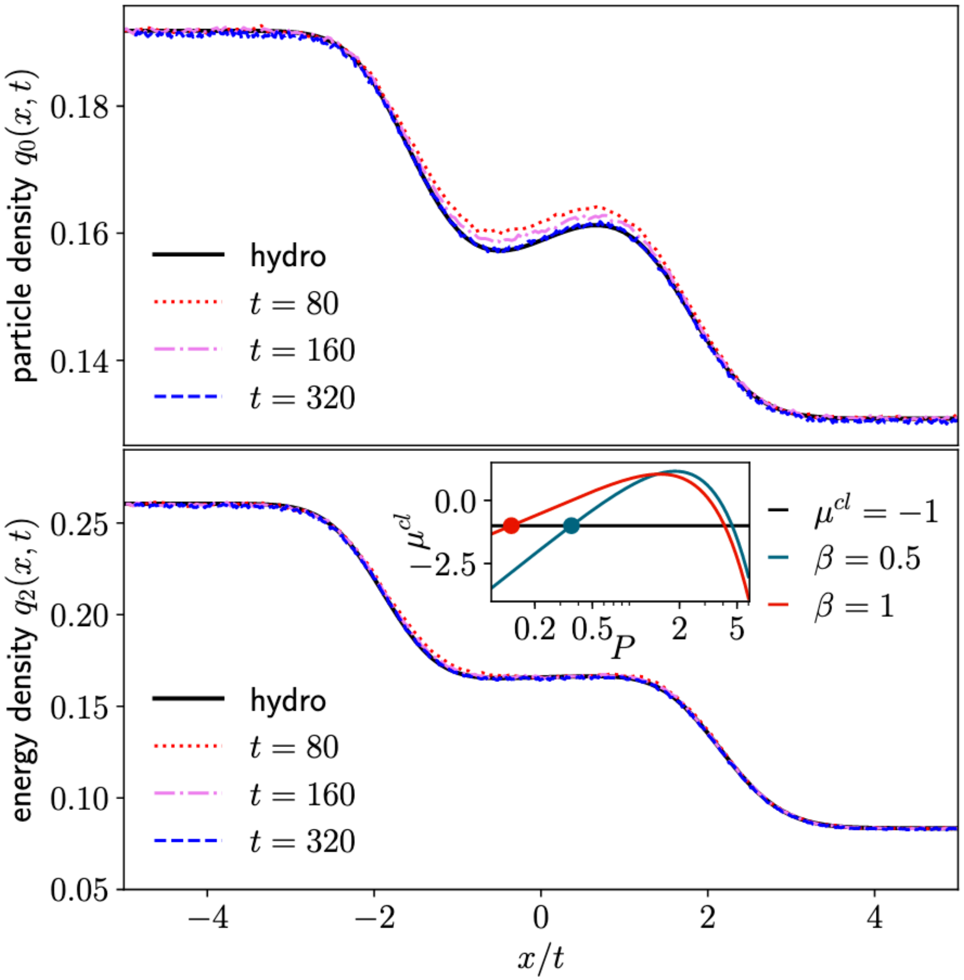
Interpretations: Particles arrive either from the left reservoir or the right reservoir depending on their effective velocity at the ray ξ.



GHD solution to Riemann problem

GHD for classical Toda [BD 2019; Bulchandani, Cao, Moore 2019].

Here Riemann problem numerics from [Bulchandani, Cao, Moore 2019].



Hydrodynamic matrices and correlation functions in GHD

The hydrodynamic matrices **follow from the expressions of average densities and currents** [BD, Spohn 2017]:

$$\mathsf{C}_{ij} = \int \mathrm{d}\theta\, \rho_{\mathrm{p}}(\theta) f(\theta) h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta)$$

$$(\mathsf{AC})_{ij} = \int \mathrm{d}\theta\, \rho_{\mathrm{p}}(\theta) f(\theta) v^{\mathrm{eff}}(\theta) h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta)$$

$$\mathsf{D}_{ij} = \int \mathrm{d}\theta\, \rho_{\mathrm{p}}(\theta) f(\theta) v^{\mathrm{eff}}(\theta)^2 h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta)$$

where

$$f(\theta) = -\frac{F''(\epsilon(\theta))}{F'(\epsilon(\theta))}$$

encodes the statistics, and the dressing operation

$$h^{\mathrm{dr}}(\theta) = h(\theta) + \int \mathrm{d}\alpha\, \varphi(\theta,\alpha) n(\alpha) h^{\mathrm{dr}}(\alpha)$$

diagonalises the flux Jacobian: $\sum_j \mathsf{A}_i^{\,j} h_j^{\mathrm{dr}}(\theta) = v^{\mathrm{eff}}(\theta) h_i^{\mathrm{dr}}(\theta)$.

Hydrodynamic matrices and correlation functions in GHD

From this, one can find expressions for **correlation functions from hydrodynamic projections** [BD 2018; Bastianello, BD, Watts, Yoshimura 2018]

$$\lim_{\substack{k\to 0,\, t\to\infty \\ kt \text{ fixed}}} \int \mathrm{d}x\, e^{\mathrm{i}kx} \langle \mathcal{O}(x,t)\mathcal{O}'(0,0)\rangle^{\mathrm{c}}_{\boldsymbol{\beta}} = \int \mathrm{d}\theta\, e^{\mathrm{i}kv^{\mathrm{eff}}(\theta)t} \rho_{\mathrm{p}}(\theta) f(\theta) V^{\mathcal{O}}(\theta) V^{\mathcal{O}'}(\theta)$$

For conserved densities:

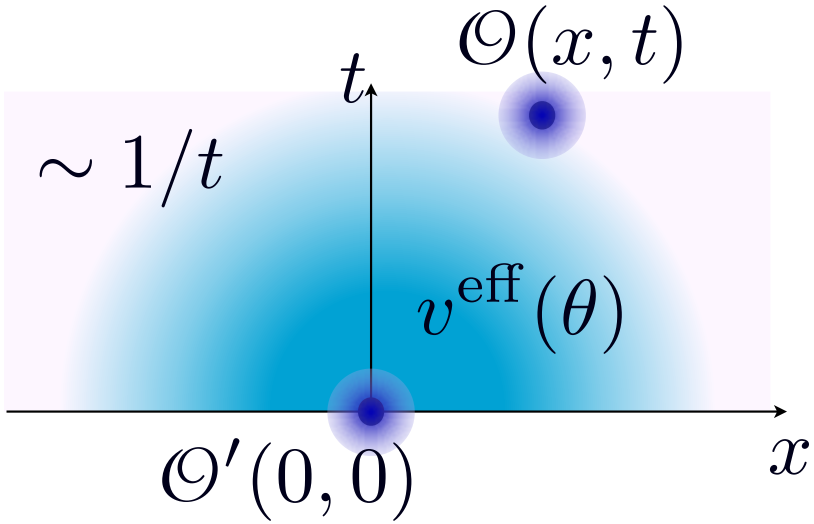
$$V^{q_i}(\theta) = h_i^{\mathrm{dr}}(\theta)$$

and for more general observables, simply differentiate GGE averages:

$$\frac{\partial}{\partial \beta^i} \langle \mathcal{O}\rangle_{\boldsymbol{\beta}} = \int \mathrm{d}\theta\, \rho_{\mathrm{p}}(\theta) f(\theta) h_i^{\mathrm{dr}}(\theta) V^{\mathcal{O}}(\theta)$$

Hydrodynamic matrices and correlation functions in GHD

As there is a continuum of effective velocities, generically **the decay is in $1/t$ everywhere**.



Hydrodynamic matrices and correlation functions in GHD

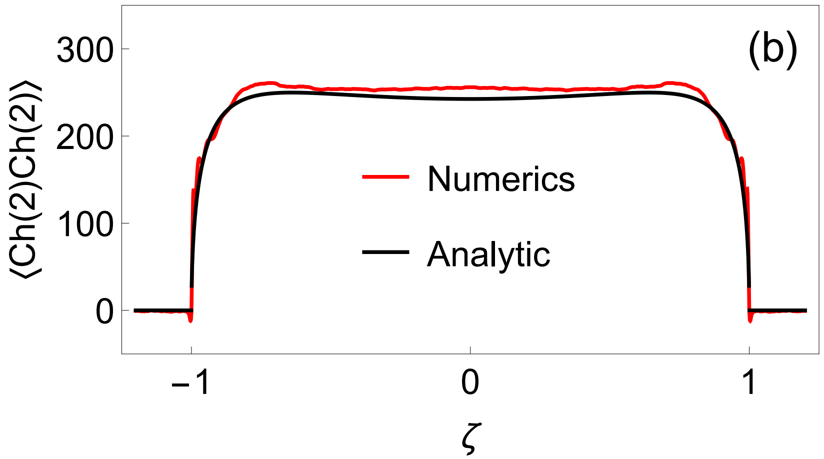
Numerical check of hydrodynamic projections. Performing Monte Carlo simulations in the **classical sinh-Gordon model** [Bastianello, BD, Watts, Yoshimura 2018]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{g^2}(\cosh(g\Phi) - 1)$$

we evaluate, in a thermal state, the two-point function

$$\langle\cosh(2g\Phi(x,t))\cosh(2g\Phi(0,0))\rangle^{\rm c}_{\rm thermal}$$

at large space-time distances on a ray $\zeta = x/t$.



Conclusion

We have a very general formalism for explicit solutions of integrable systems in nonequilibrium setups. A lot has been done:

- GHD for Lieb-Liniger model [Castro Alvaredo, BD, Yohimura 2016], classical Toda gas/chain [BD 2019; Bulchandani, Cao, Moore 2019; Spohn 2019], XXZ spin chain [Bertini, Collura, De Nardis, Fagotti 2016], Hubbard model [Ilievski, De Nardis 2017], classical hard rod gas [Boldrighini, Dobrushin, Sukhov 1982; BD, Yoshimura, Caux 2017], soliton gases [Zakharov 1971; El 2003; El, Kamchatnov 2005; El, Kamchatnov, Pavlov, Zykov 2011; BD, Yoshimura, Caux 2017], quantum sinh-Gordon model [Castro Alvaredo, BD, Yohimura 2016], classical sinh-Gordon model [Bastianello, BD, Watts, Yoshimura 2018], quantum sine-Gordon model [Bertini, Piroli, Kormos 2019].
- Exact Full counting statistics for nonequilibrium transport (largely generalising the Levitov-Lesovik formula) [Myers, Bhaseen, Harris, BD 2018; BD, Myers 2019].
- Exact exponential decay of correlation functions of twist fields [BD, Myers 2019].

- Integral-equation solution to GHD equations [BD, Spohn, Yoshimura 2017], integrability structure [El 2003; El, Kamchatnov 2005; El, Kamchatnov, Pavlov, Zykov 2011; BD, Yoshimura, Caux 2017Bulchandani 2017] and various numerical techniques [BD, Yoshimura, Caux 2017; Bulchandani, Vasseur, Karrasch, Moore 2017; BD, Dubail, Konik, Yoshimura 2018].
- Evolution in external potentials [BD, Yoshimura 2017] and solution to the famous quantum Newton cradle [Caux, BD, Dubail, Konik, Yoshimura 2018], and with slowly varying couplings [Bastianello, Alba, Caux 2019].
- Exact Drude weights [Ilievski, De Nardis 2017; BD, Spohn 2017; Bulchandani, Vasseur, Karrasch, Moore 2018] and diffusion operator [De Nardis, Bernard, BD 2018, 2019].
- Experimental verification in cold atomic gases [Schemmer, Bouchoule, BD, Dubail 2019].