Bloch theory and spectral gaps for linearized water waves

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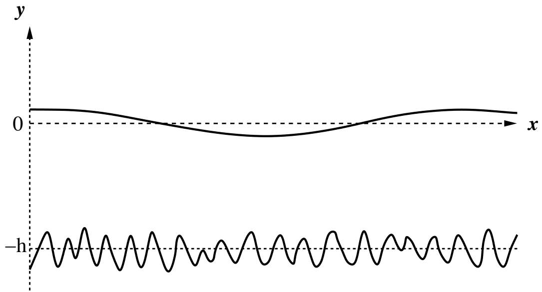
Wave propagation over variable topography



- Problem of significance in coastal regions where waves strongly affected by the bottom topography.
- Shallow water regime.
- Long-shore sandbars along gentle beaches. Narrow submarine sand ridges, lying in shallow, near-shore waters, approximately parallel to the beach.

Fig. 1. Aerial view of submarine longshore bars, Escambia Bay, Florida.

Figure: Aerial view of submarine longshore bars, Escambia Bay, Florida, (Lau-Trevis, J. Geophysical Research, 1973).

Figure: Zoom of the aerial view of submarine longshore bars.

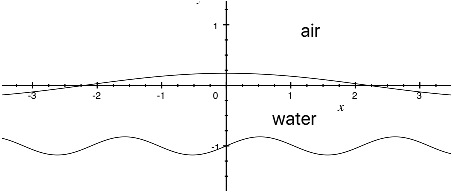
In shallow, near-shore waters, oriented approximately parallel to the beach.

Free surface water waves : Euler equation

Free boundary problem. Time-dependent 2D fluid domain:

$$\mathcal{S}[b,\eta] = \{(x,y) \in \mathbb{R}^2, -h + b(x) < y < \eta(x,t)\}$$

delimited by a fixed bottom and a free surface $y = \eta(x,t)$.



$u(x,y,t)$: velocity of a particle of fluid located at (x,y), at time t.

- Irrotational : $curl\ u = 0 \Rightarrow$ Potential flow: $u = \nabla\varphi$
- Incompressible: $div\ u = 0 \Rightarrow \quad \Delta\varphi = 0$ in $\mathcal{S}[b,\eta]$

The water wave problem as a Hamiltonian system

V.E. Zakharov (1968)

- Hamiltonian = Total energy = kinetic + potential

$$H = \int_{\mathbb{R}} \int_{-h+b(x)}^{\eta(x,t)} \frac{1}{2} |\nabla \varphi|^2 \, dy dx + \int_{\mathbb{R}} \frac{g}{2} \eta^2 \, dx$$

- Canonical variables : (η, ξ)
  $\eta =$ Surface elevation
  $\xi = \varphi(x, \eta(x,t), t)$ = Trace of velocity potential on $y = \eta$

- Hamilton's canonical equations :

$$\partial_t \xi = -\frac{\delta H}{\delta \eta}; \qquad \partial_t \eta = \frac{\delta H}{\delta \xi}$$

$$\begin{cases} \partial_t \eta - G[\eta, b]\xi = 0, \\ \partial_t \xi + g\eta + \frac{1}{2}|\partial_x \xi|^2 - \dfrac{(G[\eta, b]\xi + \partial_x \eta \cdot \partial_x \xi)^2}{2(1 + |\partial_x \eta|^2)} = 0 \,. \end{cases}$$

$\eta, \xi : (x, t) \in \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; g : acceleration due to gravity.
The operator $G[\eta, b]$ is the Dirichlet – Neumann operator:

$$G[\eta, b]\xi = \sqrt{1 + |\partial_x \eta|^2}\, \partial_n \varphi_{|_{y=\eta}} \,,$$

where φ is the solution of the elliptic boundary value problem

$$\begin{cases} \partial_x^2 \varphi + \partial_y^2 \varphi = 0 \quad \text{in} \quad \mathcal{S}(b, \eta) \,, \\ \varphi_{|_{y=\eta}} = \xi, \qquad \partial_n \varphi_{|_{y=-h+b}} = 0 \,. \end{cases}$$

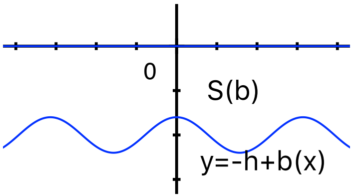
Linearized water waves

2D water wave system linearized near the stationary state $(\eta(x), \xi(x)) = (0,0)$ over bottom $y = -h + b(x)$

$$\begin{cases} \partial_t \eta - G[b]\xi = 0 \\ \partial_t \xi + g\eta = 0, \end{cases}$$

$\eta, \xi : (x,t) \in \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; g : acceleration due to gravity.
G[b] is the Dirichlet-Neumann operator for the domain $S(b) = \{(x,y), -h + b(x) < y < 0\}$:

$$\xi \ \to \ G[b]\xi = \partial_y \varphi_{|_{y=0}} \ , \quad \text{(nonlocal, 1}^{\text{st}} \text{ order operator)}$$



φ solution of Laplace

$$\begin{cases} \Delta\varphi = 0 \quad \text{in} \quad S(b) \\ \varphi_{|_{y=0}} = \xi \ , \quad \partial_n \varphi_{|_{y=-h+b}} = 0 \ . \end{cases}$$

$$\partial_{tt}\eta + gG[b]\eta = 0.$$

Initial conditions: $\eta(x,0) = \eta_0(x), \;\; \eta_t(x,0) = \eta_1(x), \;\; x \in \mathbb{R}.$

Assume: $b(x)$ is 2π-periodic ; $\int_0^{2\pi} b(x)dx = 0$; $h - b(x) \geq c_0 > 0.$

Analog of the wave equation, with $(-\partial_{xx})$ replaced by nonlocal operator $G[b]$ whose coefficients are 2π-periodic in x.

Flat bottom ($b = 0$): $\partial_{tt}\widehat{\eta}(k,t) - gk\tanh(hk)\widehat{\eta}(k,t) = 0.$

Non-flat bottom: Exact theory. Construction of time periodic solutions

$\rightarrow$ Spectrum of DN operator $G[b]$ on $\mathbb{R}$.

Bloch decomposition.

Spectral decomposition for differential operators with periodic coefficients – a classical tool to study wave propagation in periodic media.

- The principle of Bloch spectral decomposition is to parametrize the continuous spectrum and the generalized eigenfunctions of a given operator L (here $G[b]$) on $\mathbb{R}$ with a family (parametrized by θ) of spectral problems for L with *θ-periodic boundary conditions*, which, in turn, can be transformed to spectral problems with (usual) *periodic boundary conditions*.
- Our theory constructs the spectrum of $G[b]$ as a sequence of bands separated by gaps.

  It is the analog of the structure of spectral bands and gaps of the Hill's operator $-\partial_{xx} + V$, V periodic.
- It is a perturbative method with respect to bottom perturbations. Provides explicit formulas for spectral gaps, therefore intervals of 'forbidden' modes of the linearized water wave problem over periodic bottom.

Spectral problem for Dirichlet-Neumann operator $G[b]$

Find Bloch eigenvalues and eigenfunctions

$$G[b]\Phi(x,\theta) = \Lambda(\theta)\Phi(x,\theta) \tag{1}$$

with boundary conditions: $\Phi(x,\theta)$ θ-*periodic*, i.e.

$$\Phi(x+2\pi,\theta) = \Phi(x,\theta)e^{2\pi i\theta}, \quad 0 \le \theta < 1. \tag{2}$$

Let $\psi(x,\theta) = e^{-i\theta x}\Phi(x,\theta)$. Problem (1)-(2) becomes an e.v problem with *periodic boundary conditions*:

$$\begin{cases} G_\theta\psi(x,\theta) := \left(e^{-i\theta x}G[b]e^{i\theta x}\right)\psi(x,\theta) = \Lambda(\theta)\psi(x,\theta) \\ \psi(x+2\pi,\theta) = \psi(x,\theta)\ . \end{cases}$$

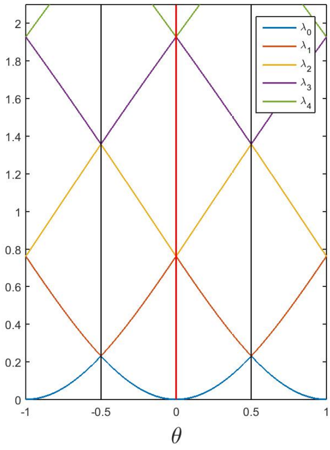
The case of a flat bottom

When $b(x) = 0$, G_θ is diagonal in Fourier space variables.
Its eigenvalues and eigenfunctions are ($n \in \mathbb{Z}, \theta \in [0,1)$)

$$\begin{cases} \Lambda_n^{(0)}(\theta) = (n+\theta)\tanh(h(n+\theta)) \\ \psi_n(x,\theta) = e^{inx} \,. \end{cases}$$

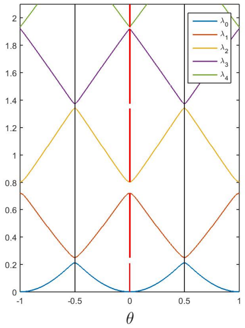
Eigenvalues are simple for : $0 < \theta < \frac{1}{2}$ and $\frac{1}{2} < \theta < 1$.
Eigenvalues have multiplicity 2 if $\theta = 0, \frac{1}{2}$.

– Relabeled in order of increasing amplitude, the e.v. are *continuous*.
– Extend them by periodicity in θ with period 1.
– With this ordering, associated e.f. $\Phi_n^{(0)}(x,\theta)$ periodic in θ, of period 1.



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
θ
λ_0
λ_1
λ_2
λ_3
λ_4

In the presence of a generic periodic bottom, spectral curves which meet when $b = 0$ typically separate, creating spectrum gaps corresponding to zones of forbidden energies.



Spectrum of $G[b]$ on $L^2(\mathbb{R})$:
union of the ranges of Bloch e.v

$$\sigma_{L^2(\mathbb{R})}(G[b]) = \cup_{n=0}^{+\infty}[\Lambda_n^-, \Lambda_n^+]$$

[red intervals] where
$\Lambda_n^- = \min_\theta \Lambda_n(\theta)$, $\Lambda_n^+ = \max_\theta \Lambda_n(\theta)$.

It is the analog of the structure of spectral bands and gaps of the Hill's operator.
The spectrum is purely continuous.

Important element of the analysis

1. Properties of Dirichlet-Neumann operator. It is self-adjoint from H^1 to L^2. ($b \in C^1$, $h - b(x) > c_0$)

   $$G[b] = G[0] + L[b], \quad G[0] = D \tanh(hD)$$

   $L[b]$: correction due to the presence of the topography.
2. Perturbation of a single e.v. : General theory of self-adjoint operators (Rellich, 1969).
3. Perturbation of a double eigenvalue. Look for a transformation that, when described in terms of Fourier modes, will reduce $G_\theta = e^{-i\theta x} G[b] e^{i\theta x}$ to a matrix operator that is block diagonal.

1. Properties of operator $L[b]$ where

$$G[b] = G[0] + L[b], \quad G[0] = D \tanh(hD)$$

(i) L[b] can be expressed in terms of integral operators

(ii) It is smoothing

(iii) It has an convergent Taylor expansion in powers of b

$$L[b] = \sum_k L^{(k)}[b]$$

with explicit formulas for $L^{(k)}[b]$.

(ii) Smoothing properties

$$M_\theta = e^{-i\theta x} L[b] e^{i\theta}$$

Proposition

Let $b \in$ ball $B_R(0) \subset C^1(\mathbb{T}^1)$, $f \in L^2(\mathbb{T}^1)$, then $M_\theta f$ is also periodic of period 2π and

$$\|M_\theta f\|_{L^2} \ \leq \ C(|b|_{C^1}) \ \|f\|_{L^2}$$

$$\|M_\theta f\|_{H^s} \ \leq \ C(|b|_{C^1}) \ \|f\|_{H^{-r}}$$

In order to *quantify the smoothing properties of (hermitian) operator M_θ*, introduce an operator norm on M_θ in terms of its action on Fourier modes: Let $(M_\theta)_{jl} = \langle e^{ijx}, M_\theta e^{ilx} \rangle$,

$$\|M_\theta\|_{\rho,r} = \sup_j \sum_l \ e^{\rho|j|} \ (M_\theta)_{jl} \ e^{\rho l} \ \langle j - l \rangle^r$$

Proposition

If $(h - b(x)) > h/2$, then

$$\|M_\theta\|_{h/2,r} < \infty$$

(iii) It has an convergent Taylor expansion in powers of b (Lannes 2013)

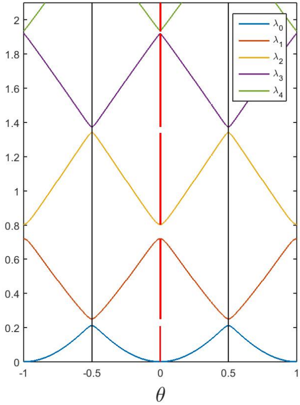
$$L[b] = \sum_k L^{(k)}[b]$$

with explicit formulas for $L^{(k)}[b]$ (Craig-Guyenne-Nicholls-S 2005).

$$L^{(1)}[b] = -D\mathrm{sech}(hD)\ b(x)\ D\mathrm{sech}(hD)$$

$$L^{(2)}[b] = -D\mathrm{sech}(hD)\, b\, D\tanh(hD)\, b\, D\mathrm{sech}(hD)$$

The spectral problem: Opening of a gap



Gap opening between e.v Λ_{2n-1} and Λ_{2n} near $\theta = 0$.

P_n: orthogonal projection in $L^2[0,2\pi]$ onto subspace $\{e^{inx}, e^{-inx}\}$.
Write $(I = P_n + (I - P_n))$

$$G_\theta := G_\theta^{(0)} + M_\theta = P_n(G_\theta^{(0)} + M_\theta)P_n + (I - P_n)(G_\theta^{(0)} + M_\theta)P_n + P_n(G_\theta^{(0)} + M_\theta)(I - P_n) + (I - P_n)(G_\theta^{(0)} + M_\theta)(I - P_n) \; .$$

A simplified, finite-dimensional model.

- Drop the last 3 terms
- Simplify M_θ by taking into account only one term in Taylor series expansion in b, i.e.

$$M_\theta \sim M_\theta^{(1)} = -e^{-i\theta x} \, D\mathrm{sech}(hD) b(x) D\mathrm{sech}(hD) \, e^{i\theta x} \; .$$

Acting on Fourier coefficients of a periodic function, the operator $P_n(G_\theta^{(0)} + M_\theta^{(1)})P_n$ is represented by a 2×2 matrix

$$A = \begin{pmatrix} g_n(\theta) & \widehat{b}_{2n} s_n(\theta) s_n(-\theta) \\ \overline{\widehat{b}}_{2n} s_n(\theta) s_n(-\theta) & g_n(-\theta) \end{pmatrix},$$

where $g_n(\theta) = (n+\theta)\tanh(h(n+\theta)),\ s_n(\theta) = (n+\theta)\mathrm{sech}(h(n+\theta))$.

$$A = O\Lambda O^*, \quad O = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = e^T \quad \text{rotation}$$

$$T = \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix},\ \Lambda = \begin{pmatrix} \Lambda_{2n-1}(\theta) & 0 \\ 0 & \Lambda_{2n}(\theta) \end{pmatrix}.$$

$\Lambda_{2n-1}, \Lambda_{2n} =$

$$\tfrac{1}{2}\big(g_n(\theta) + g_n(-\theta)\big) \pm \tfrac{1}{2}\sqrt{(g_n(\theta) - g_n(-\theta))^2 + 4|\widehat{b}_{2n}|^2 s_n(\theta)^2 s_n(-\theta)^2}.$$

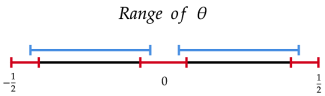
Assuming $\widehat{b}_{2n} \neq 0$, e.v split near $\theta = 0$, $\Lambda_{2n}(0) - \Lambda_{2n-1}(0) = \mathcal{O}(|\widehat{b}_{2n}|)$.

The general case

$\theta \in [-\frac{1}{2}, \frac{1}{2})$: 2 cases

(i) θ far from 0, $\pm\frac{1}{2}$: perturbation of a single e.v.

(ii) θ near 0 and $\pm\frac{1}{2}$, separation of a double e.v, opening of a gap



- Fix n. Seek a unitary transformation $O = e^T$ parametrized by operators T satisfying $T^* = -T$, such that, when acting on Fourier series, reduces the operator

$$H_\theta := e^{-T}(G_\theta + M_\theta[b])e^T \text{ to be block diagonal.}$$

Using $I = P_n + (I - P_n)$, solve

$$F_\theta^n(T_\theta^n, M_\theta) := P_n H_\theta (I - P_n) + (I - P_n) H_\theta P_n = 0$$

$F_\theta^n(0,0) = 0$.
Given b (equivalently M_θ), find T_θ^n.

Existence of such transformation obtained from Implicit Function Theorem in spaces of operators. (Nirenberg, Courant Inst. Lect. Notes 2001).

Spaces. $\mathcal{L}_{\rho,r}$: space of linear operators from h^r to itself $(r \geq 1)$ equipped with the norm

$$\|L\|_{\rho,r} = (\sup_j \sum_l e^{\rho|j|} \, |(L)_{jl}| \, e^{\rho|l|} \, \langle j-l \rangle^r)^{1/2} (\sup_l \sum_j (*))$$

$\mathcal{H}_{\rho,r}$: its subspace of Hermitian symmetric operators
$\mathcal{A}_{\rho,r}$: its subspace of anti-Hermitian operators
$\mathcal{P}_{\rho,r}$: subspace of $\mathcal{A}_{\rho,r}$ with additional property:

$$T(P_n L^2) \subseteq (I - P_n)L^2, \quad \text{and} \quad T((I - P_n)L^2) \subseteq P_n L^2 \ .$$

Proposition

(i) There exists R, such that for $b \in B_R(0) \subset C^1(\mathbb{T}^1)$*, there exists* $T_\theta^n \in \mathcal{P}_{\rho,r}$*. The neighborhood can be chosen independent of n.*
(ii)The e.v. $\Lambda_{2n-1}(\theta), \Lambda_{2n}(\theta)$ *obtained as e.v. of* 2×2 *matrix* $P_n H_\theta P_n$*.*

Statements of results

(Craig-Gazeau-Lacave-S. SIMA 2018, Craig-S. 2019)

Fix $b \in B_R(0) \subseteq C^1(\mathbb{T}^1)$, R small enough.
For $\theta \in \mathbb{T}^1$, the spectrum of $G_\theta[b]$ on $L^2(\mathbb{T}^1)$ is composed of a non-decreasing sequence of eigenvalues

$$\Lambda_0(\theta) \le \Lambda_1(\theta) \le \cdots \le \Lambda_n(\theta) \le \cdots$$

continuous and periodic in θ, continuous in b.
(i) Each parameter interval $\theta \in [\frac{1}{16}, \frac{7}{16}]$ contains only simple spectrum $\Lambda_n(\theta)$ which is analytic in (θ, b), for $n = 1, 2, \ldots$.

(ii) For $\theta \in [-\frac{1}{8} . \frac{1}{8}]$, the spectrum $\Lambda_n(\theta)$ is simple or double. The eigenvalues $\Lambda_n(\theta)$ are continuous in θ and b.
The subspace spanned by $\{\psi_{2n-1}, \psi_{2n}\}$ is analytic in (θ, b).

(iii) The lowest eigenvalue $\Lambda_0(\theta)$ is simple and it and the eigenfunction $\psi_0(x,\theta)$ are analytic in θ and b. $\Lambda_0(0) = 0$ *for any* $b(x)$*, and its corresponding eigenfunction is* $\Phi(x,0) = 1$.

Similar results are true for $\theta \in [\frac{9}{16}, \frac{15}{16}]$ *and* $\theta \in [\frac{3}{8}, \frac{5}{8}]$.

(iv) Spectrum of $G[b]$ *on* $L^2(\mathbb{R})$*: union of the ranges of Bloch e.v*

$$\sigma_{L^2(\mathbb{R})}(G[b]) = \cup_{n=0}^{+\infty}[\Lambda_n^-, \Lambda_n^+]$$

where $\Lambda_n^- = \min_{\theta\in\mathbb{T}^1} \Lambda_n(\theta)$ *and* $\Lambda_n^+ = \max_{\theta\in\mathbb{T}^1} \Lambda_n(\theta)$.

(v) There is no point spectrum to the Dirichlet-Neumann operator $G[b]$*. The entire spectrum is purely continuous.*

The example of $y = -h + b(x)$, $b(x) = \varepsilon \cos(x)$

Perturbation calculation.

- The 1st gap occurs for $\theta = \pm\frac{1}{2}$, is of order $O(\varepsilon)$

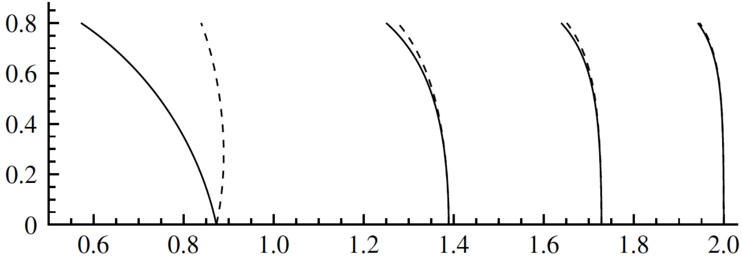
$$\Lambda_1^- - \Lambda_0^+ = \tfrac{1}{4}\mathrm{sech}^2(\frac{h}{2})\varepsilon \ .$$

- The 2nd gap occurs at $\theta = 0$. Unlike the case of the Mathieu operator, the second gap opens at order $O(\varepsilon^4)$:

$$\Lambda_2^- - \Lambda_1^+ = \tfrac{1}{12}\varepsilon^4 \mathrm{sech}^2(h)\tanh(2h).$$

- Unlike the case of the Mathieu operator, the n^{th} gap does not necessarily open at order ε^n.

Yu and Howard JFM 2012

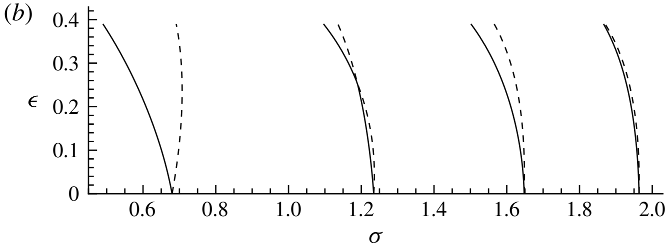


The n^{th} gap satisfies

$$\Lambda_n^- - \Lambda_{n-1}^+ \leq C(n)\varepsilon^n.$$

Gaps are not guaranteed to remain open as the size of bottom variations increase.

Yu and Howard JFM 2012



Thank you.