Bloch theory and spectral gaps for linearized water waves

Catherine Sulem

University of Toronto

Classical and Quantum Integrability
Institut de Mathématiques de Bourgogne
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Collaborators

Walter Craig (1953-2019)

Maxime Gazeau (Borealis AI, Toronto)

Christophe Lacave (Université Grenoble-Alpes)
Wave propagation over variable topography
Problem of significance in coastal regions where waves strongly affected by the bottom topography.

Shallow water regime.

Long-shore sandbars along gentle beaches. Narrow submarine sand ridges, lying in shallow, near-shore waters, approximately parallel to the beach.
That some material is also furnished by the longshore transport is regarded as incidental and not vital to offshore bar formation. Although Hunt and Johns [1963] were among the first to attempt to explain the bar formation from the point of view of the standing wave mechanism, they calculated only the Lagrangian drift velocity components at the outer edge of the boundary layer. Since the direction and magnitude of sediment motion is most probably related to the mean transport velocity in the lowermost part of the Stokes boundary layer instead of that at the outer edge of the boundary layer, Carter et al. [1972] studied the mass transport velocity components throughout the bottom boundary layer in an oscillatory flow. The result of a normal incident and partially reflected wave over constant depth is used to interpret offshore sand bar formation and sediment transport. Erosional and depositional features are presented from laboratory tank experiments over a sandy bed for the full range of wave reflectivity to substantiate the theoretical predictions of the spacing between the sandbars. However, some observed features, such as the offshore dependence of spacing of the submarine longshore bars, can only be explained when bottom topography is included in the formulation. In this paper we will extend the analysis of Carter et al. [1972] to the case of a gently sloping beach. We will study the relationship between the spacing between the sandbars, the number of bars, and the bottom gradient. Emphasis will be focused on the standing wave mechanism of

Figure: Aerial view of submarine longshore bars, Escambia Bay, Florida, (Lau-Trevis, J. Geophysical Research, 1973).
In shallow, near-shore waters, oriented approximately parallel to the beach.
Free surface water waves: Euler equation

Free boundary problem. Time-dependent 2D fluid domain:

\[ S[b, \eta] = \{(x, y) \in \mathbb{R}^2, -h + b(x) < y < \eta(x, t)\} \]

delimited by a fixed bottom and a free surface \( y = \eta(x, t) \).

\( u(x, y, t) \): velocity of a particle of fluid located at \((x, y)\), at time \(t\).

- Irrotational: \( \text{curl} \ u = 0 \Rightarrow \text{Potential flow: } u = \nabla \varphi \)
- Incompressible: \( \text{div} \ u = 0 \Rightarrow \Delta \varphi = 0 \text{ in } S[b, \eta] \)
The water wave problem as a Hamiltonian system

V.E. Zakharov (1968)

- **Hamiltonian = Total energy = kinetic + potential**

\[ H = \int_{\mathbb{R}} \int_{-h+b(x)} \eta(x,t) \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx + \int_{\mathbb{R}} \frac{g}{2} \eta^2 \, dx \]

- **Canonical variables :** \((\eta, \xi)\)
  
  - \(\eta =\) Surface elevation
  - \(\xi = \varphi(x, \eta(x, t), t) =\) Trace of velocity potential on \(y = \eta\)

- **Hamilton’s canonical equations :**
  
  \[ \partial_t \xi = - \frac{\delta H}{\delta \eta}; \quad \partial_t \eta = \frac{\delta H}{\delta \xi} \]
\[
\begin{aligned}
\partial_t \eta - G[\eta, b] \xi &= 0, \\
\partial_t \xi + g \eta + \frac{1}{2} |\partial_x \xi|^2 - \frac{(G[\eta, b] \xi + \partial_x \eta \cdot \partial_x \xi)^2}{2(1 + |\partial_x \eta|^2)} &= 0.
\end{aligned}
\]

\(\eta, \xi : (x, t) \in \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R};\) \(g:\) acceleration due to gravity. The operator \(G[\eta, b]\) is the Dirichlet – Neumann operator:

\[G[\eta, b] \xi = \sqrt{1 + |\partial_x \eta|^2} \partial_n \varphi|_{y=\eta},\]

where \(\varphi\) is the solution of the elliptic boundary value problem

\[
\begin{aligned}
\partial_x^2 \varphi + \partial_y^2 \varphi &= 0 \quad \text{in} \quad S(b, \eta), \\
\varphi|_{y=\eta} &= \xi, \\
\partial_n \varphi|_{y=\eta} &= 0.
\end{aligned}
\]
Linearized water waves

2D water wave system linearized near the stationary state
\((\eta(x), \xi(x)) = (0, 0)\) over bottom \(y = -h + b(x)\)

\[
\begin{align*}
\begin{cases}
\partial_t \eta - G[b] \xi = 0 \\
\partial_t \xi + g\eta = 0,
\end{cases}
\end{align*}
\]

\(\eta, \xi : (x, t) \in \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R};\) \(g:\) acceleration due to gravity.
\(G[b]\) is the Dirichlet-Neumann operator for the domain
\(S(b) = \{(x, y), -h + b(x) < y < 0\}:

\[
\xi \rightarrow G[b] \xi = \partial_y \varphi |_{y=0}, \quad \text{(nonlocal, 1\textsuperscript{st} order operator)}
\]

\(\varphi\) solution of Laplace

\[
\begin{align*}
\begin{cases}
\Delta \varphi = 0 \quad \text{in} \ S(b) \\
\varphi |_{y=0} = \xi, \quad \partial_n \varphi |_{y=-h+b} = 0.
\end{cases}
\end{align*}
\]
\[ \partial_{tt} \eta + gG[b] \eta = 0. \]

Initial conditions: \( \eta(x, 0) = \eta_0(x), \quad \eta_t(x, 0) = \eta_1(x), \quad x \in \mathbb{R}. \)

Assume: \( b(x) \) is 2\( \pi \)-periodic; \( \int_0^{2\pi} b(x)dx = 0 \); \( h - b(x) \geq c_0 > 0. \)

Analog of the wave equation, with \( (-\partial_{xx}) \) replaced by nonlocal operator \( G[b] \) whose coefficients are 2\( \pi \)-periodic in \( x \).

Flat bottom \( (b = 0) \): \( \partial_{tt} \hat{\eta}(k, t) - gk \tanh(hk) \hat{\eta}(k, t) = 0. \)

Non-flat bottom: **Exact theory.** Construction of time periodic solutions
\( \rightarrow \) Spectrum of DN operator \( G[b] \) on \( \mathbb{R} \).
Bloch decomposition.

Spectral decomposition for differential operators with periodic coefficients – a classical tool to study wave propagation in periodic media.

- The principle of Bloch spectral decomposition is to parametrize the continuous spectrum and the generalized eigenfunctions of a given operator $L$ (here $G[b]$) on $\mathbb{R}$ with a family (parametrized by $\theta$) of spectral problems for $L$ with $\theta$-periodic boundary conditions, which, in turn, can be transformed to spectral problems with (usual) periodic boundary conditions.

- Our theory constructs the spectrum of $G[b]$ as a sequence of bands separated by gaps. It is the analog of the structure of spectral bands and gaps of the Hill’s operator $-\partial_{xx} + V$, $V$ periodic.

- It is a perturbative method with respect to bottom perturbations. Provides explicit formulas for spectral gaps, therefore intervals of ‘forbidden’ modes of the linearized water wave problem over periodic bottom.
Spectral problem for Dirichlet-Neumann operator $G[b]$

Find Bloch eigenvalues and eigenfunctions

$$G[b] \Phi(x, \theta) = \Lambda(\theta) \Phi(x, \theta)$$  \hspace{1cm} (1)

with boundary conditions: $\Phi(x, \theta)$ $\theta$-periodic, i.e.

$$\Phi(x + 2\pi, \theta) = \Phi(x, \theta) e^{2\pi i \theta}, \hspace{0.5cm} 0 \leq \theta < 1.$$  \hspace{1cm} (2)

Let $\psi(x, \theta) = e^{-i \theta x} \Phi(x, \theta)$. Problem (1)-(2) becomes an e.v problem with \textit{periodic boundary conditions}:

\begin{equation}
\begin{cases}
G_{\theta} \psi(x, \theta) := (e^{-i \theta x} G[b] e^{i \theta x}) \psi(x, \theta) = \Lambda(\theta) \psi(x, \theta) \\
\psi(x + 2\pi, \theta) = \psi(x, \theta).
\end{cases}
\end{equation}
The case of a flat bottom

When $b(x) = 0$, $G_\theta$ is diagonal in Fourier space variables. Its eigenvalues and eigenfunctions are $(n \in \mathbb{Z}, \theta \in [0, 1))$

$$\begin{cases} 
\Lambda_n^{(0)}(\theta) = (n + \theta) \tanh(h(n + \theta)) \\
\psi_n(x, \theta) = e^{inx}.
\end{cases}$$

Eigenvalues are simple for: $0 < \theta < \frac{1}{2}$ and $\frac{1}{2} < \theta < 1$. Eigenvalues have multiplicity 2 if $\theta = 0, \frac{1}{2}$.

– Relabeled in order of increasing amplitude, the e.v. are continuous.
– Extend them by periodicity in $\theta$ with period 1.
– With this ordering, associated e.f. $\Phi_n^{(0)}(x, \theta)$ periodic in $\theta$, of period 1.
In the presence of a generic periodic bottom, spectral curves which meet when \( b = 0 \) typically separate, creating spectrum gaps corresponding to zones of forbidden energies.

Spectrum of \( G[b] \) on \( L^2(\mathbb{R}) \):
union of the ranges of Bloch e.v

\[
\sigma_{L^2(\mathbb{R})}(G[b]) = \bigcup_{n=0}^{+\infty} [\Lambda_n^-, \Lambda_n^+]
\]

[red intervals] where
\[
\Lambda_n^- = \min_\theta \Lambda_n(\theta), \Lambda_n^+ = \max_\theta \Lambda_n(\theta).
\]

It is the analog of the structure of spectral bands and gaps of the Hill’s operator. The spectrum is purely continuous.
Important element of the analysis

1. Properties of Dirichlet-Neumann operator. It is self-adjoint from $H^1$ to $L^2$. ($b \in C^1, h - b(x) > c_0$)

$$G[b] = G[0] + L[b], \quad G[0] = D \tanh(hD)$$

$L[b]$ : correction due to the presence of the topography.


3. Perturbation of a double eigenvalue. Look for a transformation that, when described in terms of Fourier modes, will reduce

$$G_\theta = e^{-i\theta x} G[b] e^{i\theta x}$$

to a matrix operator that is block diagonal.
1. Properties of operator $L[b]$ where

$$G[b] = G[0] + L[b], \quad G[0] = D \tanh(hD)$$

(i) $L[b]$ can be expressed in terms of integral operators

(ii) It is smoothing

(iii) It has a convergent Taylor expansion in powers of $b$

$$L[b] = \sum_k L^{(k)}[b]$$

with explicit formulas for $L^{(k)}[b]$. 
(ii) Smoothing properties

\[ M_\theta = e^{-i\theta x} L[b] e^{i\theta} \]

**Proposition**

Let \( b \in \text{ball } B_R(0) \subset C^1(\mathbb{T}^1) \), \( f \in L^2(\mathbb{T}^1) \), then \( M_\theta f \) is also periodic of period \( 2\pi \) and

\[
\| M_\theta f \|_{L^2} \leq C(\| b \|_{C^1}) \| f \|_{L^2} \\
\| M_\theta f \|_{H^s} \leq C(\| b \|_{C^1}) \| f \|_{H^{−r}}
\]

In order to **quantify the smoothing properties of** (hermitian) operator \( M_\theta \), introduce an operator norm on \( M_\theta \) in terms of its action on Fourier modes:

Let \( (M_\theta)_{jl} = \langle e^{ix}, M_\theta e^{ilx} \rangle \),

\[
\| M_\theta \|_{\rho,r} = \sup \sum_j e^{\rho |j|} (M_\theta)_{jl} e^{\rho l} (j - l)^r
\]

**Proposition**

If \( (h - b(x)) > h/2 \), then

\[
\| M_\theta \|_{h/2,r} < \infty
\]
(iii) It has a convergent Taylor expansion in powers of $b$ (Lannes 2013)

$$L[b] = \sum_{k} L^{(k)}[b]$$

with explicit formulas for $L^{(k)}[b]$ (Craig-Guyenne-Nicholls-S 2005).

$$L^{(1)}[b] = -D\text{sech}(hD) \, b(x) \, D\text{sech}(hD)$$

$$L^{(2)}[b] = -D\text{sech}(hD) \, b \, D \, \tanh(hD) \, b \, D\text{sech}(hD)$$
The spectral problem: Opening of a gap

Gap opening between e.v $\Lambda_{2n-1}$ and $\Lambda_{2n}$ near $\theta = 0$. 
$P_n$: orthogonal projection in $L^2[0, 2\pi]$ onto subspace $\{e^{inx}, e^{-inx}\}$.

Write $(I = P_n + (I - P_n))$

\[ G_\theta := G_\theta^{(0)} + M_\theta = P_n(G_\theta^{(0)} + M_\theta)P_n + (I - P_n)(G_\theta^{(0)} + M_\theta)P_n \]
\[ + P_n(G_\theta^{(0)} + M_\theta)(I - P_n) + (I - P_n)(G_\theta^{(0)} + M_\theta)(I - P_n). \]

A simplified, finite-dimensional model.

- Drop the last 3 terms
- Simplify $M_\theta$ by taking into account only one term in Taylor series expansion in $b$, i.e.

\[ M_\theta \sim M^{(1)}_\theta = -e^{-i\theta x} D\text{sech}(hD)b(x)D\text{sech}(hD) e^{i\theta x}. \]
Acting on Fourier coefficients of a periodic function, the operator 
\( P_n(G^{(0)}_\theta + M^{(1)}_\theta)P_n \) is represented by a \( 2 \times 2 \) matrix 
\[
A = \begin{pmatrix}
g_n(\theta) & \hat{b}_{2n}s_n(\theta)s_n(-\theta) \\
\hat{b}_{2n}s_n(\theta)s_n(-\theta) & g_n(-\theta)
\end{pmatrix},
\]
where 
\( g_n(\theta) = (n + \theta) \tanh(h(n + \theta)) \), 
\( s_n(\theta) = (n + \theta) \text{sech}(h(n + \theta)) \).

\[
A = O\Lambda O^*, \quad O = \begin{pmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{pmatrix} = e^T \text{ rotation}
\]

\[
T = \begin{pmatrix}
0 & \varphi \\
-\varphi & 0
\end{pmatrix}, \quad \Lambda = \begin{pmatrix}
\Lambda_{2n-1}(\theta) & 0 \\
0 & \Lambda_{2n}(\theta)
\end{pmatrix}.
\]

\( \Lambda_{2n-1}, \Lambda_{2n} = \)
\[
\frac{1}{2}(g_n(\theta) + g_n(-\theta)) \pm \frac{1}{2} \sqrt{(g_n(\theta) - g_n(-\theta))^2 + 4|\hat{b}_{2n}|^2s_n(\theta)^2s_n(-\theta)^2}.
\]

Assuming \( \hat{b}_{2n} \neq 0 \), e.v split near \( \theta = 0 \), 
\( \Lambda_{2n}(0) - \Lambda_{2n-1}(0) = \mathcal{O}(|\hat{b}_{2n}|) \).
The general case

\( \theta \in [-1/2, 1/2) : \) 2 cases

(i) \( \theta \) far from 0, \( \pm \frac{1}{2} \): perturbation of a single e.v.

(ii) \( \theta \) near 0 and \( \pm \frac{1}{2} \), separation of a double e.v, opening of a gap

Fix \( n \). Seek a unitary transformation \( O = e^T \) parametrized by operators \( T \) satisfying \( T^* = -T \), such that, when acting on Fourier series, reduces the operator

\[
H_\theta := e^{-T}(G_\theta + M_\theta[b])e^T
\]

to be block diagonal.
Using $I = P_n + (I - P_n)$, solve

$$
F^n_{\theta}(T^n_{\theta}, M_{\theta}) := P_n H_{\theta}(I - P_n) + (I - P_n) H_{\theta} P_n = 0
$$

$F^n_{\theta}(0, 0) = 0$. Given $b$ (equivalently $M_{\theta}$), find $T^n_{\theta}$.

**Spaces.** $\mathcal{L}_{\rho, r}$: space of linear operators from $h^r$ to itself ($r \geq 1$) equipped with the norm

$$
\|L\|_{\rho, r} = \left( \sup_j \sum_l e^{\rho |j|} |(L)_{jl}| e^{\rho |l|} \langle j - l \rangle^r \right)^{1/2} \left( \sup_l \sum_j (\ast) \right)
$$

$\mathcal{H}_{\rho, r}$: its subspace of Hermitian symmetric operators

$\mathcal{A}_{\rho, r}$: its subspace of anti-Hermitian operators

$\mathcal{P}_{\rho, r}$: subspace of $\mathcal{A}_{\rho, r}$ with additional property:

$$
T(P_nL^2) \subseteq (I - P_n)L^2, \quad \text{and} \quad T((I - P_n)L^2) \subseteq P_nL^2.
$$

**Proposition**

(i) There exists $R$, such that for $b \in B_R(0) \subset C^1(\mathbb{T}^1)$, there exists $T^n_\theta \in \mathcal{P}_{\rho, r}$. The neighborhood can be chosen independent of $n$.

(ii) The e.v. $\Lambda_{2n-1}(\theta), \Lambda_{2n}(\theta)$ obtained as e.v. of $2 \times 2$ matrix $P_nH_0P_n$. 
Statements of results

(Craig-Gazeau-Lacave-S. SIMA 2018, Craig-S. 2019)

Fix $b \in B_R(0) \subseteq C^1(\mathbb{T}^1)$, $R$ small enough.
For $\theta \in \mathbb{T}^1$, the spectrum of $G_\theta[b]$ on $L^2(\mathbb{T}^1)$ is composed of a non-decreasing sequence of eigenvalues

$$\Lambda_0(\theta) \leq \Lambda_1(\theta) \leq \cdots \leq \Lambda_n(\theta) \leq \cdots$$

continuous and periodic in $\theta$, continuous in $b$.

(i) Each parameter interval $\theta \in \left[\frac{1}{16}, \frac{7}{16}\right]$ contains only simple spectrum $\Lambda_n(\theta)$ which is analytic in $(\theta, b)$, for $n = 1, 2, \ldots$.

(ii) For $\theta \in \left[-\frac{1}{8}, \frac{1}{8}\right]$, the spectrum $\Lambda_n(\theta)$ is simple or double. The eigenvalues $\Lambda_n(\theta)$ are continuous in $\theta$ and $b$.

The subspace spanned by $\{\psi_{2n-1}, \psi_{2n}\}$ is analytic in $(\theta, b)$. 
(iii) The lowest eigenvalue \( \Lambda_0(\theta) \) is simple and it and the eigenfunction \( \psi_0(x, \theta) \) are analytic in \( \theta \) and \( b \). \( \Lambda_0(0) = 0 \) for any \( b(x) \), and its corresponding eigenfunction is \( \Phi(x, 0) = 1 \).

Similar results are true for \( \theta \in \left[ \frac{9}{16}, \frac{15}{16} \right] \) and \( \theta \in \left[ \frac{3}{8}, \frac{5}{8} \right] \).

(iv) Spectrum of \( G[b] \) on \( L^2(\mathbb{R}) \): union of the ranges of Bloch e.v

\[
\sigma_{L^2(\mathbb{R})}(G[b]) = \bigcup_{n=0}^{+\infty} [\Lambda_n^-, \Lambda_n^+] 
\]

where \( \Lambda_n^- = \min_{\theta \in \mathbb{T}^1} \Lambda_n(\theta) \) and \( \Lambda_n^+ = \max_{\theta \in \mathbb{T}^1} \Lambda_n(\theta) \).

(v) There is no point spectrum to the Dirichlet-Neumann operator \( G[b] \). The entire spectrum is purely continuous.
The example of $y = -h + b(x)$, $b(x) = \varepsilon \cos(x)$

Perturbation calculation.

- The 1st gap occurs for $\theta = \pm \frac{1}{2}$, is of order $O(\varepsilon)$

$$\Lambda_1^- - \Lambda_0^+ = \frac{1}{4} \text{sech}^2\left(\frac{h}{2}\right) \varepsilon.$$  

- The 2nd gap occurs at $\theta = 0$. Unlike the case of the Mathieu operator, the second gap opens at order $O(\varepsilon^4)$:

$$\Lambda_2^- - \Lambda_1^+ = \frac{1}{12} \varepsilon^4 \text{sech}^2(h) \tanh(2h).$$

- Unlike the case of the Mathieu operator, the $n^{th}$ gap does not necessarily open at order $\varepsilon^n$.  

The \( n^{th} \) gap satisfies

\[
\Lambda_n^- - \Lambda_{n-1}^+ \leq C(n)\varepsilon^n.
\]
Gaps are not guaranteed to remain open as the size of bottom variations increase.

Yu and Howard JFM 2012
Thank you.