Handling congestion in fluid equations
Part 1

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Fluid equations under maximal density constraint

- free boundary problem: free domain $\{\rho < \rho^*\}$ / congested domain $\{\rho = \rho^*\}$
- free-congested Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div} \mathbb{S} = f \\ 0 \leq \rho \leq \rho^*, \ (\rho^* - \rho) p = 0, \ p \geq 0 \end{cases}$$

Objectives

- motivation, applications in physics, geophysics (and biology)
- overview of theoretical (and numerical) analysis results

Application to the modeling of crowd motion

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ u = \mathrm{P}_{C_\rho}(U_{\mathrm{des}}) \end{cases}$$

space of admissible velocities $C_\rho = \left\{ v \in L^2, \ \int v \cdot \nabla q \leq 0, \ \forall q \in H^1_\rho \right\}$

space of pressures $H^1_\rho = \left\{ q \in H^1, q \geq 0 \text{ a.e.}, \ q = 0 \text{ a.e. in } \{\rho = 1\} \right\}$

↳ "$\operatorname{div} u \geq 0$ where $\rho = 1$"

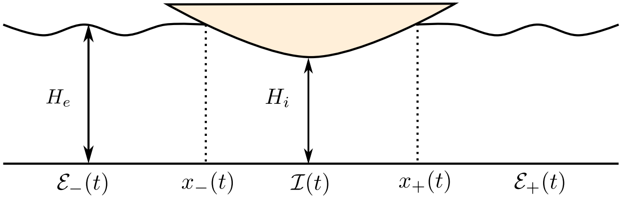
saddle-point formulation find $(u, p) \in L^2 \times H^1_\rho$ such that

$$\begin{cases} u + \nabla p = U_{\mathrm{des}} \\ \int u \cdot \nabla q \leq 0 \quad \forall q \in H^1_\rho \end{cases}$$

ref: Maury, Roudneff-Chupin, Santambrogio '10

Applications in physics - wave-structure interaction

- mixed flows in closed channels, floating structures



shallow water approximation

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{gh^2}{2} \right) = -\frac{1}{\rho} h \partial_x p \\ 0 \leq h \leq h_{\mathrm{sol}},\ (h_{\mathrm{sol}} - h)(p - p_{\mathrm{atm}}) = 0 \end{cases}$$

ref: Bourdarias, et al. '12, Lannes '17, Godlewski et al. '18

Applications in physics - modeling of two-phase flows

- 1D biphasic equations for liquid-gas mixtures, α volume fraction of the liquid

$$\begin{cases} \partial_t(\alpha\rho_l) + \partial_x(\alpha\rho_l u_l) = 0 \\ \partial_t((1-\alpha)\rho_g) + \partial_x((1-\alpha)\rho_g u_g) = 0 \\ \partial_t(\alpha\rho_l u_l) + \partial_x(\alpha\rho_l u_l^2) + \alpha\partial_x p + \tau_l = M \\ \partial_t((1-\alpha)\rho_g u_g) + \partial_x((1-\alpha)\rho_g u_g^2) + (1-\alpha)\partial_x p + \tau_g = -M \end{cases}$$

$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \mathrm{cst}, \quad p = a\rho_g^\gamma = a\bar{\rho}_g^\gamma\ (\tilde{\rho}_g)^\gamma \quad \text{where } \tilde{\rho}_g(t,x) = \frac{\rho_g(t,x)}{\bar{\rho}_g}$$

- formal limit system as $\varepsilon = \frac{\bar{\rho}_g}{\rho_l} \to 0$, $u_g = u_l + \varepsilon w$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) = 0 \end{cases}$$

pressureless equations: α can exceed 1 !!

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

Applications in physics - modeling of two-phase flows

- 1D biphasic equations for liquid-gas mixtures, α volume fraction of the liquid

$$\begin{cases} \partial_t(\alpha\rho_l) + \partial_x(\alpha\rho_l u_l) = 0 \\ \partial_t((1-\alpha)\rho_g) + \partial_x((1-\alpha)\rho_g u_g) = 0 \\ \partial_t(\alpha\rho_l u_l) + \partial_x(\alpha\rho_l u_l^2) + \alpha\partial_x p + \tau_l = M \\ \partial_t((1-\alpha)\rho_g u_g) + \partial_x((1-\alpha)\rho_g u_g^2) + (1-\alpha)\partial_x p + \tau_g = -M \end{cases}$$

$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \mathrm{cst}, \quad p = a\rho_g^\gamma = a\bar{\rho}_g^\gamma\ (\tilde{\rho}_g)^\gamma \quad \text{where } \tilde{\rho}_g(t,x) = \frac{\rho_g(t,x)}{\bar{\rho}_g}$$

- conjecture: as $\varepsilon = \frac{\bar{\rho}_g}{\rho_l} \to 0$, $u_g = u_l + \varepsilon w$, $\varepsilon\bar{\rho}_g{}^{\gamma-1}p \to \pi$ with $(1-\alpha)\pi = 0$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) + \partial_x\pi = 0 \end{cases}$$

π ensures the constraint $\alpha \leq 1$

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

hard congestion models

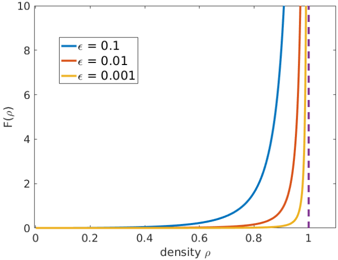
- contacts between grains in dense regime
  Signorini's conditions at micro. level
- restriction of adm. velocities $\mathrm{div}\, u \geq 0$
  $\rightarrow$ macro. potential $p \geq 0$ in $\{\rho = 1\}$

soft congestion models

- repulsive forces that prevent the contact between the grains
- $\rho < 1$ using singular constitutive laws
  $F(\rho) \underset{\rho \to 1}{\longrightarrow} +\infty$

link between the two approaches

$$F(\rho) = \varepsilon \left(\frac{\rho}{1-\rho} \right)^{\gamma}, \quad \varepsilon \to 0$$



ref: Maury '12

Congestion phenomena - some mathematical issues

soft congestion model

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2\operatorname{div}\big(\mu \mathrm{D}(u_\varepsilon)\big) = f \\ 0 \leq \rho < 1 \end{cases} \quad \xrightarrow[\varepsilon \to 0]{} \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} u) - 2\operatorname{div}\big(\mu \mathrm{D}(u)\big) = f \\ 0 \leq \rho \leq 1, \ \operatorname{spt} p \subset \{\rho = 1\}, \ p \geq 0 \end{cases}$$

hard congestion model

- existence of solutions to the hard and soft congestion systems
  - mechanisms preventing ρ to exceed 1;
  - regularity of the solutions, uniqueness;
  - explicit partially congested solutions;
- justification of the singular limit $\varepsilon \to 0$
  - compactness method based on energy estimates

Analysis of the free-congested Brinkman equations

soft congestion model

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda \operatorname{div} u_\varepsilon) \\ \qquad - 2\operatorname{div}\big(\mu\, \mathrm{D}(u_\varepsilon)\big) + r u_\varepsilon = f \\ 0 \le \rho < 1 \end{cases} \quad \xrightarrow[\varepsilon \to 0]{} \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla p - \nabla(\lambda \operatorname{div} u) \\ \qquad - 2\operatorname{div}\big(\mu\, \mathrm{D}(u)\big) + r u = f \\ 0 \le \rho \le 1,\ \operatorname{spt} p \subset \{\rho = 1\},\ p \ge 0 \end{cases}$$

hard congestion model

- we neglect the inertia in the momentum equation
- singular pressure $\quad p_\varepsilon(\rho) = \varepsilon \left(\dfrac{\rho}{1-\rho}\right)^\gamma$
- constant viscosities $\mu > 0$, $2\mu + 3\lambda \ge 0$;

Global weak sol. to the sing. compressible Brinkman eq.

Let $\Omega = \mathbb{T}^3$. Assume that $f \in L^2_{t,x}$, $\gamma > 1$ and initially that $0 \le \rho^0_\varepsilon(x) < 1$ a.e.

Then there exists a global weak solution $(\rho_\varepsilon, u_\varepsilon)$, *i.e.* for any $T > 0$:

- $\rho_\varepsilon \in \mathcal{C}([0,T]; L^q(\Omega))$, $0 \le \rho_\varepsilon < 1$ a.e., $\quad u_\varepsilon \in L^2(0,T; (H^1(\Omega))^3)$;
- $(\rho_\varepsilon, u_\varepsilon)$ satisfies the mass and momentum equations in the following weak sense:

$$\rightarrow \quad \int_0^T \int_\Omega \rho_\varepsilon \partial_t \phi + \int_0^T \int_\Omega \rho_\varepsilon u_\varepsilon \cdot \nabla \phi = - \int_\Omega \rho^0_\varepsilon \phi(0) \qquad \forall\ \phi \in \mathcal{C}^\infty_c([0,T) \times \Omega);$$

$$\rightarrow \quad - \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)\, \mathrm{div}\, \psi + 2 \int_0^T \int_\Omega \mu\, \mathrm{D}(u_\varepsilon) : \mathrm{D}(\psi) + \int_0^T \int_\Omega \lambda\, \mathrm{div}\, u_\varepsilon\ \mathrm{div}\, \psi$$
$$+ r \int_0^T \int_\Omega u_\varepsilon \cdot \psi = \int_0^T \int_\Omega f \cdot \psi \qquad \forall\ \psi \in (\mathcal{C}^\infty_c([0,T) \times \Omega))^3;$$

- the energy inequality holds with $sH'_\varepsilon(s) - H_\varepsilon(s) = p_\varepsilon(s) \rightsquigarrow H_\varepsilon(s) \underset{1}{\sim} \varepsilon(1-s)^{1-\gamma}$

$$\sup_{t \in [0,T]} \int_\Omega H_\varepsilon(\rho_\varepsilon) + \int_0^T \int_\Omega \left[\left(\frac{2}{3}\mu + \lambda \right) |\mathrm{div}\ u_\varepsilon|^2 + 2\mu \left| \mathrm{D}(u_\varepsilon) - \frac{\mathrm{div}\ u_\varepsilon}{3} \mathrm{I} \right|^2 + \frac{r}{2} |u_\varepsilon|^2 \right]$$
$$\le \int_\Omega H_\varepsilon(\rho^0_\varepsilon) + C \|f\|_{L^2_{t,x}}$$

Limit $\varepsilon \to 0$ - Idea of the proof

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2\operatorname{div}\big(\mu \mathrm{D}(u_\varepsilon)\big) + r u_\varepsilon = f \end{cases}$$

- assume that initially $\|H_\varepsilon(\rho_\varepsilon^0)\|_{L^1} \leq C$, then we get the following uniform controls:

$$(u_\varepsilon)_\varepsilon \text{ is bounded in } L^2_t H^1_x$$

$$(\rho_\varepsilon)_\varepsilon \text{ is bounded in } L^\infty_{t,x}$$

$\implies$ weak convergence (up to a subsequence) of $(\rho_\varepsilon)_\varepsilon$ and $(u_\varepsilon)_\varepsilon$ $+$ $0 \leq \rho \leq 1$ a.e.

- how to pass to the limit in the nonlinear term $\rho_\varepsilon u_\varepsilon$ $\to$ compensated-compactness

Lemma (Compensated-compactness, Lions)

Let (g_ε), (h_ε) be two sequences converging weakly respectively to g, h in $L^{r_1}_t L^{q_1}_x$ and $L^{r_2}_t L^{q_2}_x$ where $1 \leq r_1, r_2 \leq \infty$ and $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2}$.
Assume in addition that

- *$(\partial_t g_\varepsilon)$ is bounded in $L^1_t W^{-m,1}_x$ for some m independent of ε;*
- *$\|h_\varepsilon\|_{L^1_t H^s_x}$ is bounded for some $s > 0$.*

Then $(g_\varepsilon h_\varepsilon)$ converges to gh in $\mathcal{D}'((0,T) \times \Omega)$.

- limit of the pressure term ? uniform control of $p_\varepsilon(\rho_\varepsilon)$ in L^1 if $< \rho_\varepsilon^0 > \leq M^0 < 1$

Control of the pressure

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2\operatorname{div}\big(\mu \mathrm{D}(u_\varepsilon)\big) + r u_\varepsilon = f \\ 0 \le \rho_\varepsilon < 1 \end{cases}$$

- use as test function in the mom. eq.: $\psi = \nabla\Delta^{-1}(\rho_\varepsilon - <\rho_\varepsilon>)$, $\nabla\Delta^{-1} \approx \operatorname{div}^{-1}$

$$\left| \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)(\rho_\varepsilon - <\rho_\varepsilon>) \right| \le C(E^0)$$

- split the integral into two parts: $\{\rho_\varepsilon \le \frac{1+<\rho_\varepsilon>}{2}\}$ and $\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}$

  if $<\rho_\varepsilon> = <\rho_\varepsilon^0> \le M^0 < 1$ then

$$\begin{aligned} C &\ge \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)(\rho_\varepsilon - <\rho_\varepsilon>)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \\ &\ge \frac{1-<\rho_\varepsilon>}{2} \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \\ &\ge \frac{1-M^0}{2} \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \end{aligned}$$

$$\Longrightarrow \boxed{\|p_\varepsilon(\rho_\varepsilon)\|_{L^1L^1} \le C}$$

Limit pressure

- improved estimate on the pressure (inertia is neglected here)

$$\|p_\varepsilon(\rho_\varepsilon)\|_{L^2_{t,x}} \leq C$$

- $p_\varepsilon(\rho_\varepsilon) = \varepsilon \left(\dfrac{\rho_\varepsilon}{1-\rho_\varepsilon}\right)^\gamma \underset{\varepsilon\to 0}{\longrightarrow} p$ weakly in $L^2((0,T)\times\Omega)$, $\quad p \geq 0$

- $\mathrm{spt}\ p \ \subset\ \{\rho = 1\}$
  - there does not exist $A \subset \{\rho < 1\}$, $|A| \neq 0$, s.t. $\lim\limits_{\varepsilon\to 0} \rho_\varepsilon(t,x) = 1$ a.e. in A.

    write: $\displaystyle\int_0^T \int_{\mathbb{T}^3} \rho_\varepsilon\, \mathbf{1}_A \longrightarrow \int_0^T \int_{\mathbb{T}^3} \rho\, \mathbf{1}_A < |A|$
  - assume there exists a set $B \subset \{\rho < 1\}$ s.t. $p(t,x) > 0$ for $(t,x) \in B$

$$\text{then} \quad \int_0^T \int_{\mathbb{T}^3} p_\varepsilon(\rho_\varepsilon)\, \mathbf{1}_B \ \longrightarrow\ \int_0^T \int_{\mathbb{T}^3} p\, \mathbf{1}_B \ > 0.$$

  def of p_ε $\implies$ $\exists A \subset B$, $|A| \neq 0$, such that $\lim_{\varepsilon\to 0} \rho_\varepsilon(t,x) = 1$ a.e. in A

  $\to$ contradiction

Free-congested Navier-Stokes equations

soft congestion model

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2\operatorname{div}\big(\mu \mathrm{D}(u_\varepsilon)\big) = 0 \\ 0 \le \rho < 1 \end{cases} \xrightarrow[\varepsilon \to 0]{}$$

hard congestion model

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} u) - 2\operatorname{div}\big(\mu \mathrm{D}(u)\big) = 0 \\ 0 \le \rho \le 1, \ \operatorname{spt} p \subset \{\rho = 1\}, \ p \ge 0 \end{cases}$$

- pressure $\quad p_\varepsilon(\rho) = \varepsilon \left(\dfrac{\rho}{1-\rho}\right)^\gamma$ with $\gamma > 5/2$ (hyp for the construction of $(\rho_\varepsilon, u_\varepsilon)$)

Theorem (P., Zatorska 2015)

Assume that initially $0 \le \rho_\varepsilon^0 < 1$ *a.e. and*

$$< \rho_\varepsilon^0 > \le M^0 < 1, \qquad \int_\Omega \left[\frac{\rho_\varepsilon^0 |u_\varepsilon^0|^2}{2} + H_\varepsilon(\rho_\varepsilon^0)\right] \le E^0.$$

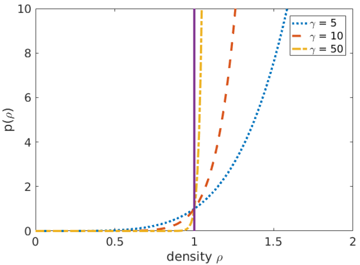
Then

- *there exists a global weak solution* $(\rho_\varepsilon, u_\varepsilon)$ *to the soft congestion system;*
- *as* $\varepsilon \to 0$, *there exists a subsequence* $(\rho_\varepsilon, u_\varepsilon, p_\varepsilon(\rho_\varepsilon))$ *which converges weakly to a global weak solution* (ρ, u, p) *to the free-congested NS equations.*

Additional remarks

- incompressible dynamics $\mathrm{div}\, u = 0$ in the congested domain
  - analogy with the low Mach number limit used for the design of num. schemes
    ref: Degond et al. '11, '17
- low regularity of the limit pressure
  - no information on the interface between the free and the congested domains
- no existing local well-posedness result for the free-congested NS eq.
  - results for compressible-incompressible NS eq with closed interface
    ref: Denisova, Solonnikov '16, Shibata '17
  - local well-posedness of the floating body pb $\sim$ free-congested Euler eq.
    ref: Iguchi, Lannes '18

- other possible approximation method: penalty method



$$p_n(\rho) = a\rho^{\gamma_n}, \ \gamma_n \to +\infty$$

ref: Lions, Masmoudi '99

see also Godlewski, Parisot, Sainte-Marie, Wahl with the approx. pressure $p_\lambda(\rho) = \frac{(\rho-1)_+^\gamma}{\lambda}$

rmk: no maximal bound on ρ_n or ρ_λ

- for $\mu, \lambda = 0$ direct existence results for the hard congestion system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ 0 \le \rho \le 1, \ \mathrm{spt}\ p \subset \{\rho = 1\}, \ p \ge 0 \end{cases}$$

  - via an approx. by sticky blocks ref: Berthelin '01 (extension to multi-d in '17)
  - via a Lagrangian approach (details tomorrow)

- existence of partially congested solutions ? what happens at the interface ?

Explicit partially congested solutions of 1D Euler equations

- reformulation in Lagrangian mass coord. (t, x) unknowns $(v = 1/\rho, u)$

$$\begin{cases} \partial_t v - \partial_{\mathrm{x}} u = 0 \\ \partial_t u + \partial_{\mathrm{x}} p = 0 \\ v \geq 1,\ \mathrm{spt}\ p \subset \{\rho = 1\},\ p \geq 0 \end{cases}$$

far field condition: $(v, u, p) \to (v_\pm, u_\pm, p_\pm)$ as $\mathrm{x} \to \pm\infty$

- congested shock travelling at speed s

$$(v, u, p)(t, \mathrm{x}) = \begin{cases} (v_-, u_-, p_-) & \text{if} \quad \mathrm{x} < st \\ (v_+, u_+, p_+) & \text{if} \quad \mathrm{x} > st \end{cases}$$

with a congested left state

$$p_- > 0, \quad p_+ = 0, \quad v_- = 1 < v_+$$

suppose that $u_- > u_+$, the Rankine-Hugoniot conditions give

$$s = \frac{u_- - u_+}{v_+ - 1} > 0, \qquad (u_- - u_+)^2 = p_-(v_+ - 1)$$

- these solutions can be recovered from sing. compressible Euler eq in the limit $\varepsilon \to 0$
ref Degond, Hua, Navoret '11

Euler system - development of congestion

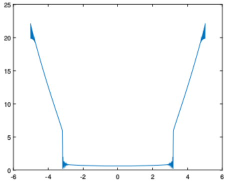
$$\begin{cases} \partial_t v - \partial_{\mathrm{x}} u = 0 \\ \partial_t u + \partial_{\mathrm{x}} p = 0 \\ v \geq 1, \ \mathrm{spt}\ p \subset \{\rho = 1\}, \ p \geq 0 \end{cases}$$

- if $v_0 > 1$, $u_0' < 0$ and $u_0(0) = 0$ then $v(t^*, 0) = 1$ at time $t^* = \frac{1 - v_0(0)}{u_0'(0)} > 0$
- assume that a congested zone $[\mathrm{x}_-, \mathrm{x}_+]$ is created at $(t^*)^+$ with shocks at $\mathrm{x}_\pm$

  Rankine-Hugoniot $\to$ natural scaling relating x, t and $\dot{\mathrm{x}}_\pm$ to the jump $v_\pm - 1$
- soft approximation $p_\varepsilon(v) = \varepsilon(v-1)^{-\gamma}$ and use of the rescaled variables

$$\begin{cases} \partial_{\tilde{t}} \tilde{v} - \partial_{\tilde{\mathrm{x}}} \tilde{u} = 0 \\ \partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{\mathrm{x}}} \left(\dfrac{1}{\tilde{v}^\gamma} \right) = 0 \end{cases}$$

$$v - 1 = \varepsilon^{1/\gamma} \tilde{v}, \ u = \varepsilon^{1/(2\gamma)} \tilde{u}$$
$$t - t^* = \varepsilon^{1/\gamma} \tilde{t}, \ \mathrm{x} = \varepsilon^{1/(2\gamma)} \tilde{\mathrm{x}}$$



numerical evidence of the apparition of shocks
but no theoretical justification known

ref: Bresch, Renardy '17

Viscous case: viscous shock profiles for the soft system

- 1D compressible NS eq. in Lagrangian mass coord. (t, x), unknowns $(v = 1/\rho, u)$

$$\begin{cases} \partial_t v_\varepsilon - \partial_\mathrm{x} u_\varepsilon = 0 \\ \partial_t u_\varepsilon + \partial_\mathrm{x} p_\varepsilon(v_\varepsilon) - \mu \partial_\mathrm{x}\left(\frac{1}{v_\varepsilon}\partial_\mathrm{x} u_\varepsilon\right) = 0 \end{cases} \qquad p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma} \text{ with } \gamma \geq 1$$

with the far field condition $(v, u) \to (v_\pm, u_\pm)$ as $\mathrm{x} \to \pm\infty$, $\quad v_\pm > 1$

Proposition

Let $1 < v_- < v_+$, $u_- > u_+$ *such that*

$$(u_+ - u_-)^2 = -(v_+ - v_-)(p_\varepsilon(v_+) - p_\varepsilon(v_-)).$$

There exists a unique (up to a shift) traveling front, that is a solution of the form

$$(v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) = (\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)(\mathrm{x} - s_\varepsilon t)$$

and the shock speed is given by

$$s_\varepsilon = \sqrt{-\frac{p_\varepsilon(v_+) - p_\varepsilon(v_-)}{v_+ - v_-}}.$$

- let $(\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)(x - s_\varepsilon t)$ be a solution, denote $\xi = x - s_\varepsilon t$

$$\begin{cases} -s_\varepsilon \mathfrak{v}_\varepsilon'(\xi) - \mathfrak{u}_\varepsilon'(\xi) = 0 \\ -s_\varepsilon \mathfrak{u}_\varepsilon'(\xi) + \big(p_\varepsilon(\mathfrak{v}_\varepsilon)\big)'(\xi) - \mu \left(\frac{\mathfrak{u}_\varepsilon'}{\mathfrak{v}_\varepsilon}\right)'(\xi) = 0 \end{cases}$$

- integrate between $\pm\infty$ and ξ using the fact that $|\mathfrak{u}_\varepsilon'| \to 0$ as $|\xi| \to \infty$

$$\begin{cases} -s_\varepsilon \mathfrak{v}_\varepsilon - \mathfrak{u}_\varepsilon = -s_\varepsilon v_\pm - u_\pm \\ -s_\varepsilon \mathfrak{u}_\varepsilon + p_\varepsilon(\mathfrak{v}_\varepsilon) - \mu \dfrac{\mathfrak{u}_\varepsilon'}{\mathfrak{v}_\varepsilon} = -s_\varepsilon u_\pm + p_\varepsilon(v_\pm) \end{cases} \implies s_\varepsilon = \pm\sqrt{-\frac{p_\varepsilon(v_+) - p_\varepsilon(v_-)}{v_+ - v_-}}$$

- ODE satisfied by $\mathfrak{v}_\varepsilon$ (using the relation $\mathfrak{u}_\varepsilon' = -s_\varepsilon \mathfrak{v}_\varepsilon'$)

$$\mathfrak{v}_\varepsilon' = \frac{\mathfrak{v}_\varepsilon}{\mu s_\varepsilon}\big(s_\varepsilon^2(v_+ - \mathfrak{v}_\varepsilon) + p_\varepsilon(v_+) - p_\varepsilon(\mathfrak{v}_\varepsilon)\big)$$

  - assume that $v_- < v_+$ and let $v_0 \in (v_-, v_+)$
    Cauchy Lipshitz $\to \exists!$ global solution $\mathfrak{v}_\varepsilon \in (v_-, v_+)$ such that $\mathfrak{v}_\varepsilon(0) = v_0$
  - use the convexity of p_ε to deduce the monotonicity of $\mathfrak{v}_\varepsilon$
  - since $v_- < v_+$ then $s_\varepsilon > 0$

Convergence towards a partially congested front

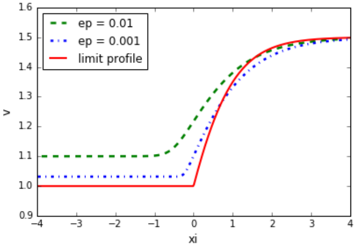
assume that $v_- = v_-^\varepsilon = 1 + \varepsilon^{1/\gamma} < v_+, \quad u_-^\varepsilon \to u_- > u_+$

$$p_\varepsilon(v_-^\varepsilon) = 1$$

$$\mathfrak{v}_\varepsilon' = \frac{\mathfrak{v}_\varepsilon}{\mu s_\varepsilon}\left(s_\varepsilon^2(v_+ - \mathfrak{v}_\varepsilon) + p_\varepsilon(v_+) - p_\varepsilon(\mathfrak{v}_\varepsilon)\right)$$

$$\mathfrak{u}_\varepsilon' = -s_\varepsilon \mathfrak{v}_\varepsilon'$$

fix the shift by imposing $\mathfrak{v}_\varepsilon(0) = 1 + \varepsilon^{\frac{1}{\gamma+1}}$



Proposition

As $\varepsilon \to 0$, up to the extraction of a subsequence, we have

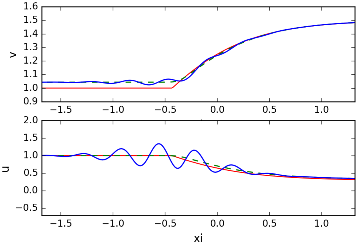
$$\mathfrak{v}_\varepsilon \to \bar{\mathfrak{v}} \quad \textit{in} \;\; \mathcal{C}(-R,R)\ \forall R > 0 \;\; \textit{and weakly-* in} \;\; W^{1,\infty}(\mathbb{R})$$

where the limit $\bar{\mathfrak{v}}$ is such that

$$\bar{\mathfrak{v}}(\xi) = \begin{cases} 1 & \textit{if } \xi < 0 \\ \dfrac{v_+}{1 + (v_+ - 1)e^{-r\xi}} & \textit{if } \xi \geq 0 \end{cases} \qquad \textit{and} \;\; r = \frac{v_+}{\mu\sqrt{v_+ - 1}}$$

Stability of the profiles $(v_\varepsilon, u_\varepsilon)$

$$\begin{cases} \partial_t v_\varepsilon - \partial_x u_\varepsilon = 0 \\ \partial_t u_\varepsilon + \partial_x p_\varepsilon(v_\varepsilon) - \mu \partial_x \left(\dfrac{1}{v_\varepsilon} \partial_x u_\varepsilon \right) = 0 \end{cases} \qquad p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma} \text{ with } \gamma \geq 1$$



$$\begin{cases} \partial_t (v - v_\varepsilon) - \partial_x (u - u_\varepsilon) = 0 \\ \partial_t (u - u_\varepsilon) + p'_\varepsilon(v_\varepsilon) \partial_x (v - v_\varepsilon) \\ \qquad - \mu \partial_x \left(\frac{1}{v_\varepsilon} \partial_x (u - u_\varepsilon) \right) = S_\varepsilon \end{cases}$$

Theorem (Dalibard, P. '19)

The profiles $(v_\varepsilon, u_\varepsilon)$ are asympt. stable under small init. perturbations with zero integral.

idea: introduction of the effective velocity $w = u - \mu \partial_x \ln v$ and use of the integrated variables (V, W) s.t. $v - v_\varepsilon = \partial_x V$, $w - w_\varepsilon = \partial_x W$ to derive weighted energy estimates

Summary, refinements and possible extensions

- singular limit soft $\to$ hard via a singular pressure $p_\varepsilon(\rho) = \varepsilon \left(\dfrac{\rho}{1-\rho} \right)^\gamma$
  - extension to more general pressure laws (possibly non monotone) provided there are sufficiently singular close to 1
  - extension to heterogeneous maximal density constraints $\rho \leq \rho^*$

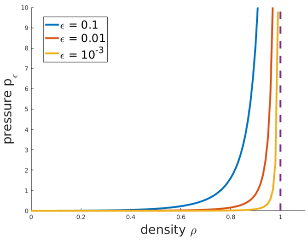
    case $\rho^*(x) \geq c > 0$, $\rho^* \in \mathcal{C}^1$ $\rightsquigarrow$ $\nabla p_\varepsilon(\rho)$ is replaced by $\rho^* \nabla p_\varepsilon \left(\frac{\rho}{\rho^*} \right)$
    ref: P., Zatorska '15

    case $\rho^*(t,x)$ with $\partial_t \rho^* + u \cdot \nabla \rho^* = 0$ $\rightsquigarrow$ ρ is replaced by $\frac{\rho}{\rho^*}$
    ref: Degond, Minakowski, Zatorska '17
- explicit 1d solutions, stability of almost congested fronts $(v_\varepsilon, u_\varepsilon)$
  - possible refined description for $(v_\varepsilon, u_\varepsilon)$ in the vicinity of the transition
    ref: Dalibard, P. '19
- other reference systems could be possible

Numerical approach

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) = 0 \end{cases}$$

- <u>singular pressure</u> $p_\varepsilon(\rho) = \varepsilon \left(\dfrac{\rho}{1-\rho} \right)^\gamma, \quad p'_\varepsilon(\rho) = \varepsilon\gamma \dfrac{\rho^{\gamma-1}}{(1-\rho)^{\gamma+1}}$



- p_ε ensures the constraint $\rho_\varepsilon < 1$
- p_ε becomes stiffer and stiffer as $\varepsilon \to 0$
- explicit treatment $\implies$ conditional stability

$$\Delta t \leq \frac{\sigma \Delta x}{\max\{|u_\varepsilon| + \sqrt{p'_\varepsilon(\rho_\varepsilon)}\}} \underset{\substack{\varepsilon\to 0 \\ \rho_\varepsilon \to 1}}{\longrightarrow} 0$$

$\rightarrow$ implicit treatment of some terms

Implicit treatment of the pressure

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \mathrm{div}\,(\rho u)^{n+1} = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \mathrm{div}\,(\rho u \otimes u)^n + \nabla p_\varepsilon(\rho^{n+1}) = 0 \end{cases}$$

- <u>reformulation</u>: $\mathrm{div}\,$(Mom eq) and insert the result into Mass eq.

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \mathrm{div}\,(\rho u)^n - \Delta t \Delta p_\varepsilon(\rho^{n+1}) - \Delta t \nabla^2 : (\rho u \otimes u)^n = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \mathrm{div}\,(\rho u \otimes u)^n + \nabla p_\varepsilon(\rho^{n+1}) = 0 \end{cases}$$

$\rightarrow$ uniform stability condition

- $\rho^{n+1} = \rho(p_\varepsilon^{n+1}) \rightarrow$ nonlinear elliptic equation on p_ε^{n+1}

1) compute p_ε^{n+1}
2) deduce the new density $\rho^{n+1} \rightarrow \rho^{n+1}$ automatically satisfies the constraint
3) compute the new momentum $(\rho u)^{n+1}$

Degond, Hua & Navoret (2011): Implicit/explicit splitting of the pressure

Time-Space discretization in 1D

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{1}{\Delta x}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right] = 0$$

$$\frac{(\rho u)_j^{n+1} - (\rho u)_j^n}{\Delta t} + \frac{1}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right] + \frac{1}{2\Delta x}\left[p_\varepsilon(\rho_{j+1}^{n+1}) - p_\varepsilon(\rho_{j-1}^{n+1})\right] = 0$$

with

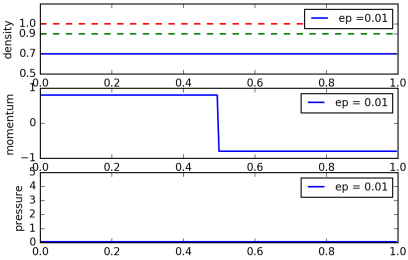
$$Q_{j+1/2}^{n+1/2} = \frac{1}{2}\left[(\rho u)_j^{n+1} + (\rho u)_{j+1}^{n+1}\right] - \frac{D_{j+1/2}^n}{2}\left(\rho_{j+1}^n - \rho_j^n\right)$$

$$F_{j+1/2}^n = \frac{1}{2}\left[\frac{\left((\rho u)_j^n\right)^2}{\rho_j^n} + \frac{\left((\rho u)_{j+1}^n\right)^2}{\rho_{j+1}^n}\right] - \frac{D_{j+1/2}^n}{2}\left((\rho u)_{j+1}^n - (\rho u)_j^n\right)$$

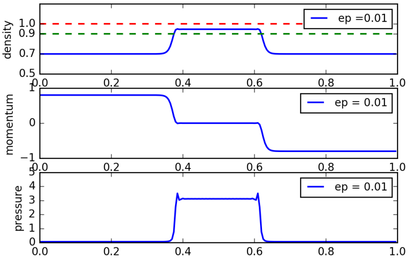
$$D_{j+1/2}^n = \max\left\{|u_j^n|, |u_{j+1}^n|\right\}$$

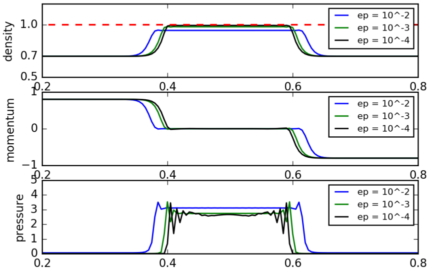
Numerical simulations: $\gamma = 2$, $\Delta t = 5.10^{-4}$, $\Delta x = 5.10^{-3}$

$t = 0$



$t = 0.04$





new numerical scheme for the congestion problem in collaboration with K. Saleh