

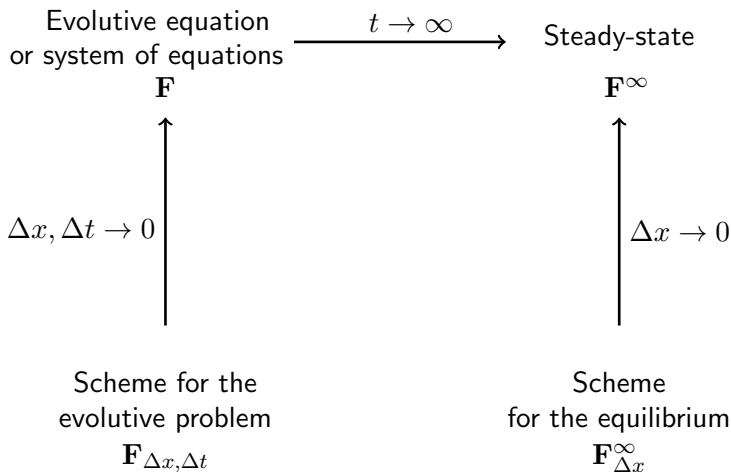
# Finite volume methods for dissipative problems : study of the long time behavior

Claire Chainais-Hillairet

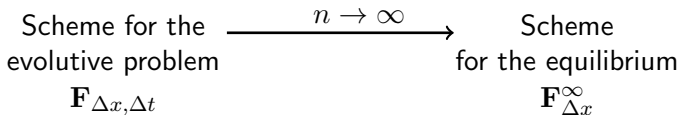
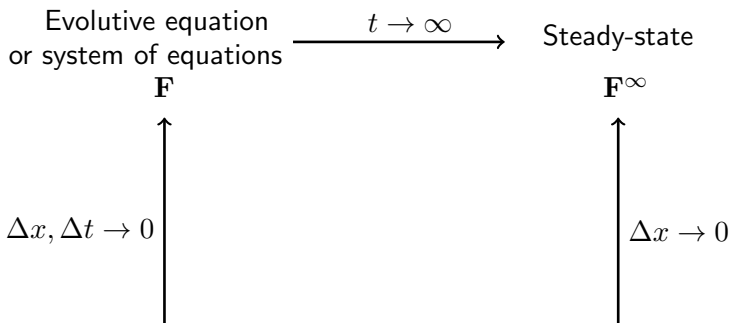
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# Overview



# Overview



## Framework : dissipative problems

$$(\mathbf{F}) \quad \begin{cases} \partial_t u + Au = 0, & t \geq 0, \\ u(0) = u_0. \end{cases} \quad (\mathbf{F}^\infty) \quad \begin{cases} u_\infty, \\ Au_\infty = 0. \end{cases}$$

### Main features

- Existence of a Lyapunov convex function  $E$  satisfying

$$\frac{d}{dt}E(u) = -\langle Au, E'(u) \rangle \leq 0.$$

- $E$  is general given by the physics : it is a physical energy or entropy, which is dissipated along time.
- $D(u) = \langle Au, E'(u) \rangle$  is the dissipation of energy/entropy.
- The steady-state is a minimizer for  $E$ .

# Dissipativity and long time behavior

$$(\mathbf{F}) \quad \begin{cases} \partial_t u + Au = 0, & t \geq 0, \\ u(0) = u_0. \end{cases} \quad (\mathbf{F}^\infty) \quad \begin{cases} u_\infty, \\ Au_\infty = 0. \end{cases}$$

## Main features

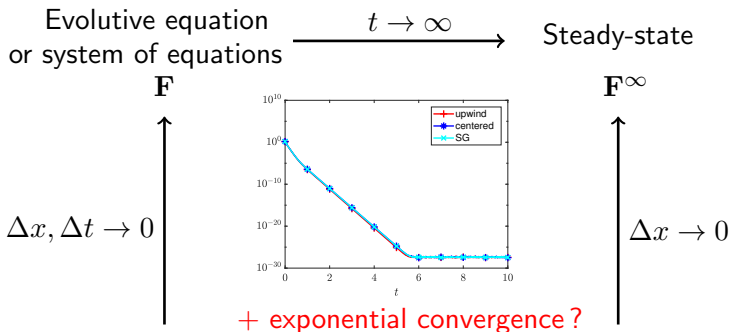
$$\frac{d}{dt}E(u) = -D(u)$$

- $D(u) \geq \lambda E(u) \implies$  exponential decay :

$$E(u) \leq E(u_0)e^{-\lambda t}.$$

- $D(u) \geq KE(u)^{1+\gamma} \implies$  polynomial decay  $\sim t^{-1/\gamma}$

# Overview



Scheme for the evolutionary problem  $\xrightarrow{n \rightarrow \infty}$  Scheme for the equilibrium

$\mathbf{F}_{\Delta x, \Delta t}$   $\mathbf{F}_{\Delta x}^\infty$

# Outline of the course

## Lecture 1 Dissipative problems and long time behavior

- 1 Some example of dissipative problems
- 2 Long time behavior of the Fokker-Planck equation
- 3 Long time behavior of the porous media equation

## Lecture 2 Finite volume schemes and long time behavior

- 1 Short introduction to TPFA finite volume schemes
- 2 Results for the porous media equation
- 3 Results for the Fokker-Planck equation

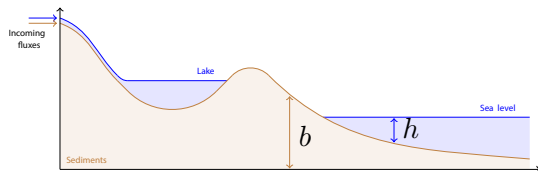
# Lecture 1 :

## Dissipative problems and long time behavior

# Outline of the lecture

- 1 Some example of dissipative problems
- 2 Long time behavior of the Fokker-Planck equation
- 3 Long time behavior of the porous media equation

# Shallow water equations with viscous terms



$h$  : water height

$b$  : ground topography

$u$  : velocity

## Shallow water equations with friction

$$\begin{cases} \partial_t h + \operatorname{div}(h\mathbf{u}) = 0 \\ \partial_t h\mathbf{u} + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) + gh\nabla(h + b) = -C|\mathbf{u}|\mathbf{u} \end{cases}$$

**Energy/dissipation**  $E(h, u) = \frac{1}{2} \int_{\mathbb{T}^2} \left( h|\mathbf{u}|^2 + g(h + b)^2 \right)$

$$D(h, u) = \int_{\mathbb{T}^2} C|\mathbf{u}|^3$$

## Study over large time scales

□ PETON, '18

$$\begin{cases} \partial_t h + \operatorname{div}(h\mathbf{u}) = 0 \\ gh\nabla(h+b) = -C|\mathbf{u}|\mathbf{u} \end{cases}$$

$$\implies \mathbf{u} = -\left(\frac{g}{C}\right)^{1/2} h^{1/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)$$

### Conservation law for the water height

$$\partial_t h + \operatorname{div}(-Kh^{1/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)) = 0$$

- degenerate 3/2-laplacian equation,
- with drift.

## Dissipative behavior

$$\begin{cases} \partial_t h + \operatorname{div}(-Kh^{1/2}|\nabla(h+b)|^{-1/2}\nabla(h+b)) = 0 \\ + \text{no-flux boundary condition} \end{cases}$$

**Energy/dissipation**  $E(h) = \frac{1}{2} \int_{\Omega} (h+b)^2$

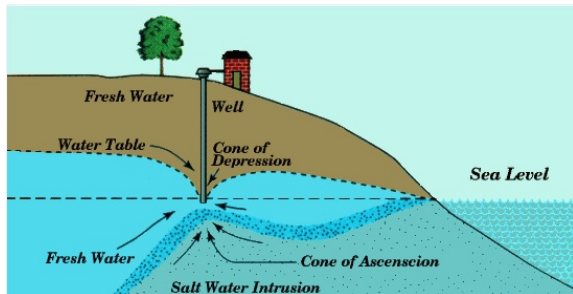
$$\begin{aligned} \frac{d}{dt} E(h) &= \int_{\Omega} \partial_t h (h+b) \\ &= - \int_{\Omega} \operatorname{div}(-Kh^{1/2}|\nabla(h+b)|^{-1/2}\nabla(h+b))(h+b) \\ &= - \int_{\Omega} Kh^{1/2}|\nabla(h+b)|^{3/2} \end{aligned}$$

## Steady states

$$h = 0 \text{ or } h + b = \text{constant.}$$

# Saltwater intrusion model

## Management of fresh water resources in costal regions



## Quantities of interest

- height of the freshwater
- height of the interface with the saltwater

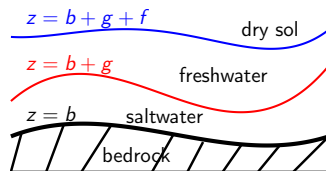
# Saltwater intrusion model

- ❑ ESCHER, LAURENÇOT, MATIOC, '12,
- ❑ LAURENÇOT, MATIOC, '14, '17
- ❑ JAZAR, MONNEAU, '14

## Assumptions

- Thin layers, sharp interfaces
- Large time scales
- Incompressible immiscible phases
- Mainly horizontal displacements

## Notations



$\rho$  : ratio of the densities,  $\rho = \frac{\rho_f}{\rho_s} < 1$ ,

$\nu$  : ratio of the kinematic viscosities,  $\nu = \frac{\nu_f}{\nu_s} \in (0, +\infty)$

# Saltwater intrusion model

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f + g + b)) = 0, \\ \partial_t g + \operatorname{div}(-g \nabla(\rho f + g + b)) = 0, \\ + \text{no-flux boundary conditions} \end{cases}$$

## Dissipative behavior

$$E(f, g) = \int_{\Omega} \frac{\rho}{2} (f + g)^2 + \frac{1 - \rho}{2} g^2 + b(\rho f + g).$$

$$\begin{aligned} \frac{d}{dt} E(f, g) &= \int_{\Omega} \partial_t f (\rho(f + g + b)) + \partial_t g (\rho f + g + b) \\ &= - \int_{\Omega} \rho \nu f |\nabla(\rho(f + g + b))|^2 + g |\nabla(\rho f + g + b)|^2 \\ &= - \int_{\Omega} \rho \nu f |\nabla \Phi_f|^2 + g |\nabla \Phi_g|^2 \end{aligned}$$

About porous media equation :  $\Omega = \mathbb{R}^d$ ,  $b = 0$ ,  $f = 0$

$$\partial_t g + \operatorname{div}(-g \nabla g) = 0$$

## Self-similar solutions

- Passage to self-similar variables :

$$\tau = \log(1+t), \quad \xi = \frac{x}{(1+t)^{1/(d+2)}}, \quad g(t, x) = e^{-\tau \frac{d}{d+2}} u(\tau, \xi)$$

- Nonlinear Fokker-Planck equation on  $u$  :

$$\partial_t u + \operatorname{div}\left(-u \nabla\left(u + \frac{|x|^2}{2(d+2)}\right)\right) = 0$$

- Self-similar solution in  $g \iff$  steady-state in  $u$
- Exponential decay in  $u$  and polynomial decay in  $g$

□ BARENBLATT, '1952

□ CARRILLO, TOSCANI, '00

# Salt water intrusion model : self-similar solutions

Initial system with  $b = 0$ ,  $\Omega = \mathbb{R}^2$

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f + g)) = 0, \\ \partial_t g + \operatorname{div}(-g \nabla(\rho f + g)) = 0, \end{cases}$$

Self-similar variables and new system

$$(f, g)(t, x) = \frac{1}{(1+t)^{1/2}} (\tilde{f}, \tilde{g}) \left( \log(1+t), \frac{x}{(1+t)^{1/4}} \right)$$

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f + g + \frac{b}{\nu})) = 0, \\ \partial_t g + \operatorname{div}(-g \nabla(\rho f + g + b)) = 0, \\ b(x) = \frac{|x|^2}{8} \end{cases}$$

## Salt water intrusion model : self-similar solutions

$$E(f, g) = \int_{\Omega} \frac{\rho}{2} (f + g)^2 + \frac{1 - \rho}{2} g^2 + b \left( \frac{\rho}{\nu} f + g \right),$$

$$D(f, g) = \int_{\Omega} \rho \nu f |\nabla \Phi_f|^2 + g |\nabla \Phi_g|^2$$

$$\text{with } \Phi_f = f + g + \frac{b}{\nu}, \quad \Phi_g = \rho f + g + b$$

### Characterization of the steady-states (self-similar solutions)

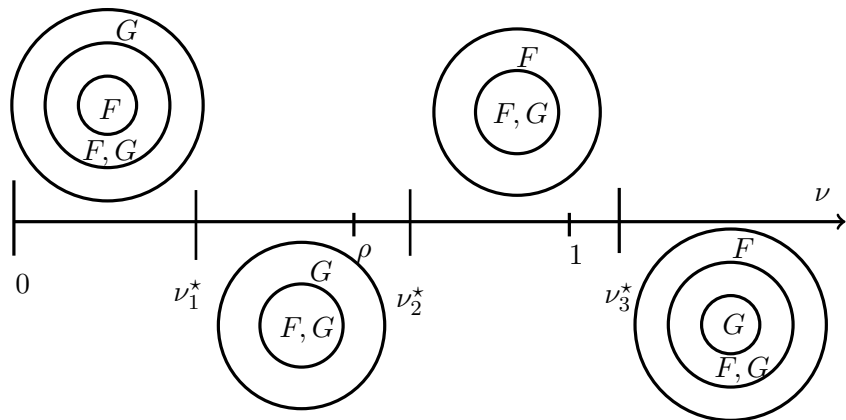
- Stationary solutions have vanishing fluxes :

$$F \nabla \Phi_F = 0, \quad G \nabla \Phi_G = 0.$$

- The minimizer of the energy is a stationary solution.
- There exists a unique minimizer of  $E$  which is radially symmetric.

- AIT HAMMOU OULHAJ, CANCÈS, C.-H., LAURENÇOT, '19
- LAURENÇOT, MATIOC, '14

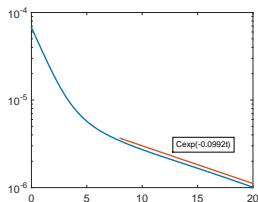
## Self-similar profiles



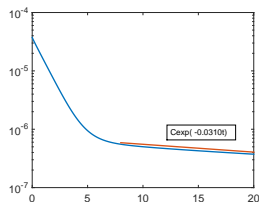
$$\nu_1^* = \frac{\rho^2 \frac{M_f}{M_g}}{1 + \rho \left( \frac{M_f}{M_g} - 1 \right)}, \quad \nu_2^* = \frac{\rho \frac{M_f}{M_g} + 1}{\frac{M_f}{M_g} + 1}, \quad \nu_3^* = 1 + (1 - \rho) \frac{M_f}{M_g}.$$

# Convergence towards the steady-state

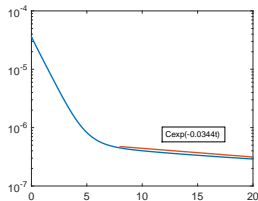
$(f(t), g(t)) \rightarrow (F, G)$  in  $L^2(\mathbb{R}^2; \mathbb{R}^2)$  as  $t \rightarrow \infty$



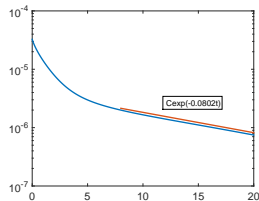
$\nu = 0.4$



$\nu = 0.9$



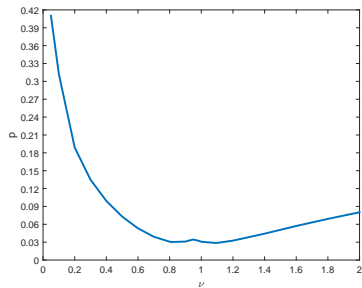
$\nu = 0.95$



$\nu = 2$

# Exponential convergence towards the steady-state?

Rate of convergence with respect to  $\nu$



# Outline of the lecture

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- 3 Long time behavior of the porous media equation

## Focus on Fokker-Planck equations

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, & \mathbf{J} = -\nabla f + \mathbf{U}f, \text{ in } \Omega \times \mathbb{R}_+ \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

### Some references

- ❑ CARRILLO, TOSCANI, '98
- ❑ ARNOLD, MARKOWICH, TOSCANI, UNTERREITER, '01
- ❑ CARRILLO ET AL., '01
- ❑ BODINEAU, LEBOWITZ, MOUHOT, VILLANI, '14
- ❑ GAJEWSKI, GRÖGER, '86, '89
- ❑ JÜNGEL, '95

## Thermal equilibrium, when $\mathbf{U} = -\nabla\Psi$

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, & \mathbf{J} = -\nabla f - \nabla\Psi f, \text{ in } \Omega \times \mathbb{R}_+ \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

$$f = \lambda e^{-\Psi} \implies \mathbf{J} = 0$$

**Existence of a thermal equilibrium**  $f^\infty = \lambda e^{-\Psi}$

- if  $\Gamma^D = \emptyset$ , with  $\lambda = \int_\Omega f_0 / \int_\Omega e^{-\Psi}$ ,
- if  $\log f^D + \Psi^D = \alpha$ , with  $\lambda = e^\alpha$ .

$$\implies \mathbf{J} = -f \nabla(\log f + \Psi) = -f \nabla \log \frac{f}{f^\infty}$$

# Entropy-dissipation property

$$\partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -f \nabla \log \frac{f}{f^\infty}$$

## Relative entropy

$$\Phi_1(x) = x \log x - x + 1$$

$$H_1(t) = \int_{\Omega} f^\infty \Phi_1\left(\frac{f}{f^\infty}\right)$$

## Dissipation of the entropy

$$\frac{d}{dt} H_1(t) = -D_1(t),$$

$$\text{with } D_1(t) = \int_{\Omega} f \left| \nabla \log \frac{f}{f^\infty} \right|^2 \geq 0$$

# Exponential decay towards thermal equilibrium

## No-flux boundary conditions

- conservation of mass :  $\int_{\Omega} f = \int_{\Omega} f_0 = \int_{\Omega} f^{\infty}$
- $H_1(t) = \int_{\Omega} f \log(f/f^{\infty})$
- $D_1(t) = \int_{\Omega} f |\nabla \log(f/f^{\infty})|^2 = 4 \int_{\Omega} f^{\infty} \left| \nabla \sqrt{f/f^{\infty}} \right|^2$
- thanks to Logarithmic Sobolev inequality :

$$0 \leq H_1(t) \leq H_1(0)e^{-\kappa t}$$

- and with Csiszar-Kullback inequality :

$$\|f(t) - f^{\infty}\|_1^2 \leq 2H_1(0)e^{-\kappa t}$$

# Exponential decay towards thermal equilibrium

## Dirichlet boundary conditions

- Upper and lower bounds on  $f$  and  $f^\infty$

- $$H_1(t) = \int_{\Omega} \Phi_1(f) - \Phi_1(f^\infty) - (f - f^\infty)\Phi_1'(f^\infty)$$

$$c\|f(t) - f^\infty\|_2^2 \leq H_1(t) \leq C\|f(t) - f^\infty\|_2^2$$

- $$D_1(t) = \int_{\Omega} f |\nabla(\log f - \log f^\infty)|^2$$

- with Poincaré inequality :

$$D_1(t) \geq C\|f(t) - f^\infty\|_2^2$$

- Conclusion :

$$c\|f(t) - f^\infty\|_2^2 \leq H_1(t) \leq H_1(0)e^{-\kappa t}$$

## General case

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, & \mathbf{J} = -\nabla f + \mathbf{U}f, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \end{cases}$$

## Steady-state

$$\begin{cases} \nabla \cdot \mathbf{J}^\infty = 0, & \mathbf{J}^\infty = -\nabla f^\infty + \mathbf{U}f^\infty, \text{ in } \Omega \times \mathbb{R}_+ \\ f^\infty = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J}^\infty \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+. \end{cases}$$

$$f = f^\infty h \implies \mathbf{J} = \mathbf{J}^\infty h - f^\infty \nabla h$$

## Exponential decay towards the steady-state

- Entropy/dissipation, with  $\Phi_2(x) = (x - 1)^2$ ,

$$H_2(t) = \int_{\Omega} f^\infty \Phi_2(h) \text{ and } D_2(t) = \int_{\Omega} f^\infty \Phi_2''(h) |\nabla h|^2$$

- Poincaré inequality + bounds on  $f^\infty$

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- 1 Some example of dissipative problems
- 2 Long time behavior of the Fokker-Planck equation
- 3 Long time behavior of the porous media equation

# Focus on the porous media equations

$$\beta > 1$$

$$\begin{cases} \partial_t f = \Delta f^\beta, & \text{in } \Omega \times \mathbb{R}_+ \\ f = f^D & \text{on } \Gamma^D \times \mathbb{R}_+ \\ \nabla f \cdot \mathbf{n} = 0 & \text{on } \Gamma^N \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

## Existence and uniqueness of a weak solution

□ VÁZQUEZ

## Steady-state

$$\begin{cases} \Delta(f^\infty)^\beta = 0, & \text{in } \Omega \\ f^\infty = f^D & \text{on } \Gamma^D, \quad \nabla f^\infty \cdot \mathbf{n} = 0 & \text{on } \Gamma^N \end{cases}$$

Existence and uniqueness of the steady state?

## Dissipative behavior, $\Gamma^D \neq \emptyset$

### Existence and uniqueness of the steady state

$$\begin{cases} \Delta(f^\infty)^\beta = 0, & \text{in } \Omega \\ f^\infty = f^D & \text{on } \Gamma^D, \quad \nabla f^\infty \cdot \mathbf{n} = 0 & \text{on } \Gamma^N \end{cases}$$

### Dissipation of the relative entropy

$$E(t) = \int_{\Omega} \frac{f^{\beta+1} - (f^\infty)^{\beta+1}}{\beta + 1} - (f^\infty)^\beta (f - f^\infty)$$

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_{\Omega} (f^\beta - (f^\infty)^\beta) \partial_t f \\ &= - \int_{\Omega} \left| \nabla (f^\beta - (f^\infty)^\beta) \right|^2 \end{aligned}$$

## Relation between entropy and dissipation, $\Gamma^D \neq \emptyset$

$$E(t) = \int_{\Omega} \frac{f^{\beta+1} - (f^{\infty})^{\beta+1}}{\beta + 1} - (f^{\infty})^{\beta}(f - f^{\infty})$$

$$D(t) = \int_{\Omega} \left| \nabla \left( f^{\beta} - (f^{\infty})^{\beta} \right) \right|^2$$

**Poincaré inequality**  $f^{\beta} - (f^{\infty})^{\beta} = 0$  on  $\Gamma^D$

$$\int_{\Omega} \left( f^{\beta} - (f^{\infty})^{\beta} \right)^2 \leq C_P \int_{\Omega} \left| \nabla \left( f^{\beta} - (f^{\infty})^{\beta} \right) \right|^2$$

### Functional inequalities

$$(z^{\beta} - 1)^2 \geq \frac{1}{\beta + 1} \left( z^{\beta+1} - (\beta + 1)z + \beta \right) \quad \forall z \geq 0. \quad (\beta \geq 1)$$

$$(x^{\beta} - y^{\beta})^2 \geq y^{\beta-1} \left( \frac{x^{\beta+1} - y^{\beta+1}}{\beta + 1} - y^{\beta}(x - y) \right) \quad \forall x, y \geq 0$$

## Exponential decay towards the steady-state, $\Gamma^D \neq \emptyset$

$$E(t) = \int_{\Omega} \frac{f^{\beta+1} - (f^{\infty})^{\beta+1}}{\beta + 1} - (f^{\infty})^{\beta}(f - f^{\infty})$$

$$D(t) = \int_{\Omega} |\nabla f^{\beta} - (f^{\infty})^{\beta}|^2$$

- Relation between entropy and dissipation :

$$D(t) \geq \frac{(m^D)^{\beta-1}}{C_P} E(t).$$

- Exponential decay of the entropy :

$$E(t) \leq E(0)e^{-\lambda t}, \text{ with } \lambda = \frac{(m^D)^{\beta-1}}{C_P}.$$

What happens when  $\Gamma^D = \emptyset$  ?

$$\begin{cases} \partial_t f = \Delta f^\beta, & \text{in } \Omega \times \mathbb{R}_+ \\ \nabla f \cdot \mathbf{n} = 0 & \text{on } \Gamma \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 \geq 0, & \text{with } \int_{\Omega} f_0 > 0. \end{cases}$$

with  $\Omega = [0, 1]^d$ , such that  $m(\Omega) = 1$ .

### Steady-state

- the associate steady-state is a constant function,
- the constant is fixed by the conservation of mass :

$$f^\infty = \frac{1}{m(\Omega)} \int_{\Omega} f_0 = \int_{\Omega} f_0$$

# Towards the long time behavior

## References

- ALIKAKOS, ROSTAMIAN, '1981
- BONFORTE, GRILLO, '05
- GRILLO, MURATORI, '13, '14
- C.-H., JÜNGEL, SCHUCHNIGG,

## Families of entropies

- Zeroth order entropies :

$$E_{\alpha}(f) = \frac{1}{\alpha + 1} \left( \int_{\Omega} f^{\alpha+1} dx - \left( \int_{\Omega} f dx \right)^{\alpha+1} \right)$$

- First order entropies :

$$F_{\alpha}[f] = \frac{1}{2} \int_{\Omega} |\nabla f^{\alpha/2}|^2 dx$$

## Dissipation of the order 0 entropies

$$\begin{cases} \partial_t f - \Delta(f^\beta) = 0, & \text{in } [0, 1]^d \times \mathbb{R}_+, \\ \nabla f \cdot n = 0 \text{ on } \Gamma \times \mathbb{R}_+, & f(0) = f_0 \\ f_\infty = \int_\Omega f_0. \end{cases}$$

- $E_\alpha(f) = \frac{1}{\alpha + 1} \left( \int_\Omega f^{\alpha+1} dx - \left( \int_\Omega f dx \right)^{\alpha+1} \right)$ 
$$\begin{aligned} \frac{d}{dt} E_\alpha(f) &= \int_\Omega f^\alpha \partial_t f = \int_\Omega f^\alpha \Delta(f^\beta) \\ &= - \int_\Omega \nabla f^\alpha \cdot \nabla f^\beta \\ &= - \frac{4\alpha\beta}{(\alpha + \beta)^2} \int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2. \end{aligned}$$
- $D_\alpha(f) = \frac{4\alpha\beta}{(\alpha + \beta)^2} \int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2.$

# Use of Poincaré-Wirtinger inequality ?

**Poincaré-Wirtinger inequality**  $u \in H^1(\Omega)$  ( $m(\Omega) = 1$ )

$$\int_{\Omega} \left( u - \frac{1}{m(\Omega)} \int_{\Omega} u \right)^2 \leq C_P^2 \|\nabla u\|_{L^2(\Omega)}^2$$

$$\|u\|_{L^2(\Omega)}^2 - \|u\|_{L^1(\Omega)}^2 \leq C_P^2 \|\nabla u\|_{L^2(\Omega)}^2$$

**Application**  $u = f^{(\alpha+\beta)/2}$

$$\begin{aligned} D_{\alpha}(f) &= \frac{4\alpha\beta}{(\alpha + \beta)^2} \int_{\Omega} |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\geq \frac{4\alpha\beta}{C_P^2(\alpha + \beta)^2} \left( \int_{\Omega} |f^{(\alpha+\beta)/2}|^2 - \left( \int_{\Omega} |f^{(\alpha+\beta)/2}| \right)^2 \right) \\ &\geq ??? K (E_{\alpha}(f))^{\kappa} \end{aligned}$$

# Beckner inequalities

## Original Beckner inequality

□ BECKNER 1989

$$\int_{\Omega} |u|^2 - \left( \int_{\Omega} |u|^{2/r} \right)^r \leq C_B^0(r) \|\nabla u\|_{L^2(\Omega)}^2, \quad 1 \leq r \leq 2$$

## Generalizations

$$\int_{\Omega} |u|^q - \left( \int_{\Omega} |u|^{1/p} \right)^{pq} \leq C_B(p, q) \|\nabla u\|_{L^2(\Omega)}^q,$$

$$0 \leq q < 2, \quad pq \geq 1.$$

$$\|u\|_{L^q(\Omega)}^{2-q} \left( \int_{\Omega} |u|^q - \left( \int_{\Omega} |u|^{1/p} \right)^{pq} \right) \leq C'_B(p, q) \|\nabla u\|_{L^2(\Omega)}^2,$$

$$0 \leq q < 2, \quad pq \geq 1$$

## Entropy/dissipation relation

$$\int_{\Omega} |u|^q - \left( \int_{\Omega} |u|^{1/p} \right)^{pq} \leq C_B(p, q) \|\nabla u\|_{L^2(\Omega)}^q, \quad 0 \leq q < 2, pq \geq 1$$

**Application :**  $u = f^{\frac{\alpha+\beta}{2}}, p = \frac{\alpha+\beta}{2}, q = \frac{2(\alpha+1)}{\alpha+\beta}$

$$\int_{\Omega} f^{\alpha+1} - \left( \int_{\Omega} f \right)^{\alpha+1} \leq C_B(p, q) \left\| \nabla f^{(\alpha+\beta)/2} \right\|_{L^2(\Omega)}^{\frac{2(\alpha+1)}{\alpha+\beta}}$$

for  $\alpha > 0, \beta > 1$ .

### Consequence

$$\begin{aligned} D_{\alpha}(f) &= \frac{4\alpha\beta}{(\alpha+\beta)^2} \int_{\Omega} |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\geq \frac{4\alpha\beta}{(\alpha+\beta)^2} \left( \frac{\alpha+1}{C_B(p, q)} \right)^{\frac{\alpha+\beta}{\alpha+1}} \left( E_{\alpha}(f) \right)^{\frac{\alpha+\beta}{\alpha+1}}. \end{aligned}$$

# Polynomial decay of the entropies

## Theorem [CCH-AJ-SS]

- $\alpha > 0, \beta > 1$
- $f_0 \in L^\infty(\Omega), \inf_\Omega f_0 \geq 0$
- $f$  positive regular solution

Then

$$E_\alpha(f(t)) \leq \frac{1}{(C_1 t + C_2)^{\frac{\alpha+1}{\beta-1}}}.$$

avec

$$C_1 = \frac{4\alpha\beta(\beta-1)}{(\alpha+1)(\alpha+\beta)^2} \left( \frac{\alpha+1}{C_B(p,q)} \right)^{(\alpha+\beta)/(\alpha+1)},$$

$$C_2 = E_\alpha(f_0)^{-(\beta-1)/(\alpha+1)}$$

## Towards the exponential decay

$$\|u\|_{L^q(\Omega)}^{2-q} \left( \int_{\Omega} |u|^q - \left( \int_{\Omega} |u|^{1/p} \right)^{pq} \right) \leq C'_B(p, q) \|\nabla u\|_{L^2(\Omega)}^2,$$
$$0 \leq q < 2, pq \geq 1$$

**Application :**  $u = f^{\frac{\alpha+\beta}{2}}, p = \frac{\alpha+\beta}{2}, q = \frac{2(\alpha+1)}{\alpha+\beta}$

$$\|f\|_{L^{\alpha+1}}^{\beta-1} \left( \int_{\Omega} f^{\alpha+1} - \left( \int_{\Omega} f \right)^{\alpha+1} \right) \leq C'_B(p, q) \left\| \nabla f^{(\alpha+\beta)/2} \right\|_{L^2(\Omega)}^2$$

**Consequence**  $(\|f\|_{L^{\alpha+1}(\Omega)} \geq \|f\|_{L^1(\Omega)} = \|f_0\|_{L^1(\Omega)})$

$$\begin{aligned} D_{\alpha}(f) &= \frac{4\alpha\beta}{(\alpha+\beta)^2} \int_{\Omega} |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\geq \frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2} \frac{\|f_0\|_{L^1(\Omega)}^{\beta-1}}{C'_B(p, q)} E_{\alpha}(f) \end{aligned}$$

# Exponential decay of the entropies

## Theorem [CCH-AJ-SS]

- $0 < \alpha \leq 1, \beta > 1$
- $f_0 \in L^\infty(\Omega), \inf_{\Omega} f_0 \geq 0$
- $f$  regular positive solution

Then

$$E_\alpha(f(t)) \leq E_\alpha(f_0)e^{-\lambda t}.$$

with

$$\lambda = \frac{4\alpha\beta(\alpha + 1)}{C'_B(p, q)(\alpha + \beta)^2} \|f_0\|_{L^1(\Omega)}^{\beta-1}$$