Modèle de Saint-Venant étendu : friction et couche visqueuse

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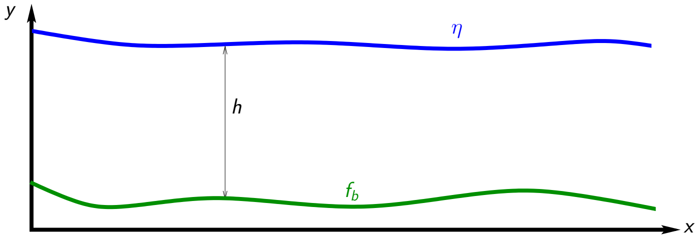
Shallow water



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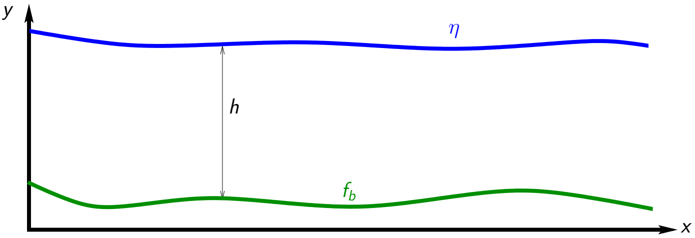

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Friction laws $S_f = C_f u$ laminar

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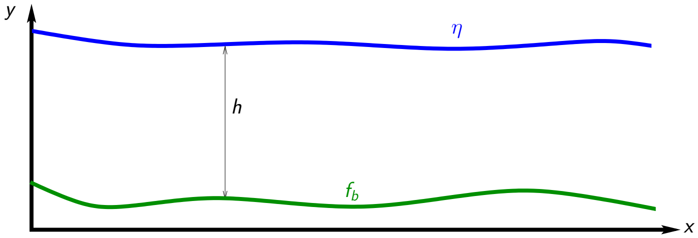

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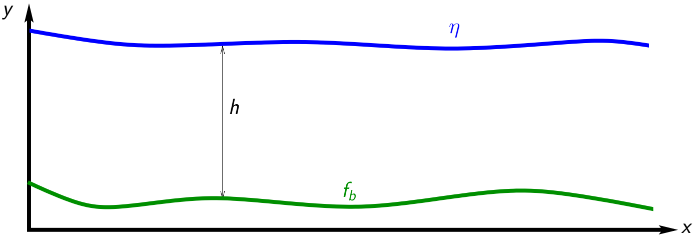
Shallow water



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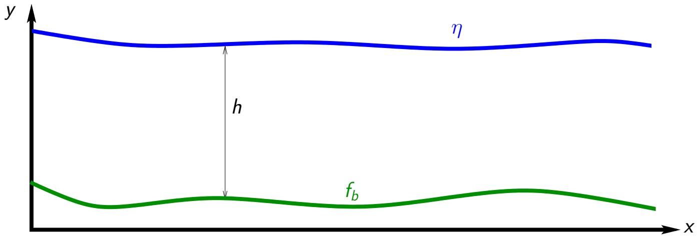
Friction laws $S_f = C_f u|u|/h^{4/3}$ Manning

Shallow water



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Friction laws $S_f =$ others (Coulomb...)

Navier-Stokes

Two-dimensional uncompressible Navier-Stokes

$$\partial_x u + \partial_y v = 0$$
$$\partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \frac{1}{Re}\Delta u$$
$$\partial_t v + u\partial_x v + v\partial_y v = -\partial_y p - \frac{1}{Fr^2} + \frac{1}{Re}\Delta v$$

where

- $U = (u, v)$ velocity vector
- $Re = \frac{u_0 h_0}{\nu}$ Reynolds
- $Fr = \frac{u_0}{\sqrt{gh_0}}$ Froude
- p pressure

Boundary conditions

Free surface $y = \eta(t, x)$

Topography $y = f_b(x)$

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$v = \partial_t \eta + u \partial_x \eta$ kinematic boundary condition

continuity of the stress tensor

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Topography $y = f_b(x)$

$$u = v = 0, \qquad \text{no-slip boundary condition}$$

Scalings

Small vertical velocity with respect to horizontal velocity

$$v = \varepsilon \tilde{v}, \quad u = \tilde{u}.$$

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$$\begin{array}{llll} x = \tilde{x} & y = \varepsilon \tilde{y} & t = \tilde{t} & \text{boundary layer} \\ x = \tilde{x}/\varepsilon & y = \tilde{y} & t = \tilde{t}/\varepsilon & \text{long wave} \end{array}$$

Long wave scaling

$$\partial_{\tilde{x}}\tilde{u} + \partial_{\tilde{y}}\tilde{v} = 0$$

$$\varepsilon\left[\partial_{\tilde{t}}\tilde{u} + \tilde{u}\partial_{\tilde{x}}\tilde{u} + \tilde{v}\partial_{\tilde{y}}\tilde{u}\right] = -\varepsilon\partial_{\tilde{x}}\tilde{p} + \frac{1}{Re}\left[\varepsilon^2\partial_{\tilde{x}}^2\tilde{u} + \partial_{\tilde{y}}^2\tilde{u}\right] \quad (1)$$

$$\varepsilon^2\left[\partial_{\tilde{t}}\tilde{v} + \tilde{u}\partial_{\tilde{x}}\tilde{v} + \tilde{v}\partial_{\tilde{y}}\tilde{v}\right] = -\frac{1}{Fr^2} - \partial_{\tilde{y}}\tilde{p} + \frac{\varepsilon}{Re}\left[\varepsilon\partial_{\tilde{x}}^2\tilde{v} + \partial_{\tilde{y}}\tilde{v}\right] \quad (2)$$

$$\tilde{u} = \tilde{v} = 0, \qquad \tilde{y} = \tilde{f}_b$$

$$\tilde{v} = \partial_{\tilde{t}}\tilde{\eta} + \tilde{u}\partial_{\tilde{x}}\tilde{\eta}, \qquad \tilde{y} = \tilde{\eta}$$

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(2) is the hydrostatic pressure $\tilde{p} = \frac{1}{Fr^2}(\tilde{\eta} - \tilde{y})$

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Effective Reynolds number $\varepsilon Re = Re_h$.
Hydrostatic Euler fluid when $Re_h \to \infty$

Integrated models (I)

"Eliminate" free surface through an equation on the water depth $h(t, x)$

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$$\begin{aligned} v(t,x,\eta) &= v(t,x,f_b) - \int_{f_b}^{\eta} \partial_x u \, dy \\ &= v(t,x,f_b) - \partial_x \left(\int_{f_b}^{\eta} u \, dy \right) + u(t,x,\eta)\partial_x \eta - u(t,x,f_b) f_b' \end{aligned}$$

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$= \partial_t \eta = \partial_t h$ by kinematic condition

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$= \partial_t \eta = \partial_t h$ by kinematic condition
$= 0$ by no-slip condition

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$= \partial_t \eta = \partial_t h$ by kinematic condition

$= 0$ by non penetration condition

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"Eliminate" free surface through an equation on the water depth $h(t, x)$

$$\int_{f_b}^{\eta} \left(\partial_x u + \partial_y v\right) dy = 0$$

$$hU = \int_{f_b}^{\eta} u \, dy$$

$$\partial_t h + \partial_x \left(hU\right) = 0$$

Integrated models (II)

Integrate the momentum equation

$$\int_{f_b}^{\eta} \left(\partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \frac{1}{Re_h}\partial_y^2 u \right) dy$$

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Integrate the momentum equation

$$\int_{f_b}^{\eta} \left(\partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \frac{1}{Re_h}\partial_y^2 u \right) dy$$

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parietal constraints

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parietal constraints
hydrostatic pressure $\partial_x p = (\partial_x h + f_b')/Fr^2$ independant of y

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parietal constraints
hydrostatic pressure $\partial_x p = (\partial_x h + f_b')/Fr^2$ independant of y
closure based on velocity profiles $u = \varphi(y)$

Integrated models (III)

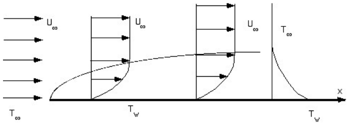
- Flat profile $\partial_y u = 0$

$$\partial_t(hU) + \partial_x\left(hU^2 + \frac{h^2}{2Fr^2}\right) = -h\frac{f_b'}{Fr^2}$$

- Nusselt-Poiseuille profile parabolic profile

$$\partial_t(hU) + \partial_x\left(\frac{6}{5}hU^2 + \frac{h^2}{2Fr^2}\right) = -h\frac{f_b'}{Fr^2} - \frac{3}{Re_h}\frac{U}{h}$$

Viscous layer (I)



ideal fluid

over

"thin" viscous layer

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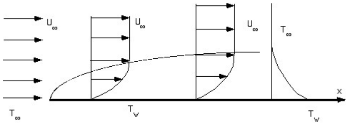

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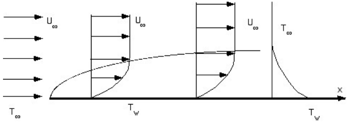

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$$\frac{1}{\overline{\delta}}\partial_{\overline{y}}\overline{p} = -\frac{1}{Fr^2}$$

$$\overline{u} = \overline{v} = 0 \qquad \text{at} \qquad \overline{y} = 0$$

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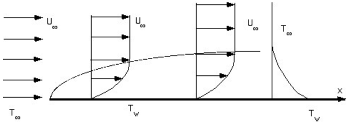

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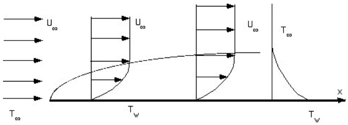

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Choice of $\overline{\delta} = \dfrac{1}{\sqrt{Re_h}}$

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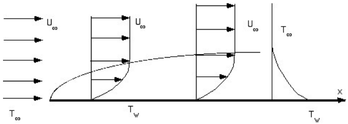

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Prandtl equation

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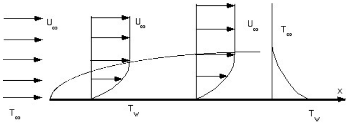

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Prandtl equation

Viscous layer (II)

Integrating Prandtl gives evolution of

- the displacement thickness δ_1 defined by

$$hU = (h - \overline{\delta}\delta_1)u_e$$

- the momentum thickness δ_2 defined by

$$\int_{f_b}^{\eta} u^2\, dy = \big(h - \overline{\delta}(\delta_1 + \delta_2)\big)\, u_e^2$$

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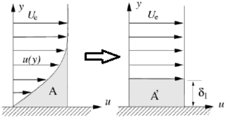

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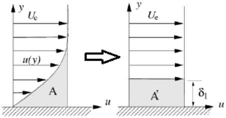

$$\partial_t \overline{u}_e + \overline{u}_e \partial_{\overline{x}} \overline{u}_e = -\partial_{\overline{x}} \overline{p} - \frac{f_b'}{Fr^2}$$

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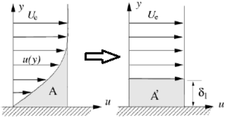

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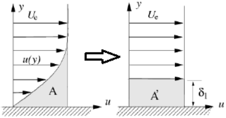


$$\partial_t(\overline{u}_e - \overline{u}) + \overline{u}_e \partial_{\overline{x}} \overline{u}_e - \overline{u}\partial_{\overline{x}}\overline{u} - \overline{v}\partial_{\overline{y}}\overline{u} = -\partial_{\overline{y}}^2 \overline{u}$$

Viscous layer (II)

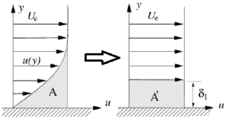
Integrating Prandtl gives evolution of

- the displacement thickness δ_1 defined by

$$hU = (h - \overline{\delta}\delta_1)u_e$$

- the momentum thickness δ_2 defined by

$$\int_{f_b}^{\eta} u^2 \, dy = \left(h - \overline{\delta}(\delta_1 + \delta_2)\right) u_e^2$$



$$\begin{aligned} \partial_t(u_e\delta_1) + u_e\delta_1\partial_x u_e + \partial_x(u_e^2\delta_2) &= \partial_{\overline{y}}\overline{u}|_{\overline{y}=0} \\ &= \tau \end{aligned}$$

von Kármán equation

Viscous layer (III)

Closure for von Kármán

Fixing a velocity profile in the viscous layer

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Fixing a velocity profile in the viscous layer

$$\frac{\overline{u}(\overline{y})}{u_e} = \varphi\left(\frac{\overline{y}}{\Delta(\overline{t},\overline{x})}\right) = \varphi(\zeta),$$

$$\varphi(\overline{t},\overline{x},0) = 0, \quad \lim_{\zeta\to+\infty} \varphi(\overline{t},\overline{x},\zeta) = 1, \quad \int_0^{\infty} (1-\varphi)\, d\zeta \equiv \alpha_2 < +\infty$$

$$\delta_1 = \Delta\alpha_2, \quad \delta_2 = \frac{\alpha_1}{\alpha_2}\delta_1 = H\delta_1, \quad \alpha_1 = \int_0^{+\infty} \varphi(1-\varphi)\, d\zeta$$

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Choosing the profile: Polhausen Falkner-Skan

f_2 and H functions of $u_e \partial_x u_e$

Towards the extended model

Going back to original variables

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- Incompressibility unchanged: yields

$$\partial_t h + \partial_x(hU) = 0, \qquad hU = \int_{f_b}^{\eta} u\, dy.$$

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$$\partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \overline{\delta}^2 \partial_y^2 u.$$

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But $\partial_y u|_{y=f_b} = \partial_{\bar{y}} \bar{u}|_{\bar{y}=0}/\bar{\delta} = \tau/\bar{\delta}$

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Flux closure: $\int_{f_b}^{\eta} u^2\, dy = \big(h - \bar{\delta}(\delta_1 + \delta_2)\big) u_e^2$

Extended Shallow Water model

$$\partial_t h + \partial_x(hU) = 0$$

$$\partial_t(hU) + \partial_x\left(\left(h - \bar{\delta}\delta_1(1 + \frac{1}{H})\right)u_e^2 + \frac{h^2}{2Fr^2}\right) = -h\frac{f_b'}{Fr^2} - \bar{\delta}\tau$$

$$\partial_t u_e + u_e\partial_x u_e = -\frac{1}{Fr^2}(\partial_x h + f_b')$$

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Two possibilities to close the system

$$hU = (h - \overline{\delta}\delta_1)u_e$$

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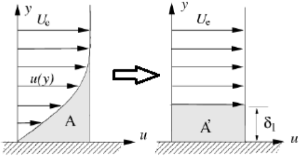
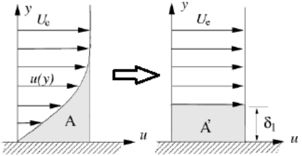
Equivalent ideal fluid I: Apparent topography



ideal fluid with velocity u_e
on height $\mathcal{H}$ such that
$hU = (h - \bar{\delta}\delta_1)u_e = \mathcal{H}u_e$

Equivalent ideal fluid I: Apparent topography



ideal fluid with velocity u_e
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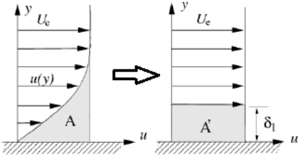
$$\partial_t \mathcal{H} + \partial_x(\mathcal{H}u_e) + \partial_t(\bar{\delta}\delta_1) = 0$$

$$\partial_t(\mathcal{H}u_e) + \partial_x(\mathcal{H}u_e^2 + \frac{\mathcal{H}^2}{2Fr^2}) + u_e\partial_t(\bar{\delta}\delta_1) = -\frac{\mathcal{H}}{Fr^2}\partial_x(\bar{\delta}\delta_1 + f_b)$$

$$\partial_t(\delta_1 u_e) + \partial_x\left((1 + \frac{1}{H})\delta_1 u_e^2\right) = u_e\partial_x(\delta_1 u_e) + \tau$$

Equivalent ideal fluid I: Apparent topography



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no-slip on $y = f_b + \bar{\delta}\delta_1$

Equivalent ideal fluid II: IBL

ideal fluid with velocity u_e on height h

Equivalent ideal fluid II: IBL

ideal fluid with velocity u_e on height h

$$\begin{cases} \partial_t h + \partial_x (h u_e - \bar{\delta}\delta_1 u_e) = 0 \\ \partial_t (h u_e) + \partial_x \left(h u_e^2 + \dfrac{h^2}{2Fr^2} \right) = -\dfrac{h f_b'}{Fr^2} + u_e \partial_x (\bar{\delta}\delta_1 u_e) \\ \partial_t (\delta_1 u_e) + \partial_x \left((1 + \dfrac{1}{H}) \delta_1 u_e^2 \right) = u_e \partial_x (\delta_1 u_e) + \tau \end{cases}$$

Equivalent ideal fluid II: IBL

ideal fluid with velocity u_e on height h

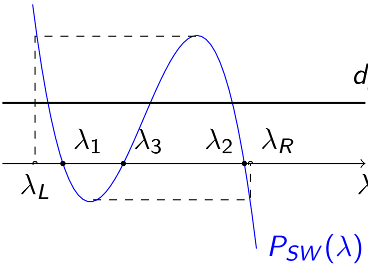
$$\begin{cases} \partial_t h + \partial_x(hu_e - \bar{\delta}\delta_1 u_e) = 0 \\ \partial_t(hu_e) + \partial_x\left(hu_e^2 + \dfrac{h^2}{2Fr^2}\right) = -\dfrac{hf_b'}{Fr^2} + u_e\partial_x(\bar{\delta}\delta_1 u_e) \\ \partial_t(\delta_1 u_e) + \partial_x\left((1+\dfrac{1}{H})\delta_1 u_e^2\right) = u_e\partial_x(\delta_1 u_e) + \tau \end{cases}$$

Transpiration boundary condition/Interactive Boundary Layer:

$$v_e|_{y=f_b} = u_e f_b' + \bar{\delta}\partial_x(\delta_1 u_e).$$

Hyperbolicity

$$P(\lambda) = -\lambda\left[(b - u_e - \lambda)\left((u_e - \lambda)^2 - Fr^{-2}h\right) - \bar{\delta}Fr^{-2}a\right]$$



Hyperbolicity

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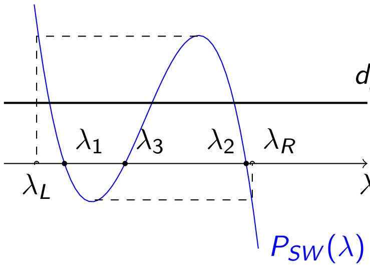
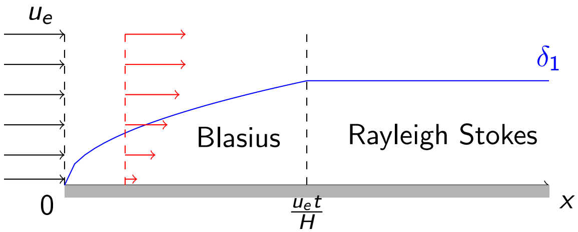


Consequences:

- conditional hyperbolicity: $\bar{\delta}$ small enough
- $\lambda_{1,2}$ shallow-water like eigenvalues, λ_3 viscous layer
- λ_3 can be nonpositive
- bounds on the eigenvalues

From Blasius to Rayleigh-Stokes



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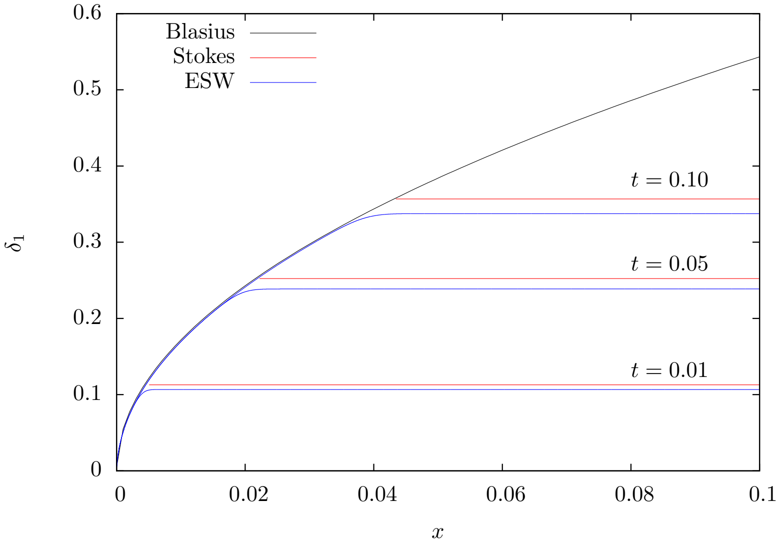
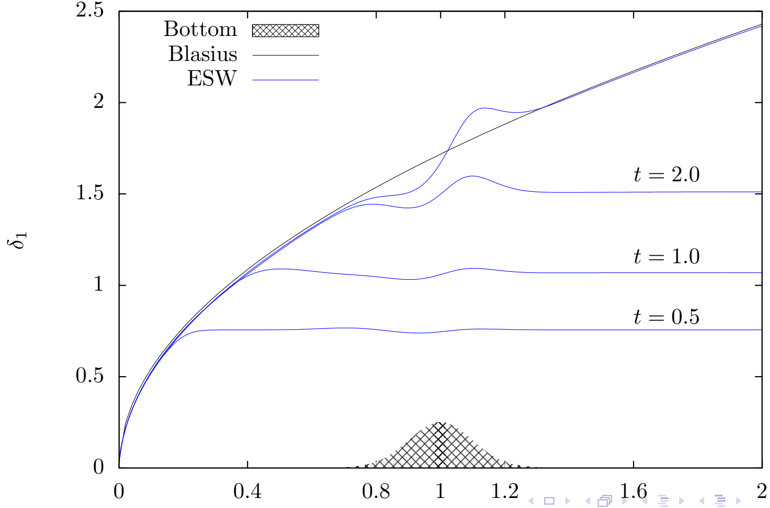
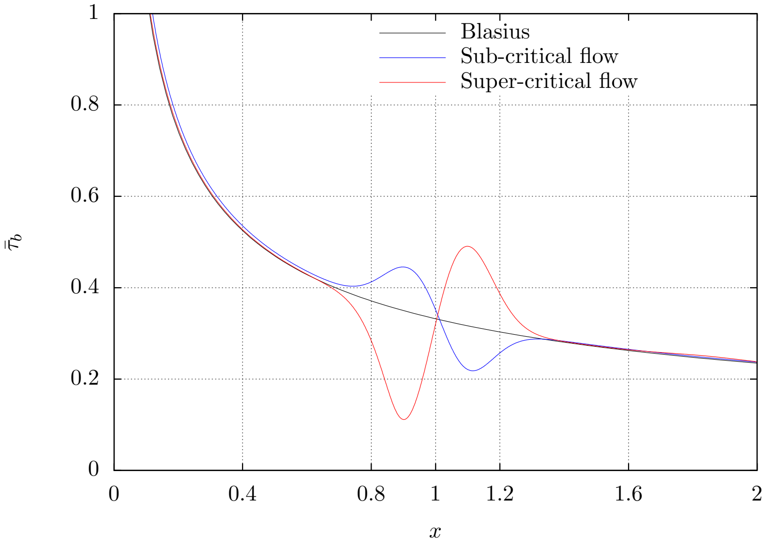


Bumps I

$$f_b = \alpha e^{-\frac{(x-1)^2}{2\sigma^2}}$$

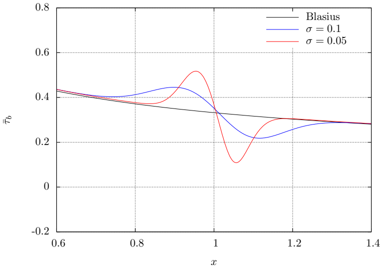


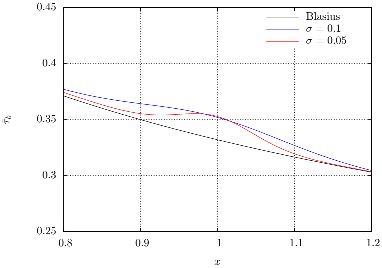
Bumps I bis



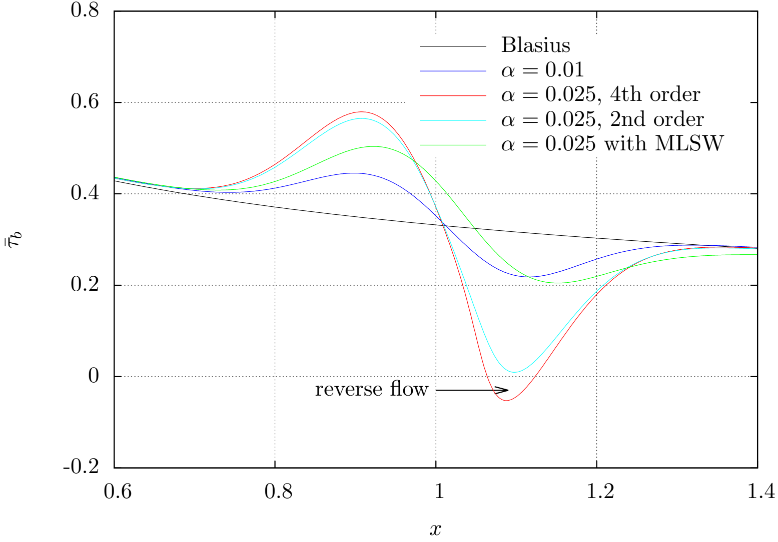
Blasius
Sub-critical flow
Super-critical flow
1
0.8
0.6
0.4
0.2
0
$\bar{\tau}_b$
0
0.4
0.8
1
1.2
1.6
2
x

Bumps II – length





Bumps III – height



That's all, folks !