Highly accurate methods for conservative two-phase flow models

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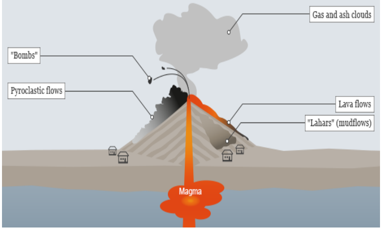
Overview

- The main purpose is to study non-equilibrium two-phase flow models that represent gas and magma mixtures arising within volcanic eruptions phenomena.



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- Mixture mass, momentum, and energy conservation equations:

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) = 0,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + P + \rho c(1-c)u_r^2) = 0,$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}\left(\rho u E + P u + \rho c(1-c)u_r\left(u u_r + (1-2c)\frac{u_r^2}{2} + \frac{\partial e}{\partial c}\right)\right) = 0.$$

- Volume and mass fractions for the gas phase and relative velocity:

$$\frac{\partial}{\partial t}(\rho\alpha) + \frac{\partial}{\partial x}(\rho u\alpha) = 0,$$

$$\frac{\partial}{\partial t}(\rho c) + \frac{\partial}{\partial x}(\rho u c + \rho c(1-c)u_r) = 0,$$

$$\frac{\partial}{\partial t}(u_r) + \frac{\partial}{\partial x}\left(u u_r + (1-2c)\frac{u_r^2}{2} + \frac{\partial e}{\partial c}\right) = \pi.$$

- Mixture velocity, density, momentum and pressure:

$$u = c_g u_g + c_m u_m, \quad \rho = \alpha\rho_g + (1-\alpha)\rho_m \quad \text{and} \quad \rho u = \alpha\rho_g u_g + (1-\alpha)\rho_m u_m.$$

- Mixture total energy and entropy:

$$E = e + c(1-c)\frac{u_r u_r}{2} \quad \text{and} \quad P = \alpha p_g + (1-\alpha)p_m.$$

$$e = c e_g(\rho_g, s) + (1-c)e_m(\rho_m, s) \quad \text{and} \quad s = c s_g + (1-c)s_m.$$

$$\alpha = \alpha_g = 1 - \alpha_m \quad \text{and} \quad c = c_g = \alpha_g \rho_g \rho^{-1}.$$

- Equations of state for perfect gas for gas and stiffened for magma:

$$e_g(\rho_g, s) = \frac{A_g}{\gamma_g - 1}\left(\frac{\rho_g}{\rho_g^0}\right)^{\gamma_g - 1} \exp\left(\frac{s}{c_V^g}\right),$$

$$e_m(\rho_m, s) = \frac{A_m}{\gamma_m - 1}\left(\frac{\rho_m}{\rho_m^0}\right)^{\gamma_m - 1} \exp\left(\frac{s}{c_V^m}\right) + A_0\frac{\rho_m^0}{\rho_m}.$$

- Relative velocity between the two phases:

$$u_r = u_g - u_m.$$

- Mass densities and pressures of each of the phases:

$$\rho_g = \frac{c\rho}{\alpha}, \qquad \rho_m = \frac{(1-c)\rho}{1-\alpha}, \qquad p_g = \rho_g^2 \frac{\partial e_g}{\partial \rho_g} \qquad \text{and} \qquad p_m = \rho_m^2 \frac{\partial e_m}{\partial \rho_m}.$$

- Source term that represents the interfacial friction between phases:

$$\pi = \kappa(u_2 - u_1) = \kappa u_r.$$

- Conservative form of the governing equations: $\dfrac{\partial \mathbf{U}}{\partial t} + \dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}).$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\alpha \\ \rho u \\ \rho c \\ u_r \\ \rho E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u \alpha \\ \rho u^2 + P + \rho c(1-c) u_r^2 \\ \rho u c + \rho c(1-c) u_r \\ u u_r + (1-2c)\dfrac{u_r^2}{2} + \dfrac{\partial e}{\partial c} \\ \rho u E + P u + \rho c(1-c) u_r \left(u u_r + (1-2c)\dfrac{u_r^2}{2} + \dfrac{\partial e}{\partial c} \right) \end{pmatrix},$$

$$\mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pi \\ 0 \end{pmatrix}.$$

- Relaxation system for the conservative form of the governing equations:

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x} &= \mathbf{S}(\mathbf{U}), \\ \frac{\partial \mathbf{V}}{\partial t} + \mathbf{B}^2 \frac{\partial \mathbf{U}}{\partial x} &= -\frac{1}{\tau}\big(\mathbf{V} - \mathbf{F}(\mathbf{U})\big). \end{aligned}$$

$\mathbf{V} \in \mathbb{R}^6$ is the relaxation variable and $\mathbf{B}^2 = \text{diag}\{B_1^2, \ldots, B_6^2\}$ is a diagonal matrix with positive diagonal elements B_k^2, $k = 1, \ldots, 6$.

$\tau \in [0, 1)$ is the relaxation rate.

- Semi-discrete relaxation scheme:

$$\frac{d\mathbf{U}_i}{dt} + \frac{\mathbf{V}_{i+1/2} - \mathbf{V}_{i-1/2}}{\Delta x} = \mathbf{S}(\mathbf{U})_i,$$

$$\frac{d\mathbf{V}_i}{dt} + \mathbf{B}_i^2 \frac{\mathbf{U}_{i+1/2} - \mathbf{U}_{i-1/2}}{\Delta x} = -\frac{1}{\tau}\left(\mathbf{V}_i - \mathbf{F}(\mathbf{U})_i.\right)$$

$\mathbf{W}_i$ ($\mathbf{W} = \mathbf{U}$ or $\mathbf{W} = \mathbf{V}$) is the space average of a generic solution $\mathbf{W}$ in the cell $[x_{i-1/2}, x_{i+1/2}]$ at time t and $\mathbf{W}_{i+1/2}$ is the numerical flux at $x = x_{i+1/2}$ and time t.

$$\mathbf{W}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{W}(x,t)dx, \qquad \mathbf{W}_{i+1/2} = \mathbf{W}(x_{i+1/2}, t).$$

- Implementation of an s-stage IMEX scheme: Step 1

For $l = 1, \dots, s$:

(1) Evaluate $\left(\hat{\mathcal{U}}_i^{(l)}, \hat{\mathcal{V}}_i^{(l)}\right)$ as:

$$\hat{\mathcal{U}}_i^{(l)} = \mathbf{U}_i^n - \Delta t \sum_{m=1}^{l-1} \tilde{a}_{lm} \frac{\hat{\mathcal{V}}_{i+1/2}^{(m)} - \hat{\mathcal{V}}_{i-1/2}^{(m)}}{\Delta x},$$

$$\hat{\mathcal{V}}_i^{(l)} = \mathbf{V}_i^n - \Delta t \mathbf{B}_i^2 \sum_{m=1}^{l-1} \tilde{a}_{lm} \frac{\hat{\mathcal{U}}_{i+1/2}^{(m)} - \hat{\mathcal{U}}_{i-1/2}^{(m)}}{\Delta x}.$$

(2) Then solve for $\left(\mathcal{U}_i^{(l)}, \mathcal{V}_i^{(l)}\right)$:

$$\mathcal{U}_i^{(l)} = \hat{\mathcal{U}}_i^{(l)} + \Delta t \sum_{m=1}^{l} a_{lm} \mathbf{S}(\mathcal{U}^{(l)})_i,$$

$$\mathcal{V}_i^{(l)} = \hat{\mathcal{V}}_i^{(l)} - \frac{\Delta t}{\tau} \sum_{m=1}^{l} a_{lm} \left(\mathcal{V}_i^{(l)} - \mathbf{F}(\mathcal{U}^{(l)})_i\right).$$

- Implementation of an s-stage IMEX scheme: Step 2

For $l = 1, \ldots, s$, update $\left(\mathbf{U}_i^{n+1}, \mathbf{V}_i^{n+1}\right)$ as:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \Delta t \sum_{l=1}^{s} \tilde{b}_l \frac{\mathcal{V}_{i+1/2}^{(l)} - \mathcal{V}_{i-1/2}^{(l)}}{\Delta x} + \Delta t \sum_{l=1}^{s} b_l \mathbf{S}(\mathcal{U}^{(l)})_i,$$

$$\mathbf{V}_i^{n+1} = \mathbf{V}_i^n - \Delta t \mathbf{B}_i^2 \sum_{l=1}^{s} \tilde{b}_l \frac{\mathcal{U}_{i+1/2}^{(l)} - \mathcal{U}_{i-1/2}^{(l)}}{\Delta x} - \frac{\Delta t}{\tau} \sum_{l=1}^{s} b_l \left(\mathcal{V}_i^{(l)} - \mathbf{F}(\mathcal{U}^{(l)})_i\right).$$

- Fully-discrete relaxation scheme

$$\frac{\partial \mathcal{F}^+}{\partial t} + \mathbf{B}\frac{\partial \mathcal{F}^+}{\partial x} = -\frac{1}{\tau}\left(\mathcal{F}^+ - \mathcal{M}^+\right),$$

$$\frac{\partial \mathcal{F}^-}{\partial t} - \mathbf{B}\frac{\partial \mathcal{F}^-}{\partial x} = -\frac{1}{\tau}\left(\mathcal{F}^- - \mathcal{M}^-\right).$$

$\mathcal{F}^+$ and $\mathcal{F}^-$ are kinetic variables (Riemann invariants) the components of which are

$$\mathcal{F}_k^+ = \frac{1}{2}\left(U_k + \frac{V_k}{B_k}\right), \qquad \mathcal{F}_k^- = \frac{1}{2}\left(U_k - \frac{V_k}{B_k}\right), \qquad k = 1, \ldots, 6,$$

whereas the local equilibrium functions (or Maxwellians) $\mathcal{M}^+$ and $\mathcal{M}^-$ are

$$\mathcal{M}_k^+ = \frac{1}{2}\left(U_k + \frac{F_k(U)}{B_k}\right), \qquad \mathcal{M}_k^- = \frac{1}{2}\left(U_k - \frac{F_k(U)}{B_k}\right), \qquad k = 1, \ldots, 6.$$

- Fully-discrete relaxation scheme: First-order reconstruction

$$\mathcal{F}^{+}_{i+1/2} = \mathcal{F}^{+}_{i}, \qquad \mathcal{F}^{-}_{i+1/2} = \mathcal{F}^{-}_{i+1}.$$

The intermediate solutions are

$$U_{i+1/2} = \frac{U_i + U_{i+1}}{2} - \frac{V_{i+1} - V_i}{2B_k}$$

$$V_{i+1/2} = \frac{V_i + V_{i+1}}{2} - B_k \frac{U_{i+1} - U_i}{2}$$

For time integration, the first-order IMEX scheme defined by the double Butcher's *tableau* is considered.

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

- Fully-discrete relaxation scheme: Second-order reconstruction

$$\mathcal{F}^{+}_{i+1/2} = \mathcal{F}^{+}_{i} + \frac{1}{2}\Delta x \sigma^{+}_{i}, \qquad \mathcal{F}^{-}_{i+1/2} = \mathcal{F}^{-}_{i+1} + \frac{1}{2}\Delta x \sigma^{-}_{i+1}.$$

σ_i^{+} and σ_i^{-} are the slope of $\mathcal{F}^{+}$ and $\mathcal{F}^{-}$ on the cell $[x_{i-1/2}, x_{i+1/2}]$.

$$\sigma^{\pm}_{i} = \frac{1}{\Delta x}\left(\mathcal{F}^{\pm}_{i+1} - \mathcal{F}^{\pm}_{i}\right)\Phi(\theta^{\pm}_{i}), \qquad \theta^{\pm}_{i} = \frac{\mathcal{F}^{\pm}_{i} - \mathcal{F}^{\pm}_{i-1}}{\mathcal{F}^{\pm}_{i+1} - \mathcal{F}^{\pm}_{i}}, \qquad \Phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}.$$

The intermediate solutions are

$$\begin{aligned} U_{i+1/2} &= \frac{U_i + U_{i+1}}{2} - \frac{V_i - V_{i+1}}{2B_k} + \frac{\sigma^{+}_{i} + \sigma^{-}_{i+1}}{4B_k}, \\ V_{i+1/2} &= \frac{V_i + V_{i+1}}{2} - B_k\frac{U_{i+1} - U_i}{2} + \frac{\sigma^{+}_{i} - \sigma^{-}_{i+1}}{4}. \end{aligned}$$

For time integration, the second-order IMEX scheme defined by the double Butcher's *tableau* is considered.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cc} -1 & -1 & 0 \\ 2 & 1 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

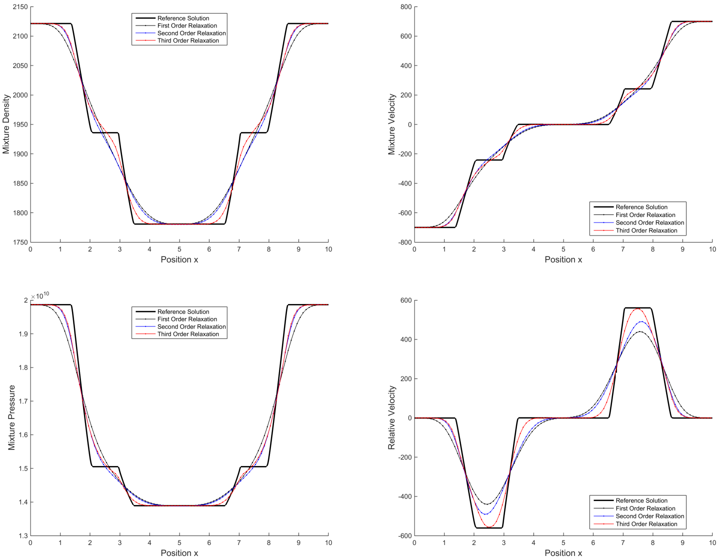
- Expansion tube problem for a mixture of gas and magma:

Magma

$$(\rho_m, u_m, s_m)_L = (3800\ kg/m^3, -700\ m/s, 7556.5895\ J/K.kg)$$
$$(\rho_m, u_m, s_m)_R = (3800\ kg/m^3, 700\ m/s, 7556.5895\ J/K.kg)$$

Gas

$$(\alpha, \rho_g, u_g, s_g)_L = (0.75, 1562\ kg/m^3, -700\ m/s, 12658.879\ J/K.kg)$$
$$(\alpha, \rho_g, u_g, s_g)_R = (0.75, 1562\ kg/m^3, 700\ m/s, 12658.879\ J/K.kg)$$



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- Shock tube problem for a mixture of gas and magma:

Magma

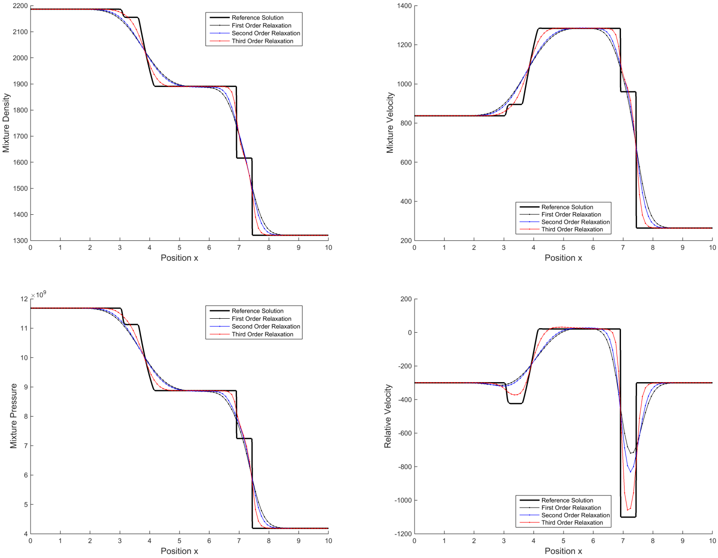
$$(\rho_m, u_m, s_m)_L = (2500\ kg/m^3, 1000\ m/s, 7556.5895\ J/K.kg)$$

$$(\rho_m, u_m, s_m)_R = (1800\ kg/m^3, 400\ m/s, 7556.5895\ J/K.kg)$$

Gas

$$(\alpha, \rho_g, u_g, s_g)_L = (0.6, 1977\ kg/m^3, 700\ m/s, 12658.879\ J/K.kg)$$

$$(\alpha, \rho_g, u_g, s_g)_R = (0.6, 1000\ kg/m^3, 100\ m/s, 12658.879\ J/K.kg)$$


Mixture Density
Position x
Reference Solution
First Order Relaxation
Second Order Relaxation
Third Order Relaxation
Mixture Velocity
Position x
Mixture Pressure
$\times 10^9$
Position x
Relative Velocity
Position x

- The relaxation method transforms the nonlinear hyperbolic system to a semi-linear model which can be solved numerically without using either Riemann solvers or linear iterations.
- The scheme can be applied to other systems of non-equilibrium two-phase flow problems such as slurry pipeline flows.

Some references

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THANK YOU !