Optimal Lagrangian control of Korteweg-de Vries: a shallow water pollution problem

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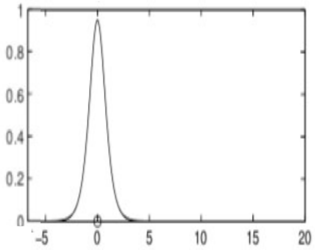
Context

The Rompsay canal, Charente-Maritime, France
(source : La Rochelle tourisme)

Dry cleaning of the Saint-Martin canal (2016), Paris, France
(source : Ouest France)

SOLITON



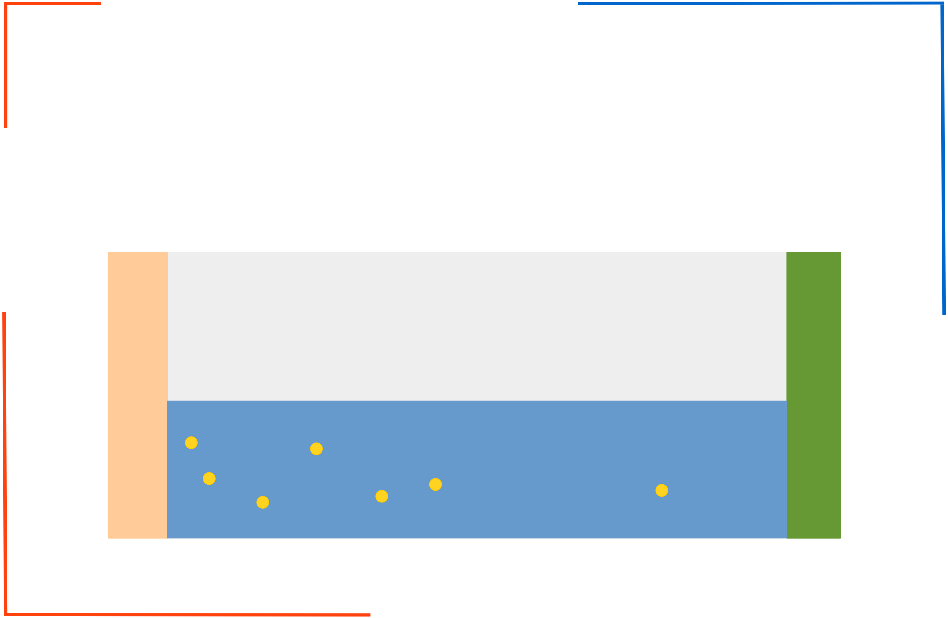
$$\eta(t,x) = \frac{\alpha^2}{2} sech^2\left(\frac{-\alpha(x-s)+\alpha^3 t}{2}\right)$$

$\alpha > 0, s \in \mathbb{R}$

- non-linear equation
- constant travelling speed
- fixed height

Soliton in water
(source : European Space Agency)

Context



Summary

- Korteweg-de Vries equation with a boundary control
  Lagrangian controllability
  Optimal control problem

- Results
  Existence for the optimal control problem
  Pontryagin’s necessary optimal conditions

Korteweg-de Vries equations :

$$\begin{cases} \partial_t\eta + \partial_x\eta + \partial_{xxx}\eta + \eta\partial_x\eta = 0, \, (x,t) \in (0,L) \times (0,T) \\ \eta(t,0) = v_1(t) \, , \, \eta(t,L) = v_2(t) \, , \, \partial_x\eta(t,L) = v_3(t), \, t \in (0,T) \\ \eta(0,x) = \eta_0(x), \, x \in (0,L) \end{cases} \tag{3.1}$$

with the velocity η and control $v = (v_1, v_2, v_3)$.

Theorem 1 (J.L. Bona, M. Sun and B.-Y. Zhang, 2003)

For a given control and a given $\eta_0(x)$, there exists a unique $\eta \in \mathcal{C}([0,T], H^1(0,L)) \cap L^2(0,T; H^2(0,L))$ solution of (3.1).

Korteweg-de Vries equations :

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We extend the velocity η in $\mathbb{R}$ as follows :

$$\widehat{\eta}(t,x) = \begin{cases} \eta(t,0) \text{ if } x < 0, \\ \eta(t,x) \text{ if } x \in [0,L], \\ \eta(t,L) \text{ if } x > L. \end{cases}$$

Definition 1 : Lagrangian controllability

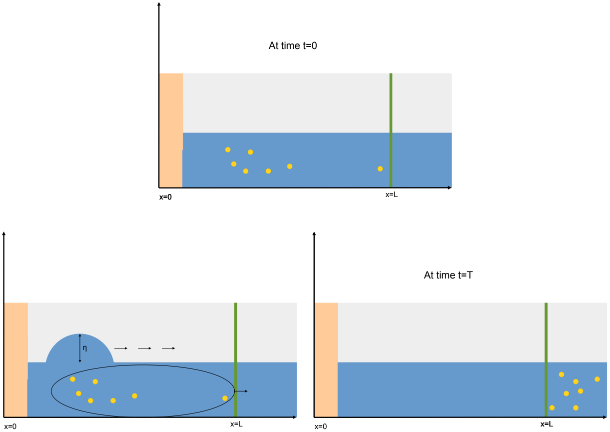
System (3.1) is small-time Lagrangian controllable if and only if, for all T, $L > 0$, there exist $v_1(t)$, $v_2(t)$, $v_3(t) \in \mathbb{R}$ such that the flow ϕ defined by

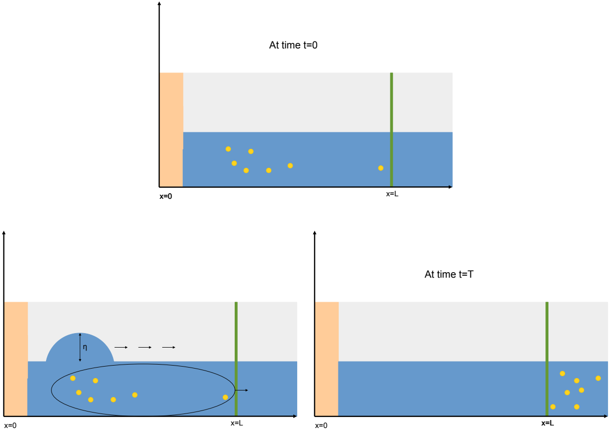
$$\begin{cases} \partial_t \phi(t,x) = \widehat{\eta}(t, \phi(t,x)) & t \in (0,T), \; x \in \mathbb{R} \\ \phi(0,x) = x & x \in \mathbb{R} \end{cases}$$

satisfies

$$\phi(T,x) \geq L, \forall x \in [0,L].$$

Model
Lagrangian controllability





Theorem 2 (L. Gagnon, 2016)

Let $T, L > 0$. Let γ small enough. If the initial data $\|\eta_0\| < \gamma$, (3.1) is Lagrangian controllable.

Let $E = \{(v_1, v_2, v_3) \in H^{\frac{4}{3}}(0,T) \times H^{\frac{4}{3}}(0,T) \times H^{\frac{1}{3}}(0,T)\}$ and $E_R = \{v \in E, \|v\|_E \leq R\}$. Let f a continuous and convex function.

Problem $(\mathcal{P})$

$$\min_{v \in E_R} J(v) = \int_0^T f(v)\, dt$$

subject to

$$\begin{cases} \partial_t \eta + \partial_x \eta + \partial_{xxx} \eta + \eta \partial_x \eta = 0, \text{ in } (0,L) \times (0,T) \\ \eta(t,0) = v_1(t)\ ,\ \eta(t,L) = v_2(t)\ ,\ \partial_x \eta(t,L) = v_3(t) \\ \eta(0,x) = \eta_0(x) \\ \partial_t \phi(t,x) = \widehat{\eta}(t, \phi(t,x)), \text{ in } (0,T) \times \mathbb{R} \\ \phi(0,x) = x, \text{ in } \mathbb{R} \\ \phi(T,x) \geq L,\ x \in [0,L] \end{cases}$$

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Existence of a solution for the optimal control problem

Let $E = \{(v_1, v_2, v_3) \in H^{\frac{4}{3}}(0, T) \times H^{\frac{4}{3}}(0, T) \times H^{\frac{1}{3}}(0, T)\}$ and $E_R = \{v \in E, \|v\|_E \leq R\}$. Let f a continuous and convex function.

Problem ($\mathcal{P}$)

$$\min_{v \in E_R} J(v, \phi) = \int_0^T f(v)\, dt + \frac{1}{2} \int_0^L \left|\phi(T, x) - \bar{\phi}(x)\right|^2 dx$$

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Theorem 3 (É.C, L. Gagnon and T. Takahashi, 2019)

There exist $(v^*, \eta^*, \phi^*) \in E_R \times L^2([0,T]; H^2(0,L)) \times \mathcal{C}([0,T]; H^1_{loc}(\mathbb{R}))$ solving Problem $(\mathcal{P})$.

Proof

Lower semi-continuity of convex functions argument

Question : uniqueness ?

Pontryagin's necessary conditions

Theorem 4 (É.C, L. Gagnon and T. Takahashi, 2019)

Let (v^*, η^*, ϕ^*) the solution of $(\mathcal{P})$.
There exist $(\lambda, \mu) \in L^2([0,T]; H^2(0,L)) \times L^\infty([0,T]; H^1(\mathbb{R}))$ such that

$$\partial_{v_1} f(v_1) = -\partial_{xx}\lambda(t,0) - \int_{-\infty}^{\phi(t,0)} \mu(t,x)\,dx$$

$$\partial_{v_2} f(v_2) = -\partial_{xx}\lambda(t,L) - \int_{\phi(t,L)}^{+\infty} \mu(t,x)\,dx$$

$$\partial_{v_3} f(v_3) = -\partial_x\lambda(t,L)$$

satisfying the terminal conditions

$$\lambda(T,x) = 0, \;\; \mu(T,x) = \bar{\phi}(x) - \phi(T,x)$$

and equations

$$\partial_t\lambda + \partial_x\lambda + \partial_{xxx}\lambda + \eta\partial_x\lambda + \mu(t,\phi^{-1})\frac{1}{\phi'(\phi^{-1})} = 0, \quad \partial_t\mu + \mu\partial_x\eta(t,\phi) = 0$$

$$\lambda(t,0) = \lambda(t,L) = \partial_x\lambda(t,0) = 0$$

Pontryagin's necessary conditions

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satisfying the terminal conditions

$$\lambda(T,x) = 0,\ \ \cancel{\mu(T,x) = \bar{\phi}(x) - \phi(T,x)}$$

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$$\lambda(t,0) = \lambda(t,L) = \partial_x\lambda(t,0) = 0$$

Kuhn-Tucker conditions

$\mu(T,x)(\phi(T,x) - L) = 0$, $\mu(T,x) \geq 0$, $\phi(T,x) \geq L$

Conclusion and Outlook

- Numerical simulations : in progress
- Lagrangian controllability : for an initial data small enough
  → what about a troubled canal ?
- Suspended particules
  → Outlook : solids on the canal bottom