Determinantal formulas for the resultant of some mixed multilinear systems

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Solving mixed multilinear systems

Objective

Solve (symbolically) square some **mixed sparse** multilinear systems

- Take into the account the sparseness
- Polynomial time wrt the number of solutions

Results

- Sylvester- and Koszul-type determinantal formula for the resultant
- Extension of the Eigenvalue criteria
- Extension of the Eigenvector criteria
- Applications to the Multiparameter Eigenvalue Problem (MEP)

The resultant

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{K}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

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Example : Resultant of linear forms = Determinant

The system $\begin{cases} a_1\, x + a_2\, y + a_3\, z = 0 \\ b_1\, x + b_2\, y + b_3\, z = 0 \\ c_1\, x + c_2\, y + c_3\, z = 0 \end{cases}$ has a solution over $\mathbb{P}^2$

$$\Updownarrow$$

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0.$$

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Example : Resultant of linear forms = Determinant

Example : Resultant of binary forms = Det of Sylvester matrix

$$\begin{cases} a_1\, x^2 + a_2\, x\, y + a_3\, y^2 = 0 \\ b_1\, x^3 + b_2\, x^2\, y + b_3\, x\, y^2 + b_4\, y^3 = 0 \end{cases} \quad \text{has a solution over } \mathbb{P}^1$$

$$\Updownarrow$$

$$\mathtt{det} \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix} = 0.$$

Sylvester-type formulas

Classical way of computing resultant → Sylvester-type formula

$$(g_0, \ldots, g_n) \mapsto \sum_{i=0}^{n} g_i\, f_i$$

Macaulay resultant matrix [Macaulay, 1916]

$$\begin{cases} f_1 := & a_1\, x^2 + a_2\, x\, y + a_3\, x\, z + \\ & a_4\, y^2 + a_5\, y\, z + a_6\, z^2 \\ f_2 := & b_1\, x + b_2\, y + b_3\, z \\ f_3 := & c_1\, x + c_2\, y + c_3\, z \end{cases}$$

	x^2	$x\,y$	$x\,z$	y^2	$y\,z$	z^2
f_1	a_1	a_2	a_3	a_4	a_5	a_6
$x\,f_2$	b_1	b_2	b_3			
$y\,f_2$		b_1		b_2	b_3	
$z\,f_2$			b_1		b_2	b_2
$y\,f_3$		c_1		c_2	c_3	
$z\,f_3$			c_1		c_2	c_3

$$\texttt{Determinant} = \texttt{Resultant} \cdot \underbrace{\texttt{ExtraFactor}}_{\text{Minor of the matrix}}.$$

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$$\texttt{Determinant} = \texttt{Resultant} \cdot \underbrace{\texttt{ExtraFactor}}_{\text{Minor of the matrix}} .$$

Determinantal formula → `ExtraFactor` is a constant.

Solving

- We want to compute the two solutions $\alpha_1, \alpha_2 \in \mathbb{P}^2$ of

$$\begin{cases} f_1 := & \mathbf{1}\, x^2 + \mathbf{-1}\, x\, y + \mathbf{4}\, x\, z + \mathbf{-2}\, y^2 + \mathbf{-5}\, y\, z + \mathbf{3}\, z^2 \\ f_2 := & \mathbf{1}\, x + \mathbf{-1}\, y + \mathbf{-1}\, z \end{cases} .$$

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- Introduce $f_0 := \mathbf{-1}\, x + \mathbf{2}\, y + \mathbf{1}\, z$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array} \right) =$$

	x^2	$x\,y$	$x\,z$	y^2	$y\,z$	z^2
f_1	**1**	**−1**	**4**	**−2**	**−5**	**3**
$x\,f_2$	**1**	**−1**	**−1**			
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- Schur complement of $M_{2,2} \leftrightarrow$ Multiplication map

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\, M_{1,1}^{-1}\, M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

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- Eigenvalues of $\widetilde{M}_{2,2} \leftrightarrow f_0(\alpha_i)$ [Lazard, 1981]

$$f_0(\alpha_1) = 2 \qquad \text{and} \qquad f_0(\alpha_2) = -2.$$

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- Eigenvectors of $\widetilde{M}_{2,2} \leftrightarrow \begin{pmatrix} y\, z \\ z^2 \end{pmatrix} (\alpha_i)$ [Auzinger & Stetter, 1988]

$$\begin{pmatrix} y\, z \\ z^2 \end{pmatrix} (\alpha_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} y\, z \\ z^2 \end{pmatrix} (\alpha_2) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Problems

We want to compute the two solutions a similar system

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The submatrix $M_{1,1}$ not invertible $\implies$ we cannot compute Schur complement.

Why? Because the `ExtraFactor` vanishes.

Solving polynomial systems using the resultant

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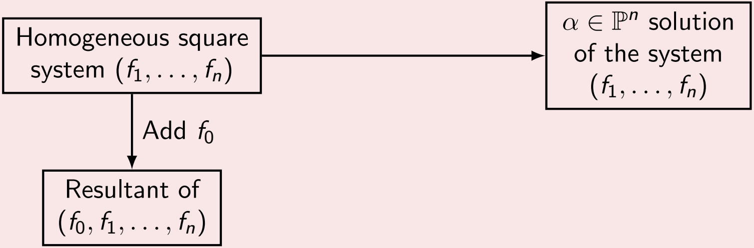

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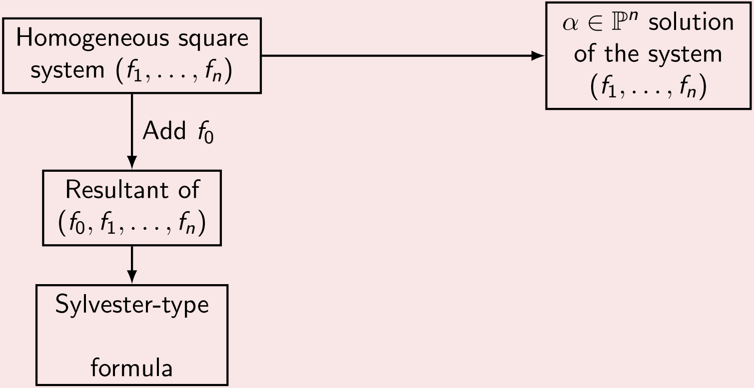

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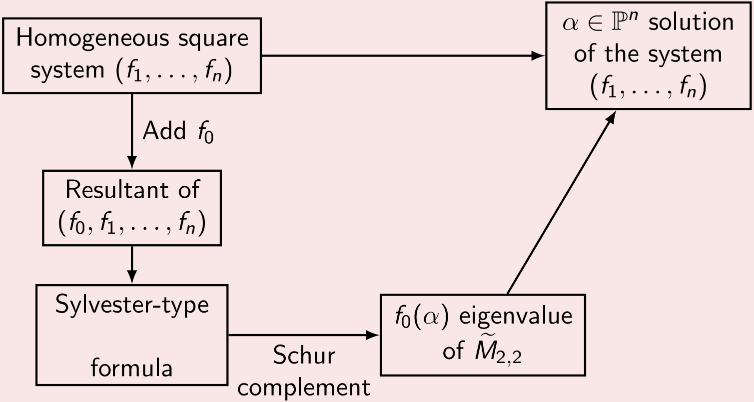

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

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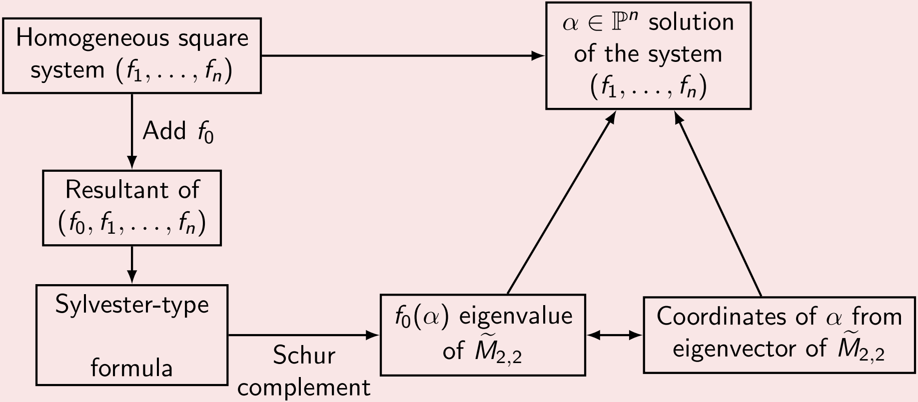

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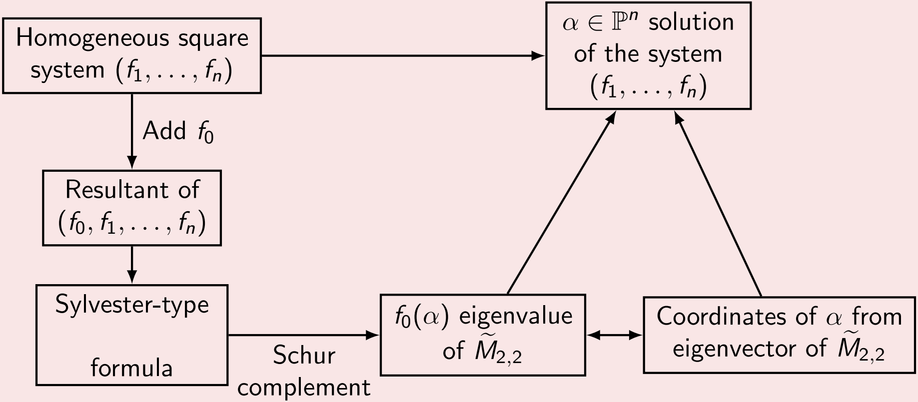

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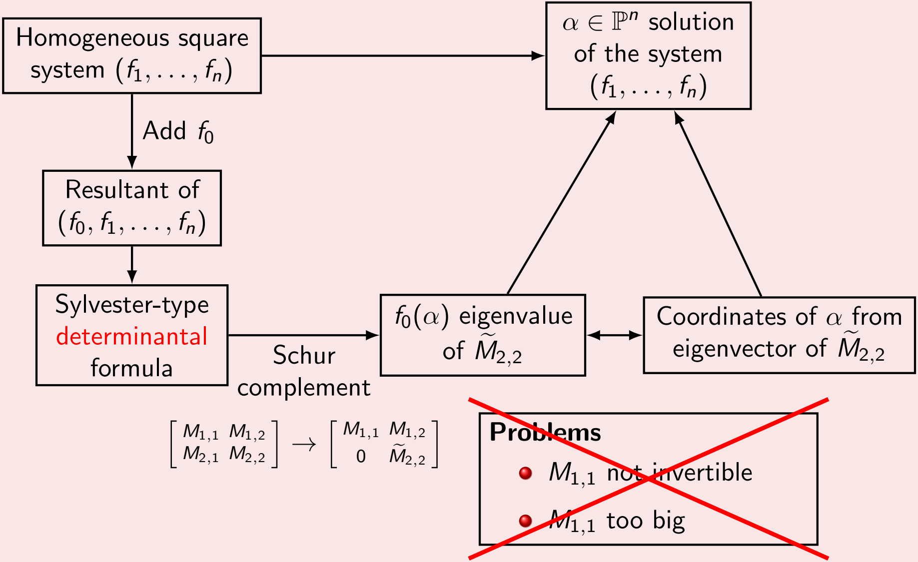
Problems

- $M_{1,1}$ not invertible
- $M_{1,1}$ too big

Solving polynomial systems using the resultant



Multihomogeneous systems

Multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \leftrightarrow$ Multihomogeneous polynomials.

Multiprojective resultant

Necessary and sufficient condition for a multihomogeneous system in $(f_0, \ldots, f_{n_1+\cdots+n_r}) \in (\mathbb{K}[x_{1,0}, \ldots, x_{1,n_1}] \otimes \cdots \otimes \mathbb{K}[x_{r,0}, \ldots, x_{r,n_r}])^{n_1+\cdots+n_r+1}$ to have solutions in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

Sylvester-type determinantal formulas

- Unmixed case (same support)
  [Sturmfels, Zelevinsky, 1994], [Weyman, Zelevinsky, 1994], [Dickenstein, Emiris, 2003], [Emiris, Mantzaflaris, 2012]
- Mixed case (different support)
  ...

We study determinantal formulas for some mixed multilinear systems.

The resultant as the determinant of a complex

Cayley method

- Compute the resultant as determinant of complex $K_\bullet$.

$$K_\bullet : 0 \to K_{n+1} \xrightarrow{\delta_{n+1}} \cdots \to K_1 \xrightarrow{\delta_1} K_0 \xrightarrow{\delta_0} \cdots \to K_{-n} \to 0.$$

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- If $(\forall i \notin \{0,1\})\ K_i = 0$, that is

$$K_\bullet : 0 \to \cdots \to 0 \to K_1 \xrightarrow{\delta_1} K_0 \to 0 \to \cdots \to 0$$

Then, determinantal formula $\to$ Determinant of $K_\bullet$ = Det. of δ_1.

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- These determinantal formula are not necessarily Sylvester-type.

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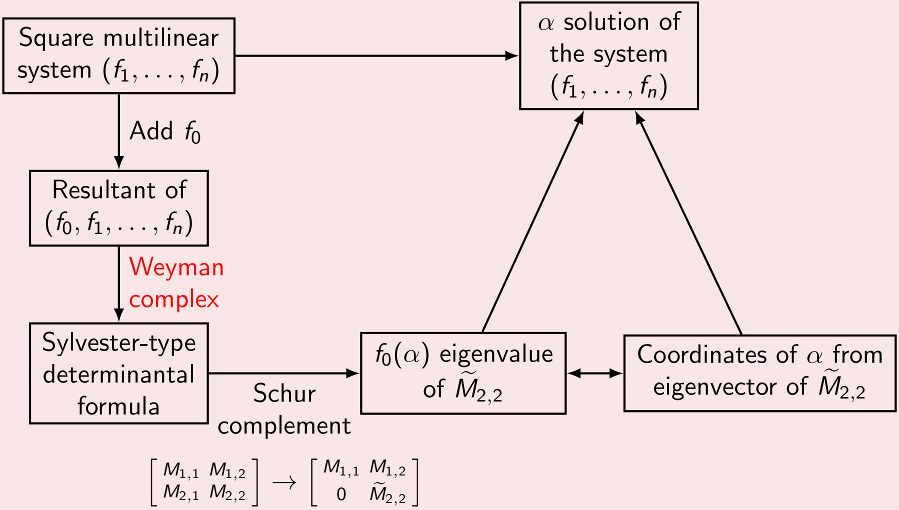
Weyman complex

- Complex parameterized by vector $\boldsymbol{m}$. Its determinant is the resultant.
- Strategy $\to$ Look for vectors $\boldsymbol{m}$ such that the Weyman complex gives a determinantal formula.

Solving polynomial systems using the resultant



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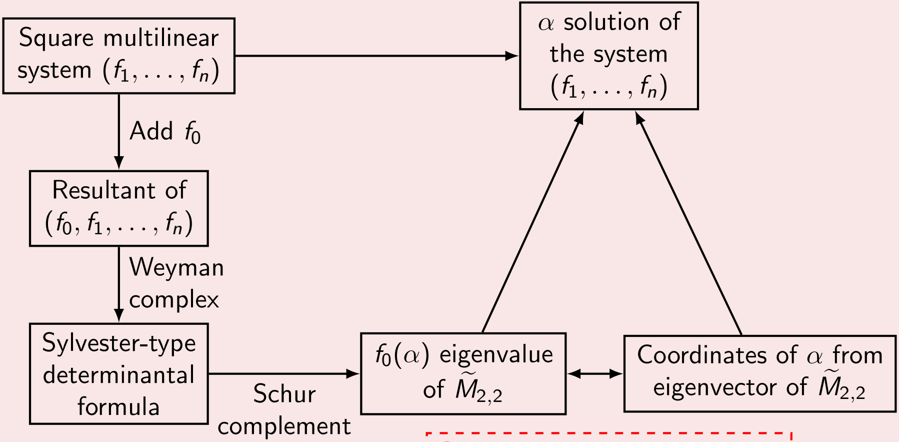

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Great, but...
Sylvester-type det. formulas does not exist in general,
Can we generalize the scheme ?

Example : Solving bilinear system with two supports

- Over $\mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$, we want to solve $(f_1, \dots, f_n)$ such that:
  - $f_1, \dots, f_r \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Y}]_1$, bilinear in the blocks $\boldsymbol{X}$ and $\boldsymbol{Y}$, and
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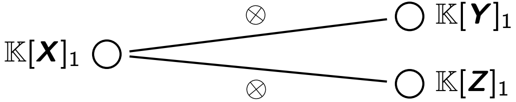

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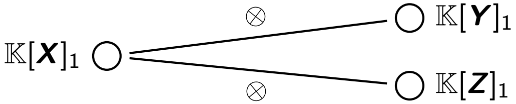

- We introduce a trilinear polynomial $f_0 \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Y}]_1 \otimes \mathbb{K}[\boldsymbol{Z}]_1$.

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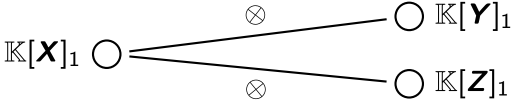
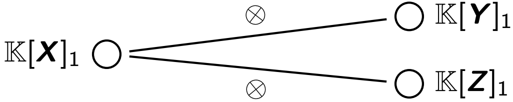


- We introduce a trilinear polynomial $f_0 \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Y}]_1 \otimes \mathbb{K}[\boldsymbol{Z}]_1$.
- Weyman complex $\rightarrow$ Koszul-type formula for the resultant.
  - Generalization of Sylvester-type formula (ie, $(g_0, \ldots, g_n) \mapsto \sum_i g_i f_i$)
  - The elements in the matrix are $\pm$ the coefficients of the polynomials.

Example : Solving bilinear system with two supports

- Over $\mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$, we want to solve $(f_1, \dots, f_n)$ such that:
  - $f_1, \dots, f_r \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Y}]_1$, bilinear in the blocks $\boldsymbol{X}$ and $\boldsymbol{Y}$, and
  - $f_{r+1}, \dots, f_n \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Z}]_1$, bilinear in the blocks $\boldsymbol{X}$ and $\boldsymbol{Z}$.



- We introduce a trilinear polynomial $f_0 \in \mathbb{K}[\boldsymbol{X}]_1 \otimes \mathbb{K}[\boldsymbol{Y}]_1 \otimes \mathbb{K}[\boldsymbol{Z}]_1$.
- Weyman complex $\rightarrow$ Koszul-type formula for the resultant.
  - Generalization of Sylvester-type formula (ie, $(g_0, \dots, g_n) \mapsto \sum_i g_i\, f_i$)
  - The elements in the matrix are $\pm$ the coefficients of the polynomials.

Number of solutions of $(f_1, \dots, f_n)$	Size of the Koszul-type matrix
$\binom{\boldsymbol{r}}{\boldsymbol{n_y}}\binom{\boldsymbol{n-r}}{\boldsymbol{n_z}}$	$(n_x+1)\binom{\boldsymbol{r}}{\boldsymbol{n_y}}\binom{\boldsymbol{n-r}}{\boldsymbol{n_z}}\frac{r\cdot(n-r)-n_y\cdot n_z+n+1}{(r-n_y+1)(n-r-n_z+1)}$

Example : Koszul-type formula

$$\begin{cases} \\ f_1 := \mathbf{7}\, x_0 y_0 + \mathbf{-8}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{2}\, x_1 y_1 \\ f_2 := \mathbf{-5}\, x_0 y_0 + \mathbf{7}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{-1}\, x_1 y_1 \\ f_3 := \mathbf{-6}\, x_0 z_0 + \mathbf{9}\, x_0 z_1 + \mathbf{-1}\, x_1 z_0 + \mathbf{-2}\, x_1 z_1 \end{cases} \quad \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}] \\ \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Z}] \end{matrix}$$

Example : Koszul-type formula

$$\begin{cases} f_0 := \mathbf{3}\, x_0 y_0 z_0 + \mathbf{-1} x_0 y_0 z_1 + \mathbf{-4}\, x_0 y_1 z_0 + \mathbf{2}\, x_0 y_1 z_1 \\ \qquad + \mathbf{1} x_1 y_0 z_0 + \mathbf{2}\, x_1 y_0 z_1 + \mathbf{2}\, x_1 y_1 z_0 + \mathbf{-2}\, x_1 y_1 z_1 \end{cases} \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]$$

$$\begin{cases} f_1 := \mathbf{7}\, x_0 y_0 + \mathbf{-8}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{2}\, x_1 y_1 \\ f_2 := \mathbf{-5}\, x_0 y_0 + \mathbf{7}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{-1}\, x_1 y_1 \end{cases} \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}]$$

$$f_3 := \mathbf{-6}\, x_0 z_0 + \mathbf{9}\, x_0 z_1 + \mathbf{-1}\, x_1 z_0 + \mathbf{-2}\, x_1 z_1 \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Z}]$$

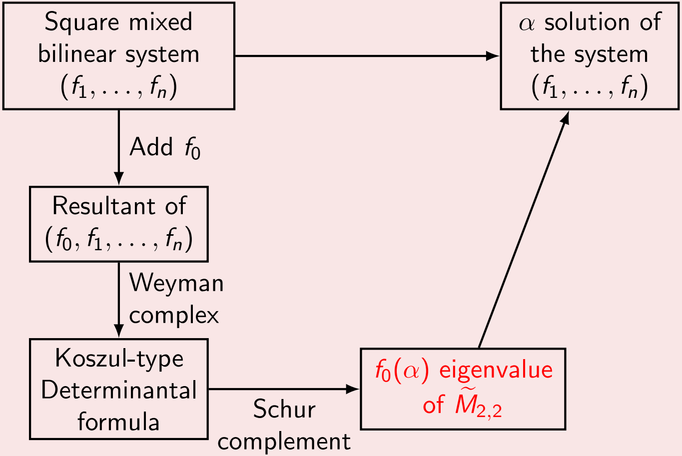
Example : Koszul-type formula

$$\begin{cases} f_0 := 3\,x_0y_0z_0 + -1x_0y_0z_1 + -4\,x_0y_1z_0 + 2\,x_0y_1z_1 \\ \qquad +1x_1y_0z_0 + 2\,x_1y_0z_1 + 2\,x_1y_1z_0 + -2\,x_1y_1z_1 \end{cases} \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}]$$

$$\begin{cases} f_1 := 7\,x_0y_0 + -8\,x_0y_1 + -1\,x_1y_0 + 2\,x_1y_1 \\ f_2 := -5\,x_0y_0 + 7\,x_0y_1 + -1\,x_1y_0 + -1\,x_1y_1 \end{cases} \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Y}]$$

$$f_3 := -6\,x_0z_0 + 9\,x_0z_1 + -1\,x_1z_0 + -2\,x_1z_1 \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Z}]$$

$$M = \begin{bmatrix} & & & 5 & -7 & 1 & 1 & & & \\ & & & 7 & -8 & -1 & 2 & & & \\ & -1 & & & & & & -1 & -5 & 7 \\ 7 & & -1 & & & & & -1 & & -5 \\ & 1 & & & & & & -2 & -7 & 8 \\ 8 & & -2 & & & & & 1 & & -7 \\ & 2 & & 9 & & -2 & & -2 & -1 & 2 \\ 2 & & -2 & & 9 & & -2 & 2 & & -1 \\ & 1 & & -6 & & -1 & & 2 & 3 & -4 \\ -4 & & 2 & & -6 & & -1 & 1 & & 3 \end{bmatrix}$$



$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} f_0 := & \boxed{3}\, \boldsymbol{x_0 y_0 z_0} + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \boldsymbol{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} f_0 := & \boxed{3}\,\mathbf{x_0 y_0 z_0} + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \boldsymbol{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

$$\widetilde{M}_{2,2} := \left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}\right) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} f_0 := & \boxed{3}\, x_0 y_0 z_0 + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \mathbf{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

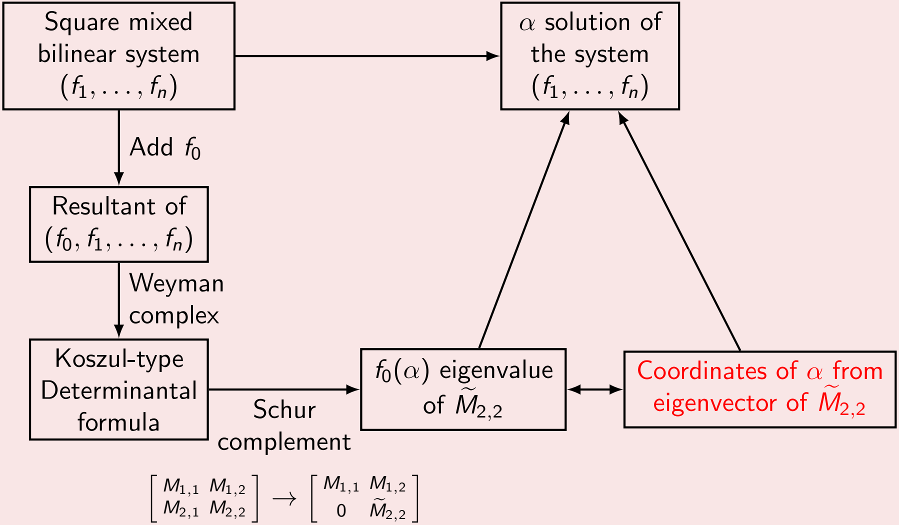
$$\widetilde{M}_{2,2} := \left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}\right) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$$

(f_1, f_2, f_3) has 2 solutions

$$\left.\begin{array}{l} (1:1\ ;\ 1:1\ ;\ 1:1) \\ (1:3\ ;\ 1:2\ ;\ 1:3) \end{array}\right\} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

Eigenvalues of $\widetilde{M}_{2,2}$

$$\frac{f_0}{x_0 y_0 z_0}((1:1\ ;\ 1:1\ ;\ 1:1)) = 3$$
$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1$$


Square mixed bilinear system $(f_1, \ldots, f_n)$
α solution of the system $(f_1, \ldots, f_n)$
Add f_0
Resultant of $(f_0, f_1, \ldots, f_n)$
Weyman complex
Koszul-type Determinantal formula
Schur complement
$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$
Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$
$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$

Example : Generalization of the eigenvector criterion

$$\frac{f_0}{x_0 y_0 z_0}((\mathbf{1}:\mathbf{3}\ ;\ \mathbf{1}:\mathbf{2}\ ;\ 1:3)) = \mathbf{1} \qquad \bar{\boldsymbol{v}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \bar{\boldsymbol{v}} = \mathbf{1} \cdot \bar{\boldsymbol{v}}$$

We can not recover $(\mathbf{1}:\mathbf{3}\ ;\ \mathbf{1}:\mathbf{2}\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example : Generalization of the eigenvector criterion

$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1 \qquad \bar{\boldsymbol{v}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \bar{\boldsymbol{v}} = 1 \cdot \bar{\boldsymbol{v}}$$

We can not recover $(1:3\ ;\ 1:2\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We extend $\bar{\boldsymbol{v}} \to \boldsymbol{v}$ s.t.

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right] \cdot \boldsymbol{v} = \frac{f_0}{\mathfrak{m}}(\alpha) \cdot \left[\begin{array}{c} 0 \\ \hline \bar{\boldsymbol{v}} \end{array}\right] \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 4 & = & 1 \cdot 2 \cdot 2 \\ 3 & = & 3 \cdot 1 \cdot 1 \\ 12 & = & 3 \cdot 2 \cdot 2 \\ 1 & = & 1 \cdot 1 \cdot 1 \\ 2 & = & 1 \cdot 1 \cdot 2 \\ 3 & = & 3 \cdot 1 \\ 6 & = & 3 \cdot 1 \cdot 2 \\ 6 & = & 3 \cdot 2 \\ 1 & = & 1 \cdot 1 \\ 2 & = & 1 \cdot 2 \end{bmatrix}$$

$$\begin{cases} (1 \cdot \partial x_0 + 3 \cdot \partial x_1) \otimes \left(1 \cdot 1 \cdot \partial y_0^2 + 1 \cdot 2 \cdot \partial y_0 \partial y_1 + 2 \cdot 2 \cdot \partial {y_1}^2\right) \otimes 1 \\ (1 \cdot \partial x_0 + 3 \cdot \partial x_1) \otimes (1 \cdot \partial y_0 + 2 \cdot \partial y_1) \otimes 1 \end{cases}$$

Solving Mixed Square Multilinear Systems

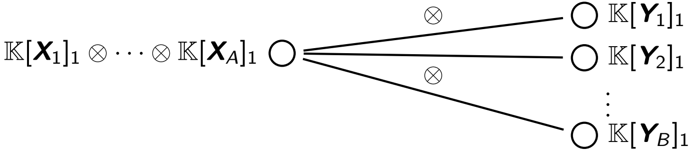
Let $(f_1, \dots, f_n) \in (\mathbb{K}[\boldsymbol{X}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{X}_A] \otimes \mathbb{K}[\boldsymbol{Y}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{Y}_B])^n$.

Solving Mixed Square Multilinear Systems

Let $(f_1, \dots, f_n) \in (\mathbb{K}[\boldsymbol{X}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{X}_A] \otimes \mathbb{K}[\boldsymbol{Y}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{Y}_B])^n$.

- **Star multilinear system**: For every f_k, there is j_k such that

$$f_k \in \mathbb{K}[\boldsymbol{X}_1]_1 \otimes \dots \mathbb{K}[\boldsymbol{X}_A]_1 \otimes \mathbb{K}[\boldsymbol{Y}_{j_k}]_1.$$



Solving Mixed Square Multilinear Systems

Let $(f_1, \dots, f_n) \in (\mathbb{K}[\boldsymbol{X}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{X}_A] \otimes \mathbb{K}[\boldsymbol{Y}_1] \otimes \cdots \otimes \mathbb{K}[\boldsymbol{Y}_B])^n$.

- **Star multilinear system**: For every f_k, there is j_k such that

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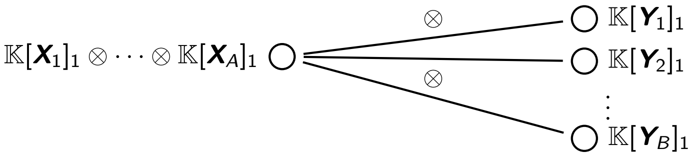
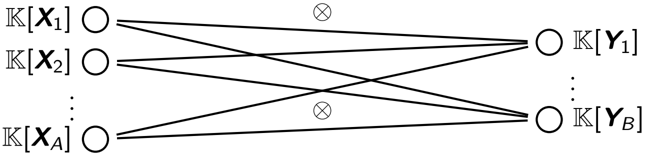


- **Bipartite bilinear system**: For every f_k, there are i_k and j_k such that

$$f_k \in \mathbb{K}[\boldsymbol{X}_{i_k}]_1 \otimes \mathbb{K}[\boldsymbol{Y}_{j_k}]_1.$$



Applications : Multiparameter Eigenvalue Problem

Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} \right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 .$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 \; .$$

Applications : Multiparameter Eigenvalue Problem

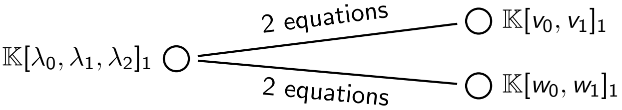
- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system

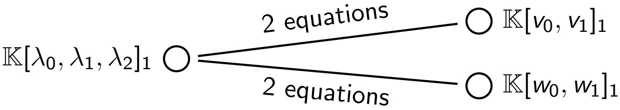


$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 .$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system

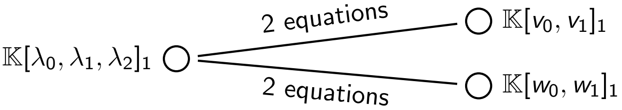


$$\begin{bmatrix} (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2) & (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2) \\ (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2) & (-2\,\lambda_0 + \lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 .$$

$$\begin{bmatrix} (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2) & (-3\,\lambda_0 - \lambda_1) \\ (4\,\lambda_0 + \lambda_1 - \lambda_2) & (\lambda_0 + 2\,\lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Applications : Multiparameter Eigenvalue Problem

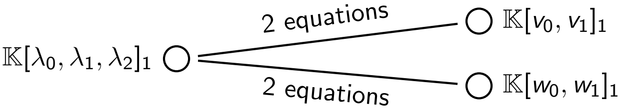
- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system



$$\begin{cases} (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 = 0 \\ (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 = 0 \\ (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 = 0 \\ (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 = 0 \end{cases} .$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system



$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases} .$$

Applications : Multiparameter Eigenvalue Problem

$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases}.$$

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To solve, we add linear $f_0 := -\lambda_0 + 5\,\lambda_1 - 3\,\lambda_2 \in \mathbb{K}[\boldsymbol{\lambda}]_1$.

Weyman complex $\rightarrow$ Sylvester-type formula

$$\delta : (\mathbb{K}[\boldsymbol{v}]_1 \otimes \mathbb{K}[\boldsymbol{w}]_1) \times (\mathbb{K}[\boldsymbol{w}]_1)^2 \times (\mathbb{K}[\boldsymbol{v}]_1)^2 \rightarrow (\mathbb{K}[\boldsymbol{\lambda}]_1 \otimes \mathbb{K}[\boldsymbol{v}]_1 \otimes \mathbb{K}[\boldsymbol{w}]_1)$$

$$(\quad g_0, \quad g_1, g_2 \quad g_3, g_4 \quad) \mapsto \quad \textstyle\sum_{i=0}^{4} g_i\, f_i$$

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$w_0 f_1$	−7		−1		12		2		−7		−3	
$w_1 f_1$		−7		−1		12		2		−7		−3
$w_0 f_2$	−7		−1		13		1		−8		−2	
$w_1 f_2$		−7		−1		13		1		−8		−2
$v_0 f_3$	−4				7	−1			−11	−3		
$v_1 f_3$			−4				7	−1			−11	−3
$v_0 f_4$	−1	−1			1	2			4	1		
$v_1 f_4$			−1	−1			1	2			4	1
$v_0 w_0 f_0$	−3				5				−1			
$v_0 w_1 f_0$		−3				5				−1		
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Summing-up

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- Weyman complex $\rightarrow$ Determinantal formula
- Sylvester- and Koszul-type formulas
- Eigenvalues/Eigenvectors
  (Evaluation of the solutions/coordinates of the solutions)

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- Sylvester- and Koszul-type determinantal formula for the resultant
- Extension of the Eigenvalue and Eigenvector criteria
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- Can we exploit structure of Koszul-type formulas?
- What can we say numerically?

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Thank you!