Characteristic polynomials of p-adic matrices.

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A first question

Determinant computation

$$\begin{bmatrix} X^5 + O(X^{10}) & 1 + O(X^{10}) & 1 + X^3 + O(X^{10}) \\ O(X^{10}) & 1 + O(X^{10}) & 1 + O(X^{10}) \\ 2X^6 + O(X^{10}) & 2X + O(X^{10}) & 2X + X^5 + O(X^{10}) \end{bmatrix}$$

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Question

What is the **precision on the determinant** ?

A little warm-up on computing determinants : expansion

An example of determinant computation

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Using Leibniz formula

If we expand directly using the expression of the determinant in terms of the coefficients, we get:

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Using Leibniz formula

If we expand directly using the expression of the determinant in terms of the coefficients, we get:

$$-2X^9 + O(X^{10}),$$

because of $1 \times 1 \times O(X^{10})$.

A little warm-up on computing determinants : row-echelon form computation

An example of determinant computation

$$\begin{bmatrix} X^5 + O(X^{10}) & 1 + O(X^{10}) & 1 + X^3 + O(X^{10}) \\ O(X^{10}) & 1 + O(X^{10}) & 1 + O(X^{10}) \\ O(X^{10}) & O(X^{10}) & -2X^4 + X^5 + O(X^{10}) \end{bmatrix}$$

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If we compute **approximate** row-echelon form, we still get:

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If we compute **approximate** SNF, we now get:

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Smith Normal Form (SNF) computation

If we compute **approximate** SNF, we now get:

$$-2X^9+X^{10}+O(X^{13}),$$

because of $1 \times X^3 \times O(X^{10}) = O(X^{13})$.

Questions for today

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Remarque

From now on, we will work over $\mathbb{Q}_p$ instead of $K[\![X]\!]$, but there is no difference in the behaviour regarding to precision.

Motivations and goal

Counting points on curves

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Motivations and goal

Counting points on curves

- Kedlaya's algorithm to count point on curves.
- One core part of Kedlaya's algorithm is the computation of the **characteristic polynomial** of the linear mapping given by the Frobenius acting on some cohomological p-adic vector space.

Today's goal

- What is the (optimal) precision on the characteristic polynomial of a matrix with p-adic entries all known at the same precision?
- How can we compute at this precision?

Table of contents

Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=l}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $l \in \mathbb{Z}$ and p a prime number.

Working with a computer, we usually only can consider the beginning of this power series expansion: we only consider elements of the form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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The **order**, or the **absolute precision** of $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$ is d.

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Exemple

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

Precision formulae

Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{\min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1} p^{a-c} + O(p^{\min(d+a-2c,b-c)})$$

In particular,
$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

The Main lemma of p-adic differential precision

Lemma (CRV14)

Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ be a (strictly) ***differentiable*** *mapping.*

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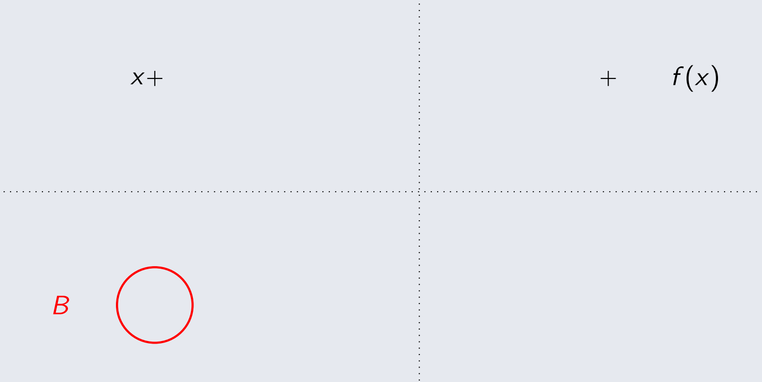
$$f(x + B) = f(x) + f'(x) \cdot B.$$

Geometrical meaning

Interpretation



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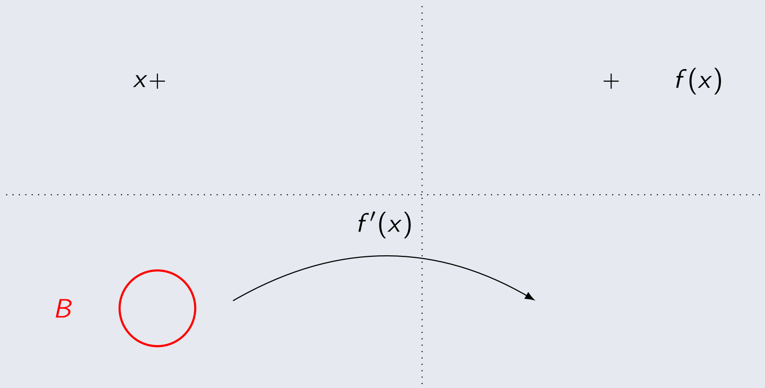

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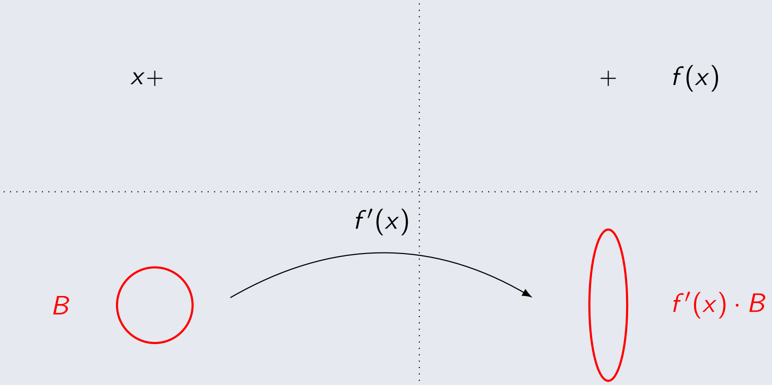

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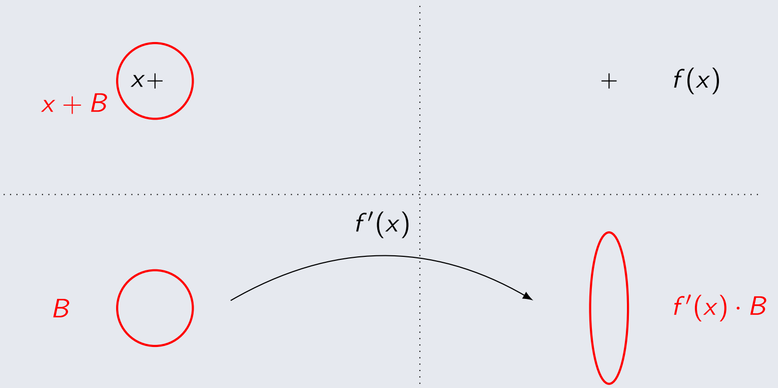

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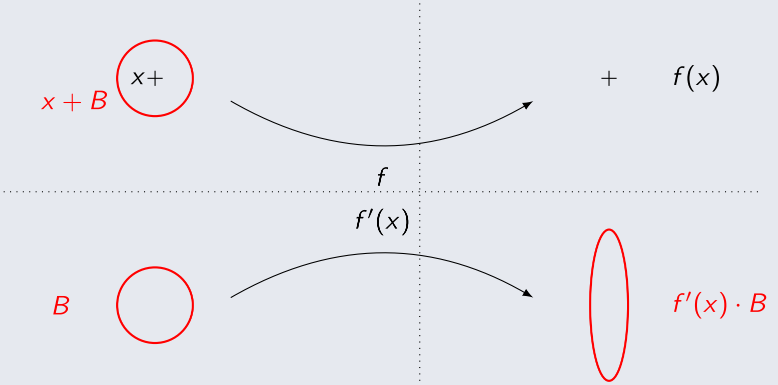

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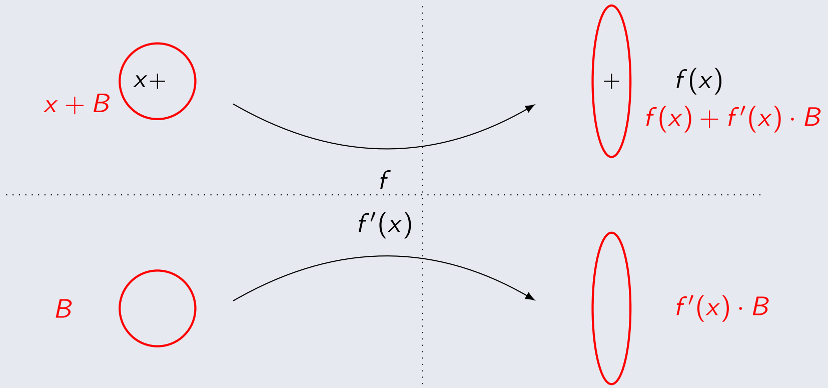

Looking back to the case of the determinant

Differential of the determinant

It is well known:

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Linear equations

One can also easily prove that SNF is optimal to solve linear equations.

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Classical ways to compute χ_M

Direct Gaussian elimination

Is $\tilde{O}(n^4)$, **with divisions**.

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Also with divisions

Fadeev-Leverrier and Berlekamp-Massey.

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What is left?

- No division, so precision is saved, can a **gain of precision** be seen?
- If we know the optimal precision. We can perform Kaltoffen-Villard at high-enough precision to get the extra digits.

$\chi'(M)$

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Derivative of χ_M

$$\chi'(M) : dM \mapsto \mathrm{Tr}(\mathrm{Adj}(XI_n - M) \cdot dM).$$

Naïve computations

Formulae

$$\chi'(M) : dM \mapsto \mathrm{Tr}(\mathrm{Adj}(XI_n - M) \cdot dM).$$

$$\mathrm{Adj}(XI_n - M) = \chi_M \times (XI_n - M)^{-1}.$$

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First idea

- Compute (approximations of) χ_M and $(XI_n - M)^{-1}$.

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First idea

- Compute (approximations of) χ_M and $(XI_n - M)^{-1}$.
- Computing $(XI_n - M)^{-1} \mod X^{n+1}$ is $\tilde{O}(n^4)$ by Gaussian elimination (+ it requires divisions).

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Invariance

Similarity

For $H = PMP^{-1}$,

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- Hessenberg?

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Enough for flat precision.

Which form?

- Jordan or trigonal? No.
- Frobenius? Too complicated?
- Hessenberg? Seems a good idea.

Hessenberg form

Hessenberg matrix

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & & m_{1,n-1} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & & m_{2,n} \\ 0 & m_{3,2} & m_{3,3} & m_{3,4} & & & & m_{3,n} \\ 0 & 0 & m_{4,3} & m_{4,4} & & & & m_{4,n} \\ & 0 & 0 & m_{5,4} & & & & \\ & 0 & & 0 & & & & \\ & & & & & m_{n-1,n-2} & m_{n-1,n-1} & m_{n-1,n} \\ 0 & 0 & 0 & 0 & & 0 & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

Remark

A companion matrix is Hessenberg.

Hessenberg form

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Remark

A companion matrix is Hessenberg. The Frobenius form is Hessenberg.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** and put it on position $(2, 1)$.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** and put it on position $(2, 1)$.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** and put it on position $(2,1)$.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** and put it on position $(2,1)$.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** and put it on position $(2, 1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2, 1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & & m_{4,n} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & & & m_{3,n} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ m_{3,1} & m_{3,4} & m_{3,3} & m_{3,2} & & & m_{3,n} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ m_{3,1} & m_{3,4} & m_{3,3} & m_{3,2} & & & m_{3,n} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ m_{3,1} & m_{3,4} & m_{3,3} & m_{3,2} & & & m_{3,n} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ m_{3,1} & m_{3,4} & m_{3,3} & m_{3,2} & & & m_{3,n} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & m_{1,4} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & m_{4,4} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ m_{2,1} & m_{2,4} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & m_{n,4} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ m_{2,1} & \widetilde{m_{2,4}} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ m_{2,1} & \widetilde{m_{2,4}} & m_{2,3} & m_{2,2} & & & m_{2,n} \\ & & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & \widetilde{m_{2,n}} \\ & & & & & \\ m_{n,1} & \widetilde{m_{n,4}} & m_{n,3} & m_{n,2} & m_{n-1,n} & m_{n,n} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ 0 & \widetilde{m_{n,4}} & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2, 1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ 0 & \widetilde{m_{n,4}} & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ 0 & \widetilde{m_{n,4}} & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & & & & & & \\ 0 & \widetilde{m_{n,4}} & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2, 1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & m_{1,3} & m_{1,2} & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & m_{4,3} & m_{4,2} & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & \widetilde{m_{3,n}} \\ 0 & \widetilde{m_{2,4}} & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & \widetilde{m_{2,n}} \\ & & & & & \\ 0 & \widetilde{m_{n,4}} & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & \widetilde{m_{1,3}} & m_{1,2} & & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & \widetilde{m_{4,3}} & m_{4,2} & & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\ 0 & 0 & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\ & 0 & & & & & \\ & 0 & & & & & \\ & & & & & & \\ 0 & 0 & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & \widetilde{m_{1,3}} & m_{1,2} & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & \widetilde{m_{4,3}} & m_{4,2} & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & \widetilde{m_{3,n}} \\ 0 & 0 & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & \widetilde{m_{2,n}} \\ & 0 & & & & \\ & 0 & & & & \\ 0 & 0 & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2, 1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

Computation of an Hessenberg form

Hessenberg reduction: modified Gaussian elimination

$$P_* M P_*^{-1} = \begin{bmatrix} m_{1,1} & \widetilde{m_{1,4}} & \widetilde{m_{1,3}} & m_{1,2} & m_{1,n} & m_{1,n} \\ m_{4,1} & \widetilde{m_{4,4}} & \widetilde{m_{4,3}} & m_{4,2} & & m_{4,n} \\ 0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & \widetilde{m_{3,n}} \\ 0 & 0 & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & \widetilde{m_{2,n}} \\ & 0 & & & & \\ & 0 & & & & \\ 0 & 0 & \widetilde{m_{n,3}} & \widetilde{m_{n,2}} & \widetilde{m_{n-1,n}} & \widetilde{m_{n,n}} \end{bmatrix}$$

We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

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We take as pivot the coefficient $m_{i,1}$ on first column with the **smallest valuation** put it on position $(2,1)$. This reflects to the columns. We pivot with the second row. It reflects on the columns. We proceed **recursively**.

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Hessenberg reduction: modified Gaussian elimination

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Hessenberg reduction: modified Gaussian elimination

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The result is Hessenberg. It required $O(n^3)$ operations on the base field.

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Hessenberg reduction: modified Gaussian elimination

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m_{1,1} & \widetilde{m_{1,4}} & \widetilde{m_{1,3}} & m_{1,2} & & m_{1,n-1} & m_{1,n} \\
m_{4,1} & \widetilde{m_{4,4}} & \widetilde{m_{4,3}} & m_{4,2} & & & m_{4,n} \\
0 & \widetilde{m_{3,4}} & \widetilde{m_{3,3}} & \widetilde{m_{3,2}} & & & \widetilde{m_{3,n}} \\
0 & 0 & \widetilde{m_{2,3}} & \widetilde{m_{2,2}} & & & \widetilde{m_{2,n}} \\
 & 0 & 0 & \widetilde{m_{5,2}} & & & \\
 & 0 & & 0 & & & \\
 & & & & m_{\widetilde{n-1,n-2}} & m_{\widetilde{n-1,n-1}} & m_{\widetilde{n-1,n}} \\
0 & 0 & 0 & 0 & 0 & m_{\widetilde{n-1,n}} & \widetilde{m_{n,n}}
\end{bmatrix}$$

The result is Hessenberg. It required $O(n^3)$ operations on the base field.
It is possible to do everything $\bmod p^N$, with no division.

Table of contents

Adjugate of $H = PMP^{-1}$

$XI_n - H$

$$\det(XI_n - H) = \chi_H.$$
$$Adj(XI_n - H) = \chi_M \times (XI_n - H)^{-1}.$$

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$I_n - XH$

$$det(I_n - XH) = \chi_H^*, \text{ reciprocal polynomial.}$$
$$\text{Adj}(I_n - XH) = \chi_M \times (I_n - XH)^{-1}.$$

Computation

An algorithm for Hessenberg matrices: computation of $(Id - XH)^{-1}$

$$P_*(I_n - XH)Q_* = \begin{bmatrix} 1 - Xh_{1,1} & Xh_{1,2} & Xh_{1,3} & & & Xh_{1,n-1} & Xh_{1,n} \\ Xh_{2,1} & 1 - Xh_{2,2} & Xh_{2,3} & & & Xh_{2,n-1} & Xh_{2,n} \\ 0 & Xh_{3,2} & 1 - Xh_{3,3} & & & & \\ 0 & 0 & Xh_{4,3} & & & & \\ 0 & 0 & 0 & & & & \\ & & & & Xh_{n-1,n-2} & 1 - Xh_{n-1,n-1} & Xh_{n-1,n} \\ 0 & 0 & 0 & & 0 & Xh_{n,n-1} & 1 - Xh_{n,n} \end{bmatrix}$$

Everything done $\bmod p^M, X^{n+1}$.

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$$P_*(I_n - XH)Q_* = \begin{bmatrix} 1 - Xh_{1,1} & Xh_{1,2} & Xh_{1,3} & & & Xh_{1,n-1} & Xh_{1,n} \\ Xh_{2,1} & 1 - Xh_{2,2} & Xh_{2,3} & & & Xh_{2,n-1} & Xh_{2,n} \\ 0 & Xh_{3,2} & 1 - Xh_{3,3} & & & & \\ 0 & 0 & Xh_{4,3} & & & & \\ 0 & 0 & 0 & & & & \\ & & & & Xh_{n-1,n-2} & 1 - Xh_{n-1,n-1} & Xh_{n-1,n} \\ 0 & 0 & 0 & & 0 & Xh_{n,n-1} & 1 - Xh_{n,n} \end{bmatrix}$$

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Everything done mod p^M, X^{n+1}.

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Everything done $\bmod p^M, X^{n+1}$. $\det(Id - XH) = \prod_i (1 - X\widetilde{h_{i,i}})$.

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Everything done $\bmod p^M, X^{n+1}$. $\det(Id - XH) = \prod_i (1 - X\widetilde{h_{i,i}})$.
$(Id - XH)^{-1}$ obtained from $Q\Delta^{-1}P$, in $\tilde{O}(n^3)$, with no division.

Conclusion on flat precision

The case of flat precision

M is given at precision $dM = O(p^N)$ on each coefficient.

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- If $P^{-1}dMP = dM$,

$$tr(\mathrm{Adj}(I_n - XM) \cdot dM) = tr(\mathrm{Adj}(I_n - XH) \cdot dM).$$

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Precision can directly be read from $\mathrm{Adj}(I_n - XH)$.

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$$tr(\mathrm{Adj}(I_n - XM) \cdot dM) = tr(\mathrm{Adj}(I_n - XH) \cdot dM).$$

Precision can directly be read from $\mathrm{Adj}(I_n - XH)$. All in all in $\tilde{O}(n^3)$, with no division.

Non-flat

Jagged precision

Precision dM on each coefficient of M may differ.

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Computing $P\,\mathrm{Adj}(XI_n - H)P^{-1}$ or $P^{-1}dMP$ is very costly.

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In $\tilde{O}(n^3)$, but with divisions to compute the factorization (Extended Euclidean Algorithm).

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Complexity and precision

In $\tilde{O}(n^3)$, but with divisions to compute the factorization (Extended Euclidean Algorithm). Enough for precision on every coefficient.

Table of contents

In practice, is it worth it?

Average precision loss on the characteristic polynomial of a random 9×9 matrix over $\mathbb{Q}_2$— results for a sample of 1000 instances.

	Average loss of accuracy	
	Optimal	Naïve, division-free
X^0 (det.)	3.17	196
X^1	2.98	161
X^2	2.75	129
X^3	2.74	108
X^4	2.57	63.2
X^5	2.29	51.6
X^6	2.07	9.04
X^7	1.64	5.70
X^8 (trace)	0.99	0.99

To sum up

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- Can know the **optimal** precision in $\tilde{O}(n^3)$ with few divisions when starting from jagged precision.
- If one allows (few) divisions, faster methods are possible.

References

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Linear Algebra

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Characteristic polynomial

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Thank you for your attention

Thanks