Fibers of multi-graded rational maps & orthogonal projection onto rational surfaces

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What are parametric curves & surfaces ?

$$\varphi := \begin{array}{rcl} \mathbb{R} & \rightarrow & \mathbb{R}^3 \\ s & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \frac{f_3(s)}{f_0(s)}\right), \end{array}$$

$$\varphi := \begin{array}{rcl} \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ (s,u) & \mapsto & \left(\frac{f_1(s,u)}{f_0(s,u)}, \frac{f_2(s,u)}{f_0(s,u)}, \frac{f_3(s,u)}{f_0(s,u)}\right), \end{array}$$

where f_0, f_1, f_2, f_3 are polynomials in s and s, u respectively over $\mathbb{R}$, then $\overline{Im(\varphi)}$ defines surface in $\mathbb{R}^3$.

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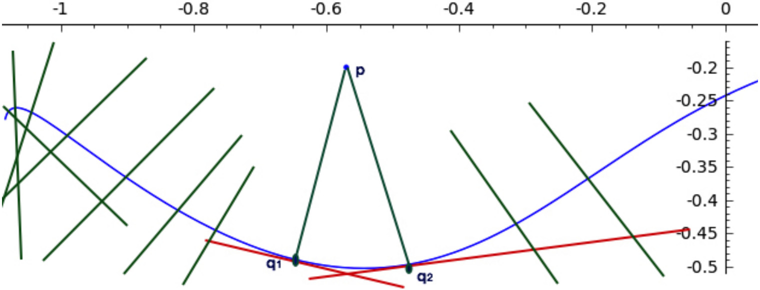
What is the distance between a point and a plane curve?

We would like to compute the distance from a point $p \in \mathbb{R}^2$ to a parametric curve $\mathcal{C}$ ($\varphi : \mathcal{R} \to \mathbb{R}^2$ such that $(s) \mapsto \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}\right)$).
For this reason, we look for the orthogonal projections of p onto $\mathcal{C}$.



Red lines : tangent lines at q_1 and q_2,
Green lines : normal lines to the curve $\mathcal{C}$.

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Parametrization for normal lines to $\mathcal{C}$:
$\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(s,t) \mapsto (\phi(s) + t\eta(s))$, where $\eta(s)$ is normal vector obtained by $\left(\frac{-d}{ds}\left(\frac{f_2(s)}{f_0(s)}\right), \frac{d}{ds}\left(\frac{f_1(s)}{f_0(s)}\right)\right)$.

Orthogonal projections of p are the pre-images of p via ψ :

$$\psi^{-1}(p) := \{(s_0, u_0) \in \mathbb{R}^2 : \psi(s_0, u_0) = p\}.$$

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Orthogonal projections of p are the pre-images of p via ψ :

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Expected number of the orthogonal projections :
Suppose $deg(f_i) = d, i = \{0,1,2\}$.

$$\begin{aligned} \text{non-rational, i.e. } f_0 = 1, &\quad 2d-1, \\ \text{rational, i.e. } f_0 \neq 1, &\quad 3d-2. \end{aligned}$$

SURFACES

Triangular surfaces

$$\begin{array}{rccl}\varphi := & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (s,u) & \mapsto & \left(\frac{f_1(s,u)}{f_0(s,u)}, \frac{f_2(s,u)}{f_0(s,u)}, \frac{f_3(s,u)}{f_0(s,u)}\right),\end{array}$$

where f_0, f_1, f_2, f_3 are polynomials of degree d in s, u over $\mathbb{R}$.
Choose a basis : monomial basis. Then, f_0, f_1, f_2, f_3 are written in basis

$$\begin{array}{lllll}\{s^d, & & & & \\ s^{d-1}, & s^{d-1}u, & & & \\ s^{d-2}, & s^{d-2}u, & s^{d-2}u^2, & & \\ \vdots & \vdots & \vdots & \vdots & \\ 1, & u, & u^2, & \cdots & u^d\}.\end{array}$$

If $f_0 = 1$, then the surface is called, non-rational triangular surface, otherwise it is called rational triangular surface.

Tensor-product surfaces

$$\varphi := \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s,u) & \mapsto & \left(\frac{f_1(s,u)}{f_0(s,u)}, \frac{f_2(s,u)}{f_0(s,u)}, \frac{f_3(s,u)}{f_0(s,u)}\right), \end{array}$$

where f_0, f_1, f_2, f_3 are polynomials of degree d_1 in s and d_2 in u over $\mathbb{R}$.

Choose a basis : monomial basis. Then, f_0, f_1, f_2, f_3 are written in basis

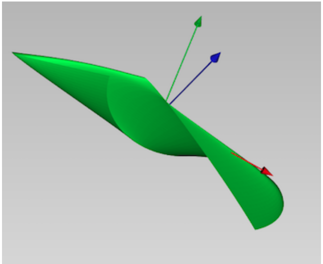
$$\begin{array}{cccc} \{s^{d_1}u^{d_2}, & s^{d_1}u^{d_2-1}, & \cdots & , \quad s^{d_1}, \\ s^{d_1-1}u^{d_2}, & s^{d_1-1}u^{d_2-1}, & \cdots & , \quad s^{d_1-1} \\ \vdots & \vdots & \vdots & \vdots \\ u^{d_2}, & u^{d_2-1}, & \cdots & , \quad u^{d_2}\}. \end{array}$$

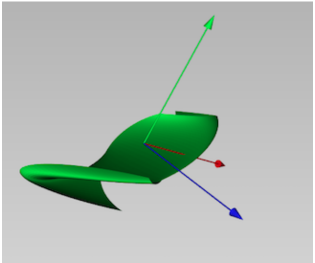
If $f_0 = 1$, then the surface is called, non-rational tensor-product surface, otherwise it is called rational tensor-product surface.

Tensor-product surfaces

Ex: $(2,2)$ tensor-product surface

$$\varphi := \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s,u) & \mapsto & \left(\begin{array}{c} \frac{-4s^2u^2-su^2-s^2+su-u^2-s+18u}{-2s^2u^2-s^2u-12s^2-8su-7u^2+2s+u-9}, \\ \frac{-s^2u^2+su^2+2s^2-2su-u^2-s+4u+1}{-2s^2u^2-s^2u-12s^2-8su-7u^2+2s+u-9}, \\ \frac{-2s^2u^2-11s^2u+5su^2+2s^2-su+3u^2-5s-5u-1}{-2s^2u^2-s^2u-12s^2-8su-7u^2+2s+u-9} \end{array} \right). \end{array}$$





Figures are done in Axl.

What are closest points?

$$\begin{array}{rcl} \varphi : \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s,u) & \mapsto & \left(\frac{f_1(s,u)}{f_0(s,u)}, \frac{f_2(s,u)}{f_0(s,u)}, \frac{f_3(s,u)}{f_0(s,u)} \right). \end{array}$$

$\overline{im(\varphi)} = \mathcal{S}$ defines a surface in $\mathbb{R}^3$, x_0 point in $\mathbb{R}^3$.

Closest points p_0's on $\mathcal{S}$ to x_0 are minimizing the distance function

$$dist_{p_0 \in \mathcal{S}}(x_0, p_0).$$

We look for the orthogonal projections

$$dist(x_0, \varphi(s,u)) = ||x_0 - \varphi(s,u)||,$$

where $||.||$ Euclidean norm. We consider

$$||x_0 - \varphi(s,u)||^2.$$

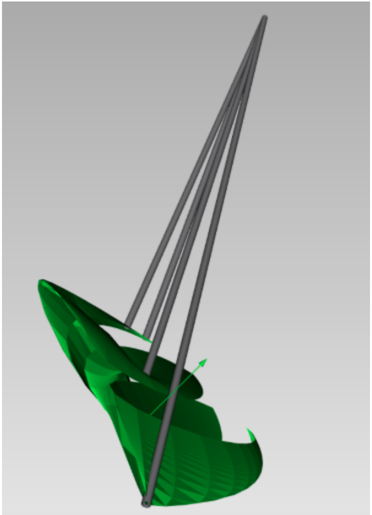
We study its extremas i.e,

$$i) \quad \frac{\partial(||x_0 - \varphi(s,u)||^2)}{\partial s} = 2(x_0 - \varphi(s,u))\frac{\partial \varphi(s,u)}{\partial s} = 0,$$

$$ii) \quad \frac{\partial(||x_0 - \varphi(s,u)||^2)}{\partial u} = 2(x_0 - \varphi(s,u))\frac{\partial \varphi(s,u)}{\partial u} = 0.$$

- *i*) and *ii*) give the orthogonality conditions.
- The solutions of *i*) and *ii*) contain the closest points.

We look for the orthogonal projections



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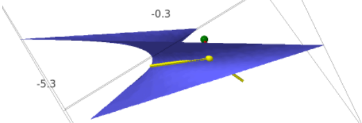
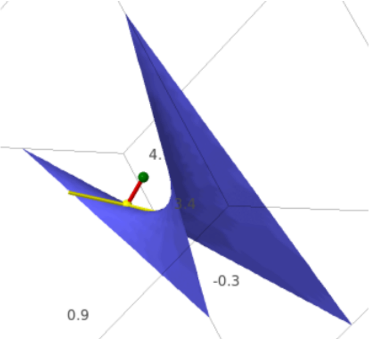
We look for the orthogonal projections





Green point : the point that we project orthogonaly on the surface,
Yellow point : orthogonal projection of green point,
Red line : normal line at yellow point,
Yellow lines : tangent lines at yellow point.

Theoretical bound for the number of orthogonal projections onto $\mathcal{S}$

Notation :

By Draisma, Horobet, Ottaviani, Sturmfels, Thomas 2014,

$$\text{EDdegree} := \text{number of the orthogonal projections.}$$

Theoretical bound for the number of orthogonal projections onto $\mathcal{S}$

$\mathcal{S}$: tensor-product surface,
ψ : parametrization of $\mathcal{S}$ of degree (d_1, d_2), then EDdegree for tensor-product surfaces $\mathcal{S}$ is

non-rational	$8d_1d_2 - 2(d_1 + d_2) + 1$,
rational	$14d_1d_2 - 6(d_1 + d_2) + 4$.

(d_1, d_2)	non-rat	rat
$(1,1)$	5	6
$(1,2)$	11	14
$(1,3)$	17	22
$(2,2)$	25	36
$(2,3)$	39	58
$(3,3)$	61	94

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Theoretical bound for the number of orthogonal projections onto $\mathcal{S}$

$\mathcal{S}$: triangular surface,
ψ : parametrization of $\mathcal{S}$ of degree d, then EDdegree for triangular surfaces $\mathcal{S}$ is

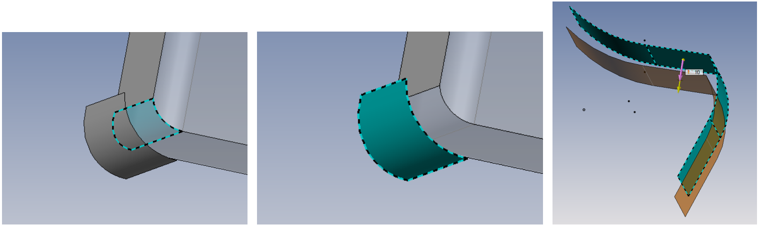
non-rational	$(2d-1)^2$,
rational	$7d^2 - 9d + 3$.

d	non-rat	rat
1	1	1
2	9	13
3	25	39
4	49	79

Where does distance problem appear in CAD ?

Applications in CAD

- Offset surface

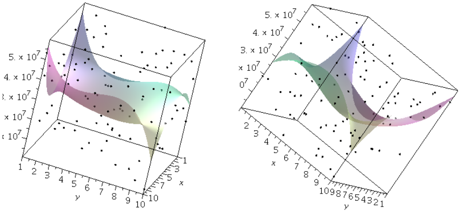


The figures are done in CAD software TopSolid.

Where does distance problem appear in CAD ?

Applications in CAD

- Offset surface
- Surface fitting



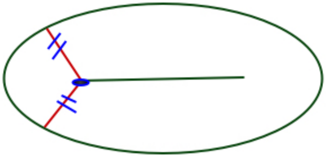
We have finite number of points p_i, for $i \in 1, \dots, n$, $n \in \mathbb{N}$ and we look for a approximate surface $\mathcal{S}$ which minimizes for instance

$$\sum_i dist(\mathcal{S}, p_i)^2.$$

Where does distance problem appear in CAD ?

Applications in CAD

- Offset surface
- Surface fitting
- Medial Axe



Existing Methods

- Iterative methods, Newton-Ralphson

  Problems

  - Initial value, convergence,
  - it does not see multiple solutions.

- Subdivision methods

  - More robust because no initial guess needed.

- Algebraic methods

  - Usually use exact data

We propose an algebraic method which is also symbolic-numeric and which get on well with approximate data.

Closest point computation using moving surfaces

Moving surface is introduced by Sederberg and Chen in 1995 for the implicitization problem.

Closest point computation using moving surfaces

What is a moving surface?

Let

$$\varphi := \begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R}^3 \\ (s,u) & \mapsto & (\varphi_1, \varphi_2, \varphi_3) \end{array}$$

be a parametrization of a given tensor-product surface. $\varphi_1, \varphi_2, \varphi_3$ are fractions of polynomials in s, u of degree d_1, d_2 respectively.
A moving surface M is

$$M = \sum_{\substack{deg_s A_i \leq d_1 \\ deg_u A_i \leq d_2 \\ \alpha_1+\alpha_2+\alpha_3 \leq r}} A_i(s,u)\, T_1^{\alpha_1}\, T_2^{\alpha_2}\, T_3^{\alpha_3},$$

where A is of degree (d_1, d_2), and r is the degree on T_1, T_2, T_3.
We say that M follows the surface if

$$\sum_{\substack{deg_s A_i \leq d_1 \\ deg_u A_i \leq d_2 \\ \alpha_1+\alpha_2+\alpha_3 \leq r}} A_i(s,u) \varphi_1(s,u)^{\alpha_1} \varphi_2(s,u)^{\alpha_2} \varphi_3(s,u)^{\alpha_3} \equiv 0.$$

Closest point computation using moving surfaces

Related work : Thomassen, Johansen, Dokken 2004

- They construct 2 moving surfaces M_1 in s and M_2 in u,

Closest point computation using moving surfaces

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- They construct 2 moving surfaces M_1 in s and M_2 in u,
- M_1, M_2 are **high** degree (with the previous notation) both in (d_1, d_2) and r,

Closest point computation using moving surfaces

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- M_1, M_2 are **high** degree (with the previous notation) both in (d_1, d_2) and r,
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- This method allows the use of numerical linear algebra,
- For degree $(2, 2)$ surface, the algorithm is accurate,
- For degree $(3, 3)$ surface, they have memory problem, no result.

Closest point computation using moving surfaces

Related work : Thomassen, Johansen, Dokken 2004

- They compute more than necessary points

deg of ψ	[TJD04]	EDdeg
$(1, 1)$	10	6
$(1, 2)$	22	14
$(1, 3)$	34	22
$(2, 2)$	52	36
$(2, 3)$	82	58
$(3, 3)$	130	94

We have a new method using AGAIN the moving surfaces

Why a new method?

- It allows using numerical linear algebra tools,
- We decrease the degrees by using moving planes, it becomes more efficient.

Our new method

We homogenize the parameterization of the surface

For a rational parametrization of a the surface $\mathcal{S}$

$$\begin{array}{rcl}\varphi : \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s,u) & \mapsto & \left(\frac{f_1(s,u)}{f_0(s,u)}, \frac{f_2(s,u)}{f_0(s,u)}, \frac{f_3(s,u)}{f_0(s,u)}\right),\end{array}$$

where f_0, f_1, f_2, f_3 are polynomials in s, u, we would like to write an homogeneous parameterization for the coungruence of normal lines to $\mathcal{S}$. We homogenize φ in either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$. Let X be either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$.

$$\begin{array}{rrcl}\Phi : & X & \to & \mathbb{P}^3 \\ & (\underline{x}) & \mapsto & (F_0, F_1, F_2, F_3)(\underline{x}).\end{array}$$

We consider the parameterization of normal lines to the surface $\mathcal{S}$ which is in form

$$\begin{array}{rcl} \Psi : X \times \mathbb{P}^1 \rightarrow & & \mathbb{P}^3 \\ (\underline{x}) \times (\lambda_0 : \lambda_1) & \mapsto & (\Psi_0 : \Psi_1 : \Psi_2 : \Psi_3). \end{array}$$

Homogeneous normal vector for a tensor product surface

$X := \mathbb{P}^1 \times \mathbb{P}^1$, $\underline{x} \in X$ and $(T_0, T_1, T_2, T_3) \in \mathbb{P}^3$. By the Jacobian matrix of Φ

$$\begin{vmatrix} \partial_{x_0} F_0 & \partial_{x_0} F_1 & \partial_{x_0} F_2 & \partial_{x_0} F_3 \\ \partial_{x_1} F_0 & \partial_{x_1} F_1 & \partial_{x_1} F_2 & \partial_{x_1} F_3 \\ \partial_{x_2} F_0 & \partial_{x_2} F_1 & \partial_{x_2} F_2 & \partial_{x_2} F_3 \\ T_0 & T_1 & T_2 & T_3 \end{vmatrix} = x_3(T_0\Delta_0(\underline{x}) + T_1\Delta_1(\underline{x}) + T_2\Delta_2(\underline{x}) + T_3\Delta_3(\underline{x})) = 0,$$

where Δ_i for $i = 0, 1, 2, 3$ are the signed minors, we characterize the normal line to $\mathcal{S}$ at $(\underline{x})$ with the projective point,

$$(0 : \Delta_1 : \Delta_2 : \Delta_3)\,.$$

Homogeneous normal vector for a triangular surface

$X := \mathbb{P}^2$, $\underline{x} \in X$ and $(T_0, T_1, T_2, T_3) \in \mathbb{P}^3$. By the Jacobian matrix of Φ

$$\begin{vmatrix} \partial_{x_0} F_0 & \partial_{x_0} F_1 & \partial_{x_0} F_2 & \partial_{x_0} F_3 \\ \partial_{x_1} F_0 & \partial_{x_1} F_1 & \partial_{x_1} F_2 & \partial_{x_1} F_3 \\ \partial_{x_2} F_0 & \partial_{x_2} F_1 & \partial_{x_2} F_2 & \partial_{x_2} F_3 \\ T_0 & T_1 & T_2 & T_3 \end{vmatrix} = T_0\Delta_0(\underline{x}) + T_1\Delta_1(\underline{x}) + T_2\Delta_2(\underline{x}) + T_3\Delta_3(\underline{x}) = 0,$$

where Δ_i for $i = 0, 1, 2, 3$ are the signed minors, we characterize the normal line to $\mathcal{S}$ at $(\underline{x})$ with the projective point,

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Lemma

Let H be a hyperplane in $\mathbb{P}^3$ *of equation* $a_0 T_0 + a_1 x_1 + a_2 T_2 + a_3 T_3 = 0$ *and L be a line in* $\mathbb{P}^3$ *that are not contained in the hyperplane at infinity* $V(T_0) \in \mathbb{P}^3$*. Then, L is orthogonal to H, in the sense that their restrictions to the affine space* $\mathbb{P}^3 \backslash V(T_0)$ *are orthogonal, iff the projective point* $(0 : a_1 : a_2 : a_3)$ *belongs to L.*

Lemma

Let H be a hyperplane in $\mathbb{P}^3$ of equation $a_0T_0 + a_1x_1 + a_2T_2 + a_3T_3 = 0$ and L be a line in $\mathbb{P}^3$ that are not contained in the hyperplane at infinity $V(T_0) \in \mathbb{P}^3$. Then, L is orthogonal to H, in the sense that their restrictions to the affine space $\mathbb{P}^3 \backslash V(T_0)$ are orthogonal, iff the projective point $(0 : a_1 : a_2 : a_3)$ belongs to L.

Proof.

Let $H_1 = \sum_{i=0}^{3} \alpha_i T_i = 0$, and $H_2 = \sum_{i=0}^{3} \beta_i T_i = 0$ are 2 hyperplanes. Suppose that $H_1 \bigcap H_2 = L$, where L is line in $\mathbb{P}^3$. We restrict then to the affine space $\mathbb{P}^3 \backslash V(T_0)$,

$$L = \left(\frac{\alpha_1}{\alpha_0} - \frac{\beta_1}{\beta_0}\right)\frac{T_1}{T_0} + \left(\frac{\alpha_2}{\alpha_0} - \frac{\beta_2}{\beta_0}\right)\frac{T_2}{T_0} + \left(\frac{\alpha_3}{\alpha_0} - \frac{\beta_3}{\beta_0}\right)\frac{T_3}{T_0} = 0.$$

Hence, L is orthogonal to H iff (a_1, a_2, a_3) is orthogonal to the both vectors $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$. Thus, $(0 : a_1, a_2, a_3)$ belongs to the H_1, H_2, then to L. □

Parameterization for the congruence of the normal lines to surface $\mathcal{S}$

For rational tensor product surface,

$$\begin{array}{cccccccc} \Psi := & \mathbb{P}^1 & \times & \mathbb{P}^1 & \times & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & (x_0 : x_1) & \times & (x_2 : x_3) & \times & (\lambda_0 : \lambda_1) & \mapsto & (\Psi_0, \Psi_1, \Psi_2, \Psi_3) \end{array}$$

$$\Psi_0 = \lambda_0 x_0^{2d_1-2} x_2^{2d_2-2} F_0(x_0, x_1; x_2, x_3),$$
$$\Psi_i = \lambda_0 x_0^{2d_1-2} x_2^{2d_2-2} F_i(x_0, x_1; x_2, x_3) + \lambda_1 \Delta_i(x_0, x_1; x_2, x_3), \;\; i = 1, 2, 3.$$

For rational triangular surface,

$$\begin{array}{cccccc} \Psi := & \mathbb{P}^2 & \times & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & (x_0 : x_1 : x_2) & \times & (\lambda_0 : \lambda_1) & \mapsto & (\Psi_0, \Psi_1, \Psi_2, \Psi_3) \end{array}$$

$$\Psi_0 = \lambda_0 x_2^{2d-3} F_0(\underline{x}),$$
$$\Psi_i = \lambda_0 x_2^{2d-3} F_i(\underline{x}) + \lambda_1 \Delta_i(\underline{x}), \;\; i = 1, 2, 3.$$

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Parameterization for the congruence of the normal lines to surface $\mathcal{S}$

Given degree d for triangular surface, or (d_1, d_2) for tensor-product surface $\mathcal{S}$, we write a parameterization for the congruence of normal lines to the surface $\mathcal{S}$ in the following degrees.

$\deg(\Psi_i)$	Triangular surface	Tensor-product surface
Non-rational	$(2d-2,1)$	$(2d_1-1, 2d_2-1, 1)$
Rational	$(3d-3,1)$	$(3d_1-2, 3d_2-2, 1)$

- (2×2) rational tensor-product surface, Ψ is of degree $(4,4,1)$,
- (3×3) rational tensor-product surface, Ψ is of degree $(7,7,1)$.

Base locus $\mathcal{B}$

For rational tensor product surface,

$$\Psi := \begin{array}{ccccccc} \mathbb{P}^1 & \times & \mathbb{P}^1 & \times & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ (x_0 : x_1) & \times & (x_2 : x_3) & \times & (\lambda_0 : \lambda_1) & \mapsto & (\Psi_0, \Psi_1, \Psi_2, \Psi_3) \end{array}$$

$$\Psi_0 = \lambda_0 x_0^{2d_1-2} x_2^{2d_2-2} F_0(x_0, x_1; x_2, x_3),$$
$$\Psi_i = \lambda_0 x_0^{2d_1-2} x_2^{2d_2-2} F_i(x_0, x_1; x_2, x_3) + \lambda_1 \Delta_i(x_0, x_1; x_2, x_3), \ \ i = 1, 2, 3.$$

Then, $\mathcal{B}$ corresponds to the ideal $(x_0^{2d_1-2} x_2^{2d_2-2}, \lambda_1)$ for $d_1 \geq 1$ and $d_2 \geq 1$.

Base locus $\mathcal{B}$

For rational triangular surface,

$$\begin{array}{ccccccc} \Psi := & \mathbb{P}^2 & \times & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & (x_0 : x_1 : x_2) & \times & (\lambda_0 : \lambda_1) & \mapsto & (\Psi_0, \Psi_1, \Psi_2, \Psi_3) \end{array}$$

$$\Psi_0 = \lambda_0 x_2^{2d-3} F_0(\underline{x}),$$
$$\Psi_i = \lambda_0 x_2^{2d-3} F_i(\underline{x}) + \lambda_1 \Delta_i(\underline{x}), \; i = 1, 2, 3.$$

Then, $\mathcal{B}$ corresponds to the ideal (x_2^{2d-3}, λ_1) for $d \geq 2$.

Thus, $\mathcal{B}$ is one-dimensional.

We study the fibers.

Why fibers ? : all pre-images of Ψ at given point $p \in \mathbb{P}^3$

Ψ : parameterization of the normal lines to the given surface,
p : point in $\mathbb{P}^3$. We consider all pre-images

$$\Psi^{-1}(p) = \{(\underline{x}_0, \underline{\lambda}_0) \in X \times \mathbb{P}^1 | \Psi(\underline{x}_0, \underline{\lambda}_0) = p\}.$$

What is the fiber of $p \in \mathbb{P}^3$?

Ψ is

for tensor-product surfaces	in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,
for triangular surfaces	in $\mathbb{P}^2 \times \mathbb{P}^1$.

X : either $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$.

$$X \times \mathbb{P}^1 \times \mathbb{P}^3 \supset \overline{\{(\underline{x}, \underline{\lambda}, \Psi(\underline{x}, \underline{\lambda})) \in X \times \mathbb{P}^1 \times \mathbb{P}^3\}}$$

$$\pi_1 : \overline{\{\ldots\}} \to X \times \mathbb{P}^1, \quad \pi_2 : \overline{\{\ldots\}} \to \mathbb{P}^3, \quad X \times \mathbb{P}^1 \overset{\Psi}{\dashrightarrow} \mathbb{P}^3$$

The fiber at $p = \Psi(\underline{x}, \underline{\lambda}) \in \mathbb{P}^3$ is $\pi_2^{-1}(p)$.

Details about fibers

X : either $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$.
$I := (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ ideal of $k[\underline{x}, \underline{\lambda}]$, where k : field.
$\mathcal{R}_I$: Rees algebra of I.
$\mathcal{S}_I$: Symmetric algebra of I.

$$\begin{array}{ccc} X \times \mathbb{P}^1 \times \mathbb{P}^3 \supset & & \mathcal{R}_I \\ & \swarrow^{\pi_1} & \downarrow^{\pi_2} \\ X \times \mathbb{P}^1 & \overset{\Psi}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

The fiber at $p \in \mathbb{P}^3$ is

$$\pi_2^{-1}(p) = Proj(\mathcal{R}_I \otimes \kappa(p)),$$

where $\kappa(p)$ denoted the residue field of p.

We study FINITE LINEAR fibers.

X : either $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$.
$I := (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ ideal of $k[\underline{x}, \underline{\lambda}]$, where k : field.
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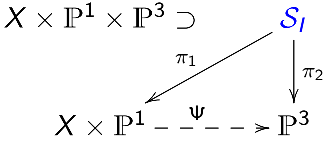

We will study the linear fiber $\mathfrak{L}_p := Proj(\mathcal{S}_I \otimes \kappa(p))$.

How is linear fiber $\mathfrak{L}_p$ is related to the syzygies of Ψ?

k : field, $k[y_0, y_1, y_2, y_3] = k[\underline{y}]$: coordinate ring of $\mathbb{P}^3$. In general setting, i.e, Ψ is a rational map of degree $(\boldsymbol{d}, e)$ over $X \times \mathbb{P}^3$. Consider the graded map

$$\begin{array}{rcl} k[\underline{x}](-\boldsymbol{d}, -e)^4 & \to & k[\underline{x}] \\ (g_0, g_1, g_2, g_3) & \mapsto & \sum_{i=0}^{3} g_i \Psi_i \end{array}$$

and denote its kernel by Z_1, which is the first module of syzygies of I. Setting $\mathcal{Z}_1 := Z_1(\boldsymbol{d}, e) \otimes k[\underline{x}][\underline{y}]$ and $\mathcal{Z}_0 = k[\underline{x}][\underline{y}]$, then the symmetric algebra $\mathcal{S}(I)$ admits the following multi-graded presentation

$$\begin{array}{rcl} \mathcal{Z}_1(-1) & \xrightarrow{\varphi} & \mathcal{Z}_0 \to \mathcal{S}(I) \to 0 \\ (g_0, g_1, g_2, g_3) & \mapsto & \sum_{i=0}^{3} g_i y_i . \end{array} \quad (1)$$

where the shift in the grading of $\mathcal{Z}_1$ is with respect to the grading of $k[\underline{y}]$. Thus, $\mathcal{S}(I) = k[\underline{x}, \underline{y}] / \sum_{i=0}^{3} g_i y_i$ such that $\sum_{i=0}^{3} g_i \Psi_i = 0$.

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We consider moving planes.

What is a moving plane?

A moving plane L is

$$L = A_0(\underline{x}) + A_1(\underline{x})T_1 + A_2(\underline{x})T_2 + A_3(\underline{x})T_3.$$

We say that L follows the surface if

$$A_0\Phi_0 + A_1\Phi_1 + A_2\Phi_2 + A_3\Phi_3 \equiv 0.$$

L is of degree 1 in T_1, T_2, T_3, with the previous notation $r = 1$.

Matrix $\mathbb{M}$ built from syzygies

(For a tensor product surface) We construct a matrix $\mathbb{M}$ by the coefficients of the family of moving planes of degree $(\boldsymbol{\mu}, 0)$ over $X \times \mathbb{P}^1 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)$,

$$\mathbb{M}_{(\boldsymbol{\mu},0)} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ L_0 & | & L_i & L_r \\ | & | & | & | \\ | & | & | & | \end{pmatrix} \text{ such that}$$

$$\left(x_1^{\mu_1}x_3^{\mu_2}, x_0x_1^{\mu_1-1}x_3^{\mu_2}, \dots, x_0^{\mu_1}x_3^{\mu_2}, x_2x_3^{\mu_2-1}x_1^{\mu_1}, \dots, x_2^{\mu_2}x_0^{\mu_1}\right)\mathbb{M}_{(\boldsymbol{\mu},0)} =$$

$$= [L_1, \dots, L_r].$$

The L_i's are the moving planes following the parametrization of the congruence normal lines to the given surface, Ψ.

$\mathbb{M}$ is built from the syzygies of $\Psi_0, \Psi_1, \Psi_2, \Psi_3$.

$\mathbb{M}_{(\boldsymbol{\mu},0)}$ is of form :

$$\mathbb{M}_{(\boldsymbol{\mu},0)} = \mathbb{M}_0 T_0 + \mathbb{M}_1 T_1 + \mathbb{M}_2 T_2 + \mathbb{M}_3 T_3,$$

where $\mathbb{M}_0, \mathbb{M}_1, \mathbb{M}_2, \mathbb{M}_3$ are matrix of coefficients in corresponding field.

For $p = \Psi(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r}; \lambda_0 : \lambda_1) \in \mathbb{P}^3$, for $i = 0, \ldots, r$ where $(x_{0_r} : x_{1_r})$, $(x_{2_r} : x_{3_r})$ and $(\lambda_0 : \lambda_1)$ are homogeneous coordinates on $\mathbb{P}^1$, we have

$$\left(x_{0_r}^{\mu_1} x_{2_r}^{\mu_1}, x_{0_r}^{\mu_1} x_{2_r}^{\mu_2-1} x_{3_r}, \ldots, x_{0_r}^{\mu_1} x_{3_r} x_{2_r}^{\mu_2-1}, \ldots, x_{3_r}^{\mu_2} x_{1_r}^{\mu_1}\right) \mathbb{M}_{(\boldsymbol{\mu},0)}(p) =$$

$$= [L_1(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r}), \cdots L_r(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r})] = [0, \cdots 0].$$

What is the degree of moving planes?

We construct $\mathbb{M}_{(\boldsymbol{\mu},\nu)}$ for $(\boldsymbol{\mu},\nu) \geq (\boldsymbol{\mu_1},\nu_1)$ component wisely.

$\boldsymbol{\mu_1},\nu_1$	Triangular surface	Tensor-product surface
Non-rational	$(6d-8,0)$	$(6d_1-4,5d_2-3,0)$ or $(5d_1-3,6d_2-4,0)$
Rational	$(9d-11,0)$	$(9d_1-7,7d_2-5,0)$ or $(7d_1-5,9d_2-7,0)$

- For (2×2) rational tensor-product surface, we consider $\mathbb{M}_{(11,9,0)}$,
- for (3×3) rational tensor-product surface, we consider $\mathbb{M}_{(20,17,0)}$.

Example

$$
\begin{aligned}
f_0(x_1, x_3) &= 1,\\
f_1(x_1, x_3) &= 0.664201612386595 x_1 x_3 - 0.696180693615241 x_1 + 0.988296384882165 x_3 + 0.906977337706699,\\
f_2(x_1, x_3) &= -0.915727734023933 x_1 x_3 + 0.988108228974431 x_1 - 0.225588687085695 x_3 - 0.621331435911471,\\
f_3(x_1, x_3) &= -0.576270958213199 x_1 x_3 - 0.954839048406471 x_1 - 0.891823661638540 x_3 + 0.362088586549061,
\end{aligned}
$$

$$
\mathbb{M}_{(2,2,0)} = \begin{pmatrix}
-0.425473294 & 5.05572860e^{-16} & -5.81997375e^{-17} & -1.70665475e^{-16} & -1.17323489e^{-16}\\
3.00831969e^{-1} & 1.73381600e^{-2} & -1.67834812e6-1 & 2.96500346e^{-1} & 1.82465261e^{-1}\\
-2.28128628e^{-1} & 5.73232481e^{-1} & 4.00966940e^{-1} & -1.29780618e^{-1} & 1.16129762e^{-1}\\
-5.17916656e^{-1} & -1.97990289e^{-1} & -8.79439603e^{-2} & -1.43415029e^{-1} & 2.92373481e^{-1}\\
6.80006794e^{-2} & 2.30338581e-1 & -1.97601814e^{-1} & 6.51978060e^{-1} & 1.53557241e^{-2}\\
1.50506778e^{-1} & 1.13101614e^{-1} & 1.46102809e^{-1} & -2.90327510e^{-1} & 5.19740790e^{-1}\\
-2.41574959e^{-1} & -3.61304919e^{-1} & -1.79427626e^{-1} & -1.29288486e^{-1} & 1.80889760e^{-1}\\
7.50543392e^{-2} & 3.86886648e^{-1} & 3.35420328e^{-3} & 3.91383654e^{-1} & 2.23246797e^{-1}\\
-2.17370476e^{-2} & -1.37540252e^{-1} & -2.03740588e^{-2} & -7.03859526e^{-2} & -1.95658132e^{-1}
\end{pmatrix}
$$

9×5 size of matrix $\mathbb{M}_{(2,2,0)}$ is computed in 1.859 ms. Its rank at randomly choosen 1000 points with 16 digits precision is equal to 4. The corank of $\mathbb{M}_{(2,2,0)}$ is $9 - 4 = 5 =$ *EDdegree*.

Difficulty

- Base locus $\mathcal{B}$ of Ψ contains curves, i.e. $dim(\mathcal{B}) = 1$.

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- Base locus $\mathcal{B}$ of Ψ contains curves, i.e. $dim(\mathcal{B}) = 1$.
- There is no related existing work.
- There exist the sections of positive degree.
- It's necessary to study the sections.

Sections

Definition

X, Y : 2 topological space.
$\pi : X \to Y$ be a continuous map.
Then, a section σ is a continuous map

$$\sigma : U \to \pi^{-1}U \text{ such that } \pi(\sigma(u)) = u, \quad \forall u \in U,$$

where U is an open subset of Y.

Example

Consider

$$\pi : \begin{array}{ll} [0,1] \times [0,1] & \to [0,1] \\ (x,y) & \mapsto x. \end{array}$$

Then, there are plenty of sections examples. For instance, $\sigma(y) = (y, y)$ or $\sigma(y) = (y, c)$ where c is constant in $[0, 1]$.

Sections

X	$\boldsymbol{a}$
$\mathbb{P}^2$	a,
$\mathbb{P}^1 \times \mathbb{P}^1$	(a_1, a_2).

Definition

The curve $\mathcal{C} \subset X \times \mathbb{P}^1$ is said to have no section in degree $< (\boldsymbol{a}, b)$ if it has no global section of degree $(\boldsymbol{\alpha}, \beta)$ such that $\boldsymbol{\alpha} < \boldsymbol{a}$ and $\beta < e$, where e is the degree over $\mathbb{P}^1$.

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Main theorems

Theorem

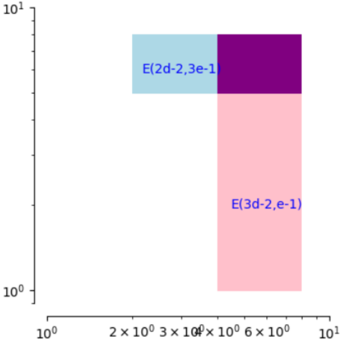
Ψ: rational map of degree $(\mathbf{d},e)$ on $X \times \mathbb{P}^1$,
$\dim(\mathcal{B}) = 1$,
$\mathcal{C}$ has no section in degree $< (0,e)$ and $I^{sat} = I'^{sat}$ where
$I = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ and I' is an ideal generated by three general linear combinations of the polynomials $\Psi_0, \dots, \Psi_3$.
Then, for any point p in $\mathbb{P}^3$ such that the fiber over p is finite we have that

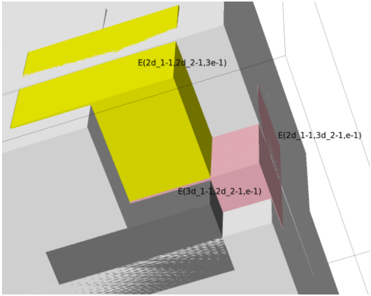
$$\operatorname{corank} M_{(\boldsymbol{\mu},\nu)}(p) = \deg(\mathfrak{L}_p)$$

for any $(\boldsymbol{\mu},\nu)$ on such that

- *if $X = \mathbb{P}^2$, then $(\mu,\nu) \in \mathbb{E}(3d-2, e-1) \cup \mathbb{E}(2d-2, 3e-1)$.*
- *if $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $(\boldsymbol{\mu},\nu) \in \mathbb{E}(3d_1-1, 2d_2-1, e-1) \cup \mathbb{E}(2d_1-1, 3d_2-1, e-1) \cup \mathbb{E}(2d_1-1, 2d_2-1, 3e-1)$.*

Main theorems





Main theorems

Theorem

Assume that $\dim(\mathcal{B}) = 1$ *and that* $\mathcal{C}$ *has no section in degree* $< (\mathbf{0}, e)$. *Moreover, assume that there exists an homogeneous ideal* $J \subset R$ *generated by a regular sequence* (g_1, g_2) *such that* $I \subset J$ *and* $(I : J)$ *defines a finite subscheme in* $X \times \mathbb{P}^1$. *Denote by* $(\boldsymbol{m}_1, n_1)$, *resp.* $(\boldsymbol{m}_2, n_2)$, *the degree of* g_1, *resp.* g_2, *set* $\eta := \max(e - n_1 - n_2, 0)$ *and let* p *be a point in* $\mathbb{P}^3$ *such that its fiber is finite. Then,*

$$\textit{corank}\, \mathbb{M}_{(\boldsymbol{\mu},\nu)}(p) = \deg(\mathfrak{L}_p)$$

for any degree $(\boldsymbol{\mu}, \nu)$ *such that*

- *if* $X = \mathbb{P}^2$, *then* $(\mu, \nu) \in \mathbb{E}(3d - 2, e - 1 + \eta) \cup \mathbb{E}(2d - 2 + d - \min\{m_1, m_2\}, 3e - 1)$.
- *if* $X = \mathbb{P}^1 \times \mathbb{P}^1$, *then* $(\boldsymbol{\mu}, \nu) \in \mathbb{E}(3d_1 - 1, 2d_2 - 1 + \tau_2, e - 1 + \eta) \cup \mathbb{E}(2d_1 - 1 + \tau_1, 3d_2 - 1, e - 1 + \eta) \cup \mathbb{E}(2d_1 - 1 + \tau_1, 2d_2 - 1 + \tau_2, 3e - 1)$ *where* $\tau_i := d_i - \min\{2m_{1,i} + m_{2,i}, m_{i,1} + 2m_{2,i}, d_i\} \geq 0, \; i = 1, 2.$

Coordinates of the orthogonal projections of p onto $\mathcal{S}$

Inversion (for tensor product surfaces)

p : point in $\mathbb{P}^3$,
Ψ : parameterization of the normal lines to the surface $\mathcal{S}$.
For $p = \Psi(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r}; \underline{\lambda})$, for $i = 0, \ldots, r$, we study

$$\left(x_{0_r}^{\mu_1} x_{2_r}^{\mu_1}, x_{0_r}^{\mu_1} x_{2_r}^{\mu_2 - 1} x_{3_r}, \ldots, x_{0_r}^{\mu_1} x_{3_r} x_{2_r}^{\mu_2 - 1}, \ldots, x_{3_r}^{\mu_2} x_{1_r}^{\mu_1} \right) \mathbb{M}_{(\boldsymbol{\mu},0)}(p) =$$

$$= [L_1(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r}), \cdots L_r(x_{0_r}, x_{1_r}; x_{2_r}, x_{3_r})] = [0, \cdots 0]$$

to compute the $(x_{0_r} : x_{1_r}; x_{2_r} : x_{3_r})$ coordinates for $i = 0, \cdots, r$.
For that purpose, we apply generalized eigenvalues, eigenvectors computation.

Inversion on an example

Randomly choosen 1 × 1 non-rational tensor product surface given by the coefficients in real field with 16 digits precision having $\mathbb{M}_{(2,2,0)}{}^{(-0.485218132066873,\ -0.632830215539379,\ -0.197871354840995)}$ of size 9 × 5 of corank 5 is

basis					
$x_1^2x_3^2$	-0.563208773	$-6.92655116e^{-17}$	$1.71937220e^{-16}$	$-1.76600478e-16$	$-2.43106876e^{-16}$
$x_1^2x_2x_3$	0.227574516	$0.3.29036312$	0.260535933	0.141610383	-0.265968167
$x_1^2x_2^2$	$-0.1.56471787$	-0.230029242	-0.0446379992	0.137481068	0.216298448
$x_0x_1x_3^2$	0.116809828	0.226320033	0.381572466	0.222591859	0.0564381284
$x_0x_1x_2x_3$	-0.0829320358	-0.235511289	-0.291276362	-0.208014459	-0.0135706199
$x_0x_1x_2^2$	-0.203506639	0.0475665548	0.00588636853	0.440521728	0.161882726
$x_0^2x_3^2$	-0.204232457	0.155130132	0.269307765	-0.388216643	-0.198295637
$x_0^2x_2x_3$	-0.155997218	0.260296192	-0.297575166	0.267006583	-0.171761130
$x_0^2x_2^2$	0.0311495026	0.185255444	0.240372238	0.162035702	0.0136607157

Inversion on an example

The cokernel of $\mathbb{M}_{2,2}$ is of size 9×5 is

basis					
$x_1^2 x_3^2$	-0.394942464	-0.241340484	-0.154618886	-0.36191343	-0.0425225374
$x_1^2 x_2 x_3$	-0.19623994	-0.0705268049	-0.0526279486	-0.630521355	-0.0740395822
$x_1^2 x_2^2$	-0.575201381	-0.420409847	0.409022593	-0.218509405	0.0837608474
$x_0 x_1 x_3^2$	-0.000230827929	-0.485981396	-0.204531394	0.0897759632	-0.536673093
$x_0 x_1 x_2 x_3$	-0.339622991	-0.167131632	-0.698813688	0.154667543	-0.211634739
$x_0 x_1 x_2^2$	-0.228638292	0.641957817	-0.165082076	-0.0572197227	-0.259437124
$x_0^2 x_3^2$	-0.494202215	0.199657677	0.162109402	-0.413107384	-0.144011460
$x_0^2 x_2 x_3$	-0.0578991045	-0.197705395	0.208193449	0.620336113	0.0419302763
$x_0^2 x_2^2$	-0.243816066	-0.697492118	-0.4230533	0.0439976225	0.750387735

red+purple rows = A,

purple+blue rows =B.

Then we compute the generalized eigenvalues and eigenvectors, i.e. $det(A - \lambda B) = 0$.

Inversion on an example

There is only one real valued eigenvalue,

$$-1.4256434878498954 \text{ for } \frac{x_1}{x_0}.$$

Its corresponding eigenvector is

$$(-0.37708551, -0.23906032 - 0.51589436i, -0.23906032 + 0.51589436i, 0.17327369 + 0.10186342i, 0.17327369 - 0.10186342i.)$$

After multiplying it by B and by taking the proportion of first two terms, we obtain the value

$$0.287755100169109 \text{ for } \frac{x_3}{x_2}.$$

Computations over real field (time in milliseconds)

For tensor-product surfaces

	non-rational				rational			
deg(Φ)	matrix size	time (ms) over ℝ	EDdeg	time (ms) Inversion	matrix size	time (ms) over ℝ	EDdeg	time (ms) Inversion
$(1,1)$	9×5	1.133	5	1.394	9×4	0.912	6	1.369
$(1,2)$	24×16	4.244	11	1.743	30×20	6.408	14	1.887
$(1,3)$	39×27	11.28	17	3.318	51×36	20.97	22	2.745
$(2,2)$	72×59	43.50	25	4.185	120×108	157.0	36	10.12
$(2,3)$	117×98	141.1	39	14.18	204×188	662.3	58	28.52
$(3,3)$	195×169	574.5	61	75.59	357×340	3353	94	136.6

Computations are done in SageMath.

- We find EDdegree for general tensor-product surfaces.
- A general (2×2) rational tensor-product surface $\mathbb{M}_{(11,9,0)}$ computation takes (in average) 157 ms.
- A general (3×3) rational tensor-product surface, $\mathbb{M}_{(20,16,0)}$ computation takes (in average) 3353 ms.

Computations over real field (time in milliseconds)

For triangular surfaces

	non-rational				rational			
deg(Φ)	matrix size	time (ms) over $\mathbb{R}$	EDdeg	time (ms) Inversion	matrix size	time (ms) over $\mathbb{R}$	EDdeg	time (ms) Inversion
2	15×7	2.441	9	3.143	36×29	15.14	13	3.448
3	66×51	41.87	25	4.746	153×150	300.9	39	13.02
4	153×132	314.2	49	18.82	351×363	2952	79	113.2

Computations are done in SageMath.

- We find EDdegree for general triangular surfaces.
- A general cubic rational triangular surface $\mathbb{M}_{(16,0)}$ computation takes (in average) 300.9 ms.
- A general degree 4 rational triangular surface, $\mathbb{M}_{(25,0)}$ computation takes (in average) 2952 ms.

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Exact computation, over rational field.

For tensor-product surfaces

	non-rational			rational		
deg(Φ)	matrix size	time (ms) over $\mathbb{R}$	time (ms) over $\mathbb{Q}$	matrix size	time (ms) over $\mathbb{R}$	time (ms) over $\mathbb{Q}$
$(1,1)$	9×5	1.133	6.164	9×4	0.912	7.309
$(1,2)$	24×16	4.244	32.16	30×20	6.408	124.8
$(1,3)$	39×27	11.28	135.9	51×36	20.97	1082
$(2,2)$	72×59	43.50	1460	120×108	157.0	31182
$(2,3)$	117×98	141.1	10867	204×188	662.3	–
$(3,3)$	195×169	574.5	96704	357×340	3353	–

Computations over $\mathbb{Q}$ are done in M2.

Exact computation, over rational field.

For triangular surfaces

	non-rational			rational		
$\deg(\Phi)$	matrix size	time (ms) over $\mathbb{R}$	time (ms) over $\mathbb{Q}$	matrix size	time (ms) over $\mathbb{R}$	time (ms) over $\mathbb{Q}$
2	15×7	2.441	18.54	36×29	15.14	266.4
3	66×51	41.87	886.5	153×150	300.9	28090
4	153×132	314.2	32473	351×363	2952	–

Computations over $\mathbb{Q}$ are done in M2.

Height of the coefficients of $\mathbb{M}$

Notation:

$h_\infty :=$ height with respect to $|.|_\infty$,,
$h_p :=$ height with respect to $|.|_p$, and
$v \quad := \{\infty, p :$ p is prime$\}$.

Height of the coefficients of $\mathbb{M}$

Definition

$f = \Sigma_\alpha a_\alpha x^\alpha$. Then,

$$|f|_v := \max_\alpha\{|a_\alpha|_v\} \text{ and } h_v(f) := \max\{0, log|f|_v\}.$$

Proposition

$h_v(\Psi) := \max\{h_v(\Psi_0), h_v(\Psi_1), h_v(\Psi_2), h_v(\Psi_3)\}$.

$$\begin{array}{lll} X = \mathbb{P}^2, & deg(\Psi) = d, & r = (\boldsymbol{\mu} + d + 1)(\boldsymbol{\mu} + d + 2), \\ X = \mathbb{P}^1 \times \mathbb{P}^1, & deg(\Psi) = (d_1, d_2), & r = 2(\mu_1 + d_1 + 1)(\mu_2 + d_2 + 1). \end{array}$$

The height of the $\mathbb{M}_{(\boldsymbol{\mu},0)}$ (where $\boldsymbol{\mu} = (\mu_1, \mu_2)$, for $X = \mathbb{P}^1 \times \mathbb{P}^1$) is bounded by

1. $h_\infty(\mathbb{M})) \leq r((r-1)h_\infty(\Psi) + log(r-1)! + h_\infty(\Psi) + logr) + logr!$,
2. $h_p(\mathbb{M}) \leq r^2 h_p(\Psi)$.

T.Krick, L.M.Pardo, M.Sombra, 1999
C.d'Andrea, T.Krick, M.Sombra, 2012.

Thanks !

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