Generalized Sparse Matrices and Applications to Decoding and Cryptography

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Reminder about Error Correcting Codes

Error correcting codes are used to transmit informations (satellites, DVD, telecommunications, ...) but also for cryptographic purpose.

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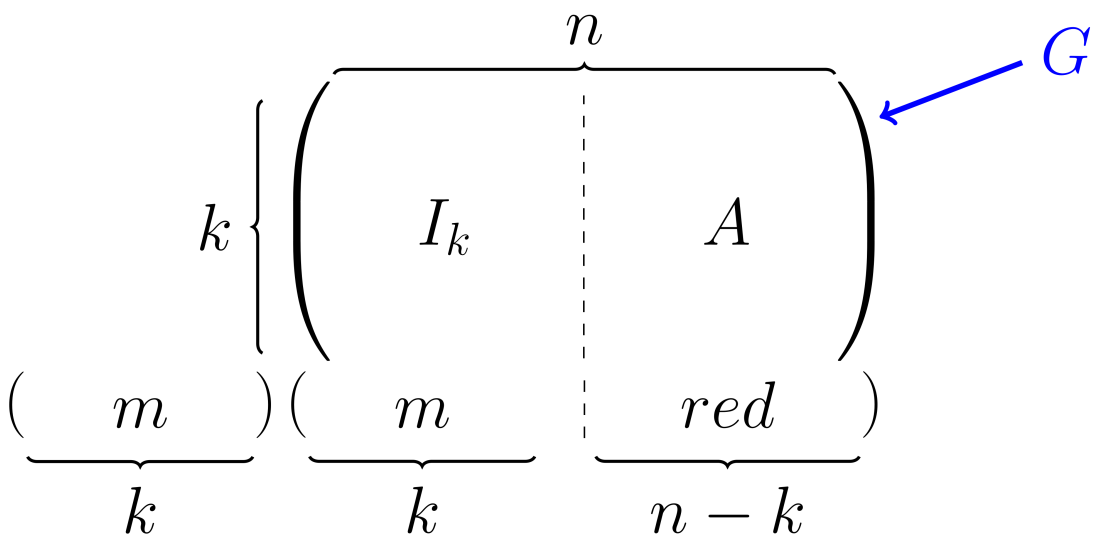
$$\begin{array}{rcl} \mathcal{E} : GF(q)^k & \longrightarrow & GF(q)^n \\ m & \longmapsto & mG \end{array}$$



Reminder about Error Correcting Codes

Parity Check Matrix

H is a parity check matrix for the code $\mathcal{C}$ if for every word $c \in GF(q)^n$:

$$c \in \mathcal{C} \iff Hc^T = 0_{n-k}$$

Hard Problems in Coding Theory

Decoding Problem (computational)

- Let G be a matrix $k \times n$ over a field $\mathbb{K}$,
  y a vector of length n (with coefficients in $\mathbb{K}$)
  and $\omega \in \mathbb{N}$.
- Find $m \in \mathbb{K}^k$ such that $weight(y - mG)$ for a given metric is smaller or equal to ω.

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  and $\omega \in \mathbb{N}$.
- Find $e \in \mathbb{K}^n$ with weight smaller or equal to ω for a given metric such that $He^t = s$?

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- Proven NP-complete with Hamming metric in 1978 by Berlekamp, McEliece and Tilborg.
- Proven to be probabilistically NP-complete with rank metric in 2017 by Gaborit and Zémor.

Decoding codes with sparse parity check matrix

- We will study two codes (MDPC and LRPC) for which one uses the sparsity of their **parity check matrix** to decode.

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- The notion of sparsity one uses depends on the chosen metric.

MDPC

- For MDPC codes sparse means : majority of zeros in the matrix (only $\mathcal{O}(\sqrt{n})$ 1's per row)

MDPC

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- Only H is sparse $\Longrightarrow$ "Moderate Density Parity Check".

Gallager's algorithm

Reminder : only the error contributes to the syndrome.

$$\begin{pmatrix}
1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0
\end{pmatrix}
\begin{pmatrix}
y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\\y_{11}\\y_{12}\\y_{13}\\y_{14}\\\dots\\y_{20}
\end{pmatrix}
=
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0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0
\end{pmatrix}
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0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0
\end{pmatrix}
\begin{pmatrix}
y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\\y_{11}\\y_{12}\\y_{13}\\y_{14}\\\dots\\y_{20}
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0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0
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#(common 1's)= {1

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1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
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0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0\\
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0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
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0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
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0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0
\end{pmatrix}
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y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\\y_{11}\\y_{12}\\y_{13}\\y_{14}\\\dots\\y_{20}
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0\\1\\0\\1\\1\\1\\0\\1\\1\\1\\0\\0
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Let $\tau = 3$ be our threshold.

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1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0
\end{pmatrix}
\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8\\ y_9\\ y_{10}\\ y_{11}\\ y_{12}\\ y_{13}\\ y_{14}\\ \dots\\ y_{20} \end{pmatrix}
=
\begin{pmatrix} 0\\1\\0\\1\\1\\1\\0\\1\\1\\1\\0\\0 \end{pmatrix}$$

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0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
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\end{pmatrix}
\begin{pmatrix}
y_1\\y_2\\y_3\\y_4+1\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\\y_{11}\\y_{12}\\y_{13}\\y_{14}\\\dots\\y_{20}
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0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0\\
0&0&1&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0\\
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\end{pmatrix}
\begin{pmatrix}
y_1\\ y_2\\ y_3\\ y_4+1\\ y_5\\ y_6\\ y_7\\ y_8+1\\ y_9\\ y_{10}\\ y_{11}\\ y_{12}\\ y_{13}\\ y_{14}\\ \dots\\ y_{20}
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0\\1\\0\\1\\1\\1\\0\\1\\1\\1\\0\\0
\end{pmatrix}$$

#(common 1's)= {1 , 1 , 1 , 3, 2, 2, 2, 3, 1, 1, 1, 1, 2, 2, 0, 1, 2, 1, 1, 2}
Let $\tau = 3$ be our threshold.

Gallager's algorithm

One receives a word $y := mG + e = codeword + error$.

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   - **a certain number of iterations is reached** $\Longrightarrow$ **RETURN FAIL**

Rank Decoding Problem

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Reminder about finite fields

$GF(q^m)$ $\longleftarrow$ finite field with q^m elements

$|$

$|$ m

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$GF(q^m)$ is a $GF(q)$-vector space of dimension m

Rank metric

$m = 4$, and let $e \in GF(2^4)^4$

$$e = (1 + \alpha^2, \alpha, 1, \alpha^2)$$

$$M = \begin{matrix} 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \end{matrix} \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

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We define $Rank(e) = Rank(M) = 3$

Rank metric

Notation : all presentation long, q is a power of a prime p and $m, n \in \mathbb{N}^*$.

Rank of a $GF(q^m)^n$ word

Let $e = (e_1, e_2, \ldots, e_n)$ be a vector of $GF(q^m)^n$.
Given a basis $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m)$ of $GF(q^m)$ over $GF(q)$, one gets :
$\forall i \in \{1, .., n\}, \exists!\ (e_{i,1}, e_{i,2}, \ldots, e_{i,m}) \in GF(q)^m$ such that
$e_i = \sum_{j=1}^{m} e_{i,j} \cdot \mathcal{B}_j$.
The rank of e, noted $Rank(e)$, is defined by the rank of the matrix $(e_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$.

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Rank distance

$$\begin{array}{rccc} d : & GF(q^m)^n \times GF(q^m)^n & \rightarrow & \mathbb{N} \\ & (e_1, e_2) & \mapsto & Rank(e_1 - e_2) \end{array}$$

is a distance (or a metric) over $GF(q^m)^n$.

Code properties in Rank metric

What are the differences between code with Hamming and Rank metric ?

Minimal distance of a code with rank metric

Let C be a $[n, k, d]$-code over $GF(q^m)$ for the rank metric
As long as C is additive :

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Support of a word in C

The support of an element $x = (x_1, \ldots, x_n) \in GF(q^m)^n$ is the $GF(q)$-vector space of $GF(q)^m$ generated by the coordinates x_i of x.

Support of a word

$$e = (1+\alpha^2, \alpha, 1, \alpha^2) \in GF(2^4)^4 \quad \text{and} \quad Rank(e) = 3$$

$$support(e) = < \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

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$support(e)$ is a vector space E of $GF(2)^4$ of dimension 3

LRPC

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- The sparse matrix is still $H \Longrightarrow$ "Low Rank Parity Check".

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- This vector space is a subset of the product space $< EF >$ of dimension at most rd.
- Since $rd < (n-k)$, it is very likely that $S =< EF >$.

Decoding LRPC

Let F_i be the i^{th} element of a basis of F, let's consider the vector space

$$F_i^{-1}S = F_i^{-1} < \begin{matrix} E_1F_1, & E_2F_1, & \dots & E_rF_1, \\ E_1F_2, & E_2F_2, & \dots & E_rF_2, \\ \dots & \dots & \dots & \dots \\ E_1F_i, & E_2F_i, & \dots & E_rF_i, \\ \dots & \dots & \dots & \dots \\ E_1F_d, & E_2F_d, & \dots & E_rF_d, \end{matrix} >$$

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So we have $E \subset F_i^{-1}S, \quad \forall\ i \in \{1, \dots, d\}$.

Decoding LRPC

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- Knowing the support of the error, one has to solve a linear system with more equations than unknowns to find the error and so to decode y.

Decoding LRPC

- This decoding is probabilistic, but the failure probability decreases exponentially according to the parameters.

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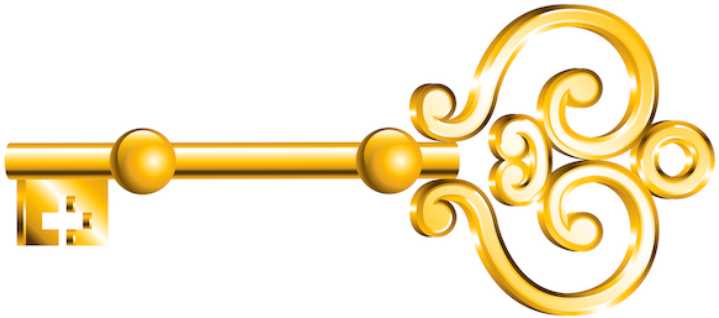
- For instance, with parameters $[n = 94, k = 47, N = 47, d = r = 5]$ for a binary LRPC, $\mathbb{P}(\textit{failure}) = 2^{-23}$.

Asymmetric Cryptography

Asymmetric Cryptography (or Public Key Cryptography)

Public Key

Private Key



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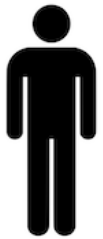

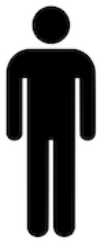

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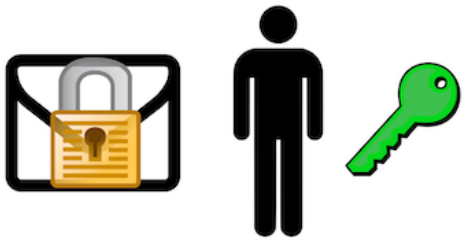

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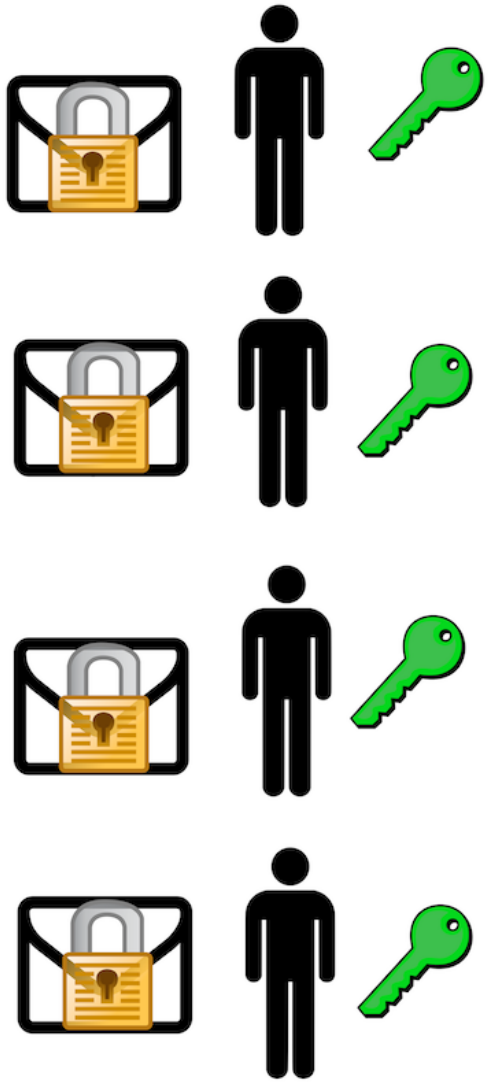

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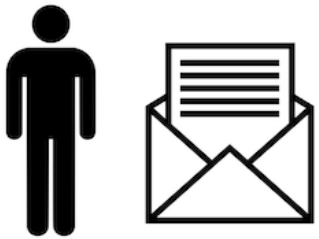

McEliece cryptosystem

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McEliece cryptosystem (or setting)

Let $\mathcal{C}$ be a linear code $[n, k, d]$-code, decodable up to $t = (d-1)/2$ errors in polynomial time with an algorithm $\mathcal{D}$.

Private Key : the generator matrix G of $\mathcal{C}$ (usually in systematic form), S a non-singular $k \times k$ matrix and P a $n \times n$ permutation matrix.
Public Key : $G' = SGP$

Encryption : let m be the plaintext (of length k), the cipher text is $y = mG' + e$ with $w(e) \leq t$ (e random).
Decryption :

- Compute $yP^{-1} = (mS)G + eP^{-1}$
- Decode $mSG = \mathcal{D}(yP^{-1})$
- Recover mS and then m (using S^{-1})

Application to Cryptography : hiding the structure

- Attack : recovering the plaintext
  $\Longrightarrow$ Decoding Random Code Problem **under the assumption that $\mathcal{C}'$ is indistinguishable from a random code**.

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- This is not the case if $\mathcal{C}$ has a "strong" algebraic structure (Reed-Solomon or Gabidulin for instance).
- One wants to get rid of the "scrambling" step.

MDPC cryptosystem

New variant of McEliece cryptosystem [MTSB13] et [GMRZ13]

Let $\mathcal{C}$ be an **MDPC** $[n, k, d]$-code, decodable up to $t = (d-1)/2$ errors in polynomial time with a **probabilistic** decoding algorithm $\mathcal{D}_H$ (using its sparse parity check matrix $H = (h_0|h_1)$).

Private Key : the parity check matrix H of $\mathcal{C}$
Public Key : the generator matrix $G = (I|(h_0 h_1^{-1})^T)$ of $\mathcal{C}$ (systematic form).

Encryption : let m be the plaintext (of length k), the cipher text is $y = mG + e$ with $w(e) \leq t$ (e random).
Decryption :

- Decode $mG = \mathcal{D}_H(y)$
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Disclaimer : this system has to be modified to be used in practice, since the systematic form would lead to leaks of the original message.

Hiding the structure

Advantages of MDPC and LRPC :

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- Almost no algebraic structure.
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- Almost no algebraic structure.
  Thus the security doesn't rely on masking the structure but only on the hardness of decoding random codes **under the assumption of indistinguability of MDPC and LRPC**.
- Thus, the "scrambling" step is replaced by going from the **sparse** H **to the dense** G.

Hiding the structure

Few visual examples : M of size 20×20,
coefficients in $GF(2^{20}) \approx 1$ million elements, $dim(F) = 2$.

```
##################
Matrice M=
[ w^362254         0  w^362254  w^362254         0  w^362254  w^471106  w^362254  w^362254         0         0  w^362254  w^362254         0  w^362254  w^733990  w^733990  w^471106         0         0]
[ w^362254  w^733990  w^471106         0  w^733990  w^362254  w^362254  w^362254  w^471106  w^733990         0  w^362254  w^733990  w^362254  w^733990  w^471106  w^471106         0  w^362254  w^471106]
[         0         0  w^471106  w^733990         0  w^733990  w^733990         0  w^733990  w^471106         0  w^362254         0  w^471106         0  w^362254  w^733990  w^733990  w^471106         0]
[ w^471106  w^471106  w^733990  w^471106  w^362254  w^733990  w^362254         0  w^471106         0  w^733990  w^362254  w^471106         0  w^733990  w^471106  w^471106         0  w^362254  w^362254]
[         0  w^362254  w^471106         0  w^471106  w^733990  w^733990  w^362254  w^471106  w^362254  w^471106         0  w^471106         0  w^362254         0  w^733990  w^471106  w^733990  w^362254]
[ w^362254  w^362254         0  w^471106  w^733990         0  w^362254  w^471106  w^733990  w^471106         0  w^733990         0  w^471106  w^733990         0  w^733990         0  w^733990  w^733990]
[ w^362254  w^733990         0  w^471106  w^471106         0  w^733990  w^733990  w^362254  w^471106  w^733990         0  w^362254  w^471106  w^362254  w^471106  w^733990         0  w^362254]
[ w^733990  w^362254         0  w^471106         0         0         0  w^471106  w^471106         0         0  w^733990  w^733990  w^471106  w^362254         0  w^471106         0         0  w^362254]
[ w^733990         0  w^733990  w^471106         0         0  w^471106  w^471106         0         0  w^362254  w^362254  w^471106  w^362254         0  w^362254         0         0  w^471106  w^733990]
[         0  w^733990  w^471106         0  w^362254  w^471106         0         0         0         0  w^471106  w^471106         0         0  w^362254  w^362254  w^733990         0         0]
[ w^471106         0  w^471106  w^733990  w^471106  w^733990  w^733990  w^471106  w^362254  w^733990         0  w^733990  w^733990         0  w^733990         0  w^471106         0  w^471106         0]
[ w^471106  w^733990  w^733990  w^733990  w^733990  w^362254  w^362254  w^471106  w^733990  w^362254  w^362254         0  w^733990         0  w^733990  w^362254  w^733990  w^471106  w^362254  w^733990]
[         0  w^733990  w^471106  w^362254  w^733990  w^471106  w^733990         0  w^471106  w^471106  w^733990         0         0  w^362254  w^362254  w^733990  w^733990         0  w^471106  w^362254]
[         0  w^471106         0  w^471106  w^733990         0  w^362254  w^362254  w^733990         0  w^471106  w^733990  w^471106  w^362254  w^471106  w^471106         0  w^733990  w^362254  w^733990]
[ w^471106  w^471106  w^471106         0  w^471106  w^471106  w^471106  w^471106  w^733990  w^733990  w^362254  w^471106  w^733990  w^471106  w^362254  w^733990         0  w^362254  w^362254  w^362254]
[ w^471106  w^471106  w^733990  w^733990  w^471106  w^362254  w^471106  w^471106  w^362254  w^733990  w^362254  w^733990  w^362254  w^362254         0  w^733990  w^471106  w^471106         0  w^362254]
[ w^362254  w^362254  w^362254  w^733990  w^471106  w^733990         0  w^471106  w^362254  w^471106  w^362254  w^733990  w^471106  w^471106         0  w^471106  w^471106  w^362254  w^471106  w^362254]
[         0         0         0  w^362254         0  w^471106  w^362254  w^471106  w^733990         0  w^362254  w^362254  w^362254  w^362254  w^733990         0  w^362254  w^471106  w^733990  w^733990]
[ w^362254  w^733990  w^733990         0         0         0  w^733990         0         0  w^733990  w^362254         0         0  w^362254         0         0  w^733990  w^733990  w^733990  w^471106]
[ w^733990  w^471106         0  w^733990  w^471106  w^471106         0  w^733990  w^362254  w^362254         0  w^733990  w^733990  w^733990         0  w^362254  w^733990         0  w^362254  w^362254]
##################
Matrice M^(-1)=
[  w^80448  w^988123  w^863025  w^898705  w^139452  w^623210  w^594037  w^864752  w^105343   w^28264 w^1045033  w^888787  w^617636  w^265974  w^169205   w^94741  w^801759   w^52909  w^607058  w^699552]
[ w^333957  w^942554  w^780678  w^618924 w^1000929  w^440446  w^964460  w^719946 w^1046953  w^959233  w^760287  w^824678  w^776259  w^356585  w^304173  w^637539  w^637281  w^111491  w^774659  w^902573]
[ w^147497  w^781692  w^115424  w^811426  w^649856  w^728662  w^156367  w^142081  w^106927  w^633380   w^63598  w^639272  w^220801  w^862548  w^417167  w^412288  w^982778  w^644957  w^218985  w^704658]
[ w^311167   w^23150  w^296538  w^567087  w^831522  w^221167  w^288209  w^468301  w^297189  w^764454  w^537310  w^720237   w^99700  w^695907  w^253634  w^691839  w^524141  w^551022  w^223229  w^875046]
[ w^993583   w^93992  w^478192  w^810565  w^141922  w^942306  w^117904  w^334543  w^819778    w^5759  w^855073  w^181661  w^888342  w^243147  w^447472  w^465924  w^763151  w^897715  w^278322  w^361146]
[ w^841810  w^756762  w^127936  w^273301  w^547058  w^172200  w^340363  w^344845  w^263618   w^13923  w^595539   w^57782  w^614572  w^312900 w^1005808  w^757103  w^536479  w^283298  w^351206  w^195139]
[ w^590967  w^706720  w^225921  w^381198  w^114839   w^44222  w^812786  w^910129  w^804021  w^155814  w^408812  w^545490  w^158519  w^588939  w^795331  w^495753  w^517364  w^930985  w^340051  w^443794]
[ w^334420  w^692708  w^546219  w^342348  w^443793  w^163160   w^31191  w^596068   w^44804  w^933333  w^234422  w^977827  w^797738  w^777330  w^244160  w^826510  w^425208  w^916518 w^1010002   w^28974]
[ w^727007  w^537798  w^387187  w^622828   w^90354  w^175592  w^578483  w^758515  w^624990 w^1021246  w^715397  w^345528  w^915186  w^455894  w^868470  w^886395  w^913260  w^347761   w^32665   w^99167]
[ w^262635  w^384261  w^266480  w^587969  w^540688  w^268454  w^260662  w^767743  w^566267  w^201909 w^1009597  w^577204  w^549731  w^230250  w^522123  w^491763  w^994584  w^766994  w^652170   w^19346]
[ w^787261  w^787284  w^577188  w^944193   w^76009   w^72114  w^168346  w^188787 w^1007068   w^78632  w^543358  w^734432   w^33820  w^617557  w^755605  w^123401  w^848074  w^717020  w^239070  w^458273]
[ w^653534  w^254962  w^613708 w^1029328  w^731945  w^737731  w^257473  w^402995  w^202381  w^918616  w^295557  w^889336  w^351942  w^380277  w^125269  w^100947   w^12813   w^69492  w^816486 w^1003168]
[ w^820747  w^472469  w^375392  w^928977  w^812964  w^318570  w^966941  w^935842 w^1027964  w^619944  w^850971  w^296038  w^603907  w^423541  w^104752  w^654707  w^379296  w^732234  w^948721  w^361762]
[ w^110971  w^680696  w^923340  w^214480  w^747262  w^926785  w^489888  w^952228  w^445776  w^423943  w^649072   w^22829  w^356036  w^706505  w^939812  w^920048  w^394764 w^1024427  w^670614  w^723510]
[ w^625737  w^796362  w^516193  w^823319  w^407960  w^792252  w^704791  w^442520  w^954309  w^636815  w^499427  w^651800  w^325780  w^982706  w^259759  w^704340  w^738285  w^423883  w^788285  w^864859]
[ w^637454 w^1005277  w^634523  w^597556  w^899547   w^50212  w^771166  w^176848  w^739499 w^1012145  w^894991  w^297408  w^401139  w^445943  w^860647  w^905019  w^586967  w^555279   w^11908  w^393060]
[ w^491731  w^105205  w^838924  w^523104  w^133971  w^621235  w^376245  w^675466  w^253911  w^239901 w^1012958  w^945009  w^722834 w^1047441  w^191877  w^127222  w^163984  w^206531 w^1005897  w^628569]
[ w^525867  w^931830  w^738973  w^215016  w^658662  w^195917  w^891334  w^903857  w^441472  w^344280  w^417181  w^577925  w^705133   w^41732  w^482903  w^837857  w^465912  w^268532  w^478086  w^988553]
[  w^93536  w^751673  w^536066  w^729970  w^123721  w^756989  w^668630  w^636987  w^454234   w^51525  w^574024  w^308632  w^461191  w^137995  w^836411  w^667454  w^403991  w^906525  w^553448  w^160312]
[ w^269807  w^696612  w^755090  w^178200  w^828283  w^733597  w^859384  w^569348  w^146891  w^116635 w^1004028  w^759077  w^155720  w^809492  w^303886 w^1014978  w^790097  w^325774  w^228381  w^278762]
##################
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Hiding the structure

Few visual examples : M of size 20×20,
coefficients in $GF(2^{20}) \approx 1$ million elements, $dim(F) = 2$.
$dim(support(M^{-1})) = 19$.

```
##################
Matrice M=
[ w^362254        0  w^362254  w^362254        0  w^362254  w^471106  w^362254  w^362254        0        0  w^362254  w^362254        0  w^362254  w^733990  w^733990  w^471106        0        0]
[ w^362254  w^733990  w^471106        0  w^733990  w^362254  w^362254  w^362254  w^471106  w^733990        0  w^362254  w^733990  w^362254  w^733990  w^471106  w^471106        0  w^362254  w^471106]
[       0        0  w^471106  w^733990        0  w^733990  w^733990        0  w^733990  w^471106        0  w^362254        0  w^471106        0  w^362254  w^733990  w^733990  w^471106        0]
[ w^471106  w^471106  w^733990  w^471106  w^362254  w^733990  w^362254        0  w^471106        0  w^733990  w^362254  w^471106        0  w^733990  w^471106  w^471106        0  w^362254  w^362254]
[       0  w^362254  w^471106        0  w^471106  w^733990  w^733990  w^362254  w^471106  w^362254  w^471106        0  w^471106        0  w^362254        0  w^733990  w^471106  w^733990  w^362254]
[ w^362254  w^362254        0  w^471106  w^733990        0  w^362254  w^471106  w^733990  w^471106        0  w^733990        0  w^471106  w^733990        0  w^733990        0  w^733990  w^733990]
[ w^362254  w^733990        0  w^471106  w^471106        0  w^733990  w^362254  w^471106  w^733990        0  w^362254  w^471106  w^362254  w^471106  w^733990        0  w^362254]
[ w^733990  w^362254        0  w^471106        0        0        0  w^471106  w^471106        0        0  w^733990  w^733990  w^471106  w^362254        0  w^471106        0        0  w^362254]
[ w^733990        0  w^733990  w^471106        0        0  w^471106  w^471106        0        0  w^362254  w^362254  w^471106  w^362254        0  w^362254        0        0  w^471106  w^733990]
[       0  w^733990  w^471106        0  w^362254  w^471106        0        0        0        0  w^471106  w^471106        0        0  w^362254  w^362254  w^733990        0        0]
[ w^471106        0  w^471106  w^733990  w^471106  w^733990  w^733990  w^471106  w^362254  w^733990        0  w^733990  w^733990        0  w^733990        0  w^471106        0  w^471106        0]
[ w^471106  w^733990  w^733990  w^733990  w^733990  w^362254  w^362254  w^471106  w^733990  w^362254  w^362254        0  w^733990        0  w^733990  w^362254  w^733990  w^471106  w^362254  w^733990]
[       0  w^733990  w^471106  w^362254  w^733990  w^471106  w^733990        0  w^471106  w^471106  w^733990        0        0  w^362254  w^362254  w^733990  w^733990        0  w^471106  w^362254]
[       0  w^471106        0  w^471106  w^733990        0  w^362254  w^362254  w^733990        0  w^471106  w^733990  w^471106  w^362254  w^471106  w^471106        0  w^733990  w^362254  w^733990]
[ w^471106  w^471106  w^471106        0  w^471106  w^471106  w^471106  w^471106  w^733990  w^733990  w^362254  w^471106  w^733990  w^471106  w^362254  w^733990        0  w^362254  w^362254  w^362254]
[ w^471106  w^471106  w^733990  w^733990  w^471106  w^362254  w^471106  w^471106  w^362254  w^733990  w^362254  w^733990  w^362254  w^362254        0  w^733990  w^471106  w^471106        0  w^362254]
[ w^362254  w^362254  w^362254  w^733990  w^471106  w^733990        0  w^471106  w^362254  w^471106  w^362254  w^733990  w^471106  w^471106        0  w^471106  w^471106  w^362254  w^471106  w^362254]
[       0        0        0  w^362254        0  w^471106  w^362254  w^471106  w^733990        0  w^362254  w^362254  w^362254  w^362254  w^733990        0  w^362254  w^471106  w^733990  w^733990]
[ w^362254  w^733990  w^733990        0        0        0  w^733990        0        0  w^733990  w^362254        0        0  w^362254        0        0  w^733990  w^733990  w^733990  w^471106]
[ w^733990  w^471106        0  w^733990  w^471106  w^471106        0  w^733990  w^362254  w^362254        0  w^733990  w^733990  w^733990        0  w^362254  w^733990        0  w^362254  w^362254]
##################
Matrice M^(-1)=
[  w^80448  w^988123  w^863025  w^898705  w^139452  w^623210  w^594037  w^864752  w^105343   w^28264 w^1045033  w^888787  w^617636  w^265974  w^169205   w^94741  w^801759   w^52909  w^607058  w^699552]
[ w^333957  w^942554  w^780678  w^618924 w^1000929  w^440446  w^964460  w^719946 w^1046953  w^959233  w^760287  w^824678  w^776259  w^356585  w^304173  w^637539  w^637281  w^111491  w^774659  w^902573]
[ w^147497  w^781692  w^115424  w^811426  w^649856  w^728662  w^156367  w^142081  w^106927  w^633380   w^63598  w^639272  w^220801  w^862548  w^417167  w^412288  w^982778  w^644957  w^218985  w^704658]
[ w^311167   w^23150  w^296538  w^567087  w^831522  w^221167  w^288209  w^468301  w^297189  w^764454  w^537310  w^720237   w^99700  w^695907  w^253634  w^691839  w^524141  w^551022  w^223229  w^875046]
[ w^993583   w^93992  w^478192  w^810565  w^141922  w^942306  w^117904  w^334543  w^819778    w^5759  w^855073  w^181661  w^888342  w^243147  w^447472  w^465924  w^763151  w^897715  w^278322  w^361146]
[ w^841810  w^756762  w^127936  w^273301  w^547058  w^172200  w^340363  w^344845  w^263618   w^13923  w^595539   w^57782  w^614572  w^312900 w^1005808  w^757103  w^536479  w^283298  w^351206  w^195139]
[ w^590967  w^706720  w^225921  w^381198  w^114839   w^44222  w^812786  w^910129  w^804021  w^155814  w^408812  w^545490  w^158519  w^588939  w^795331  w^517364  w^930985  w^340051  w^443794]
[ w^334420  w^692708  w^546219  w^342348  w^443793  w^163160   w^31191  w^596068   w^44804  w^933333  w^234422  w^977827  w^797738  w^777330  w^244160  w^826510  w^425208  w^916518 w^1010002   w^28974]
[ w^727007  w^537798  w^387187  w^622828   w^90354  w^175592  w^578483  w^758515  w^624990 w^1021246  w^715397  w^345528  w^915186  w^455894  w^868470  w^886395  w^913260  w^347761   w^32665   w^99167]
[ w^262635  w^384261  w^266480  w^587969  w^540688  w^268454  w^260662  w^767743  w^566267  w^201909 w^1009597  w^577204  w^549731  w^230250  w^522123  w^491763  w^994584  w^766994  w^652170   w^19346]
[ w^787261  w^787284  w^577188  w^944193   w^76009   w^72114  w^168346  w^188787 w^1007068   w^78632  w^543358  w^734432   w^33820  w^617557  w^755605  w^123401  w^848074  w^717020  w^239070  w^458273]
[ w^653534  w^254962  w^613708 w^1029328  w^731945  w^737731  w^257473  w^402995  w^202381  w^918616  w^295557  w^889336  w^351942  w^380277  w^125269  w^100947   w^12813   w^69492  w^816486 w^1003168]
[ w^820747  w^472469  w^375392  w^928977  w^812964  w^318570  w^966941  w^935842 w^1027964  w^619944  w^850971  w^296038  w^603907  w^423541  w^104752  w^654707  w^379296  w^732234  w^948721  w^361762]
[ w^110971  w^680696  w^923340  w^214480  w^747262  w^926785  w^489888  w^952228  w^445776  w^423943  w^649072   w^22829  w^356036  w^706505  w^939812  w^920048  w^394764 w^1024427  w^670614  w^723510]
[ w^625737  w^796362  w^516193  w^823319  w^407960  w^792252  w^704791  w^442520  w^954309  w^636815  w^499427  w^651800  w^325780  w^982706  w^259759  w^704340  w^738285  w^423883  w^788285  w^864859]
[ w^637454 w^1005277  w^634523  w^597556  w^899547   w^50212  w^771166  w^176848  w^739499 w^1012145  w^894991  w^297408  w^401139  w^445943  w^860647  w^905019  w^586967  w^555279   w^11908  w^393060]
[ w^491731  w^105205  w^838924  w^523104  w^133971  w^621235  w^376245  w^675466  w^253911  w^239901 w^1012958  w^945009  w^722834 w^1047441  w^191877  w^127222  w^163984  w^206531 w^1005897  w^628569]
[ w^525867  w^931830  w^738973  w^215016  w^658662  w^195917  w^891334  w^903857  w^441472  w^344280  w^417181  w^577925  w^705133   w^41732  w^482903  w^837857  w^465912  w^268532  w^478086  w^988553]
[  w^93536  w^751673  w^536066  w^729970  w^123721  w^756989  w^668630  w^636987  w^454234   w^51525  w^574024  w^308632  w^461191  w^137995  w^836411  w^667454  w^403991  w^906525  w^553448  w^160312]
[ w^269807  w^696612  w^755090  w^178200  w^828283  w^733597  w^859384  w^569348  w^146891  w^116635 w^1004028  w^759077  w^155720  w^809492  w^303886 w^1014978  w^790097  w^325774  w^228381  w^278762]
##################
```

Hiding the structure

```
mathis-130:beamer bros$ magma
Magma V2.23-1 (STUDENT)   Wed May 22 2019 10:06:37    [Seed = 1608908847]
Type ? for help.  Type <Ctrl>-D to quit.
> load "H.magma";
Loading "H.magma"
n =
250
r =
10
affichage =
0
Trial 1          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 2          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 3          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 4          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 5          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 6          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 7          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 8          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 9          => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
Trial 10         => dimension support(M) = 10, dimension support(M^(-1)) = 250 (max=250)
```

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- **Nevertheless, in cryptography, we prefer to use MDPC codes :**
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  - For instance, attacks on LDPC codes would be too easy due to a small private key size.

Algebraic Attack against RSD

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```
E_1_5*e_4_1 + E_1_1*e_5_1 + E_1_2*e_5_
E_1_2*e_4_1 + E_1_4*e_4_1 + E_1_2*e_5_
E_1_1*e_4_1 + E_1_3*e_4_1 + E_1_5*e_4_
E_1_6*e_4_1 + E_1_6*e_5_1 + 1,
E_1_1*e_4_1 + E_1_6*e_4_1 + E_1_1*e_5_
E_1_2*e_4_1 + E_1_2*e_5_1 + 1,
E_1_3*e_4_1 + E_1_6*e_4_1 + E_1_3*e_5_
E_1_4*e_4_1 + E_1_6*e_4_1 + E_1_4*e_5_
E_1_5*e_4_1 + E_1_5*e_5_1,
E_1_2*e_4_1 + E_1_3*e_4_1 + E_1_6*e_4_
E_1_2*e_4_1 + E_1_4*e_4_1 + E_1_6*e_4_
E_1_3*e_4_1 + E_1_5*e_4_1 + E_1_1*e_5_
```

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- Usually $n = N, k = n/2$ and $r = \mathcal{O}(\sqrt{n})$, so $2n^{3/2}$ unknowns for $(1/2)n^2$ equations.
- **One way to solve this system is to use Gröbner basis computation, but their complexity is poorly known for system which are not semi-regular.**

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  $\Longrightarrow$ drop in complexity in comparison to random systems.

Algebraic Attack against RSD

$[unknowns, equations]$	d_{reg} *random system*	d_{reg} *for RSD systems*
$[48, 48]$	8 (*Bardet*)	$[8, 8, 2, 3] \Longrightarrow 5$
$[48, 48]$	8 (*Bardet*)	$[12, 12, 8, 2] \Longrightarrow 4$
$[58, 70]$	7 *or* 8 $(*)$	$[10, 10, 2, 4] \Longrightarrow 5$

$(*)$ 7 is a prediction using my conjecture for over-constrained system whereas 8 would be the result of Bardet's method for semi-regular system.

:-)

Merci !
Thank you !

References : [McE78], [MTSB13], [GMRZ13], [Bar04].

References

Magali Bardet.

Étude des systèmes algébriques surdéterminés. Applications aux codes correcteurs et à la cryptographie.
Theses, Université Pierre et Marie Curie - Paris VI, 2004.

Philippe Gaborit, Gaétan Murat, Olivier Ruatta, and Gilles Zémor.
Low rank parity check codes and their application to cryptography.
In *Proceedings of the Workshop on Coding and Cryptography WCC*, volume 2013, 2013.

Robert J McEliece.
A public-key cryptosystem based on algebraic coding theory.
Coding Thv, 4244:114–116, 1978.

Rafael Misoczki, Jean-Pierre Tillich, Nicolas Sendrier, and Paulo SLM Barreto.
MDPC-McEliece: New McEliece variants from moderate density parity-check codes.
In *2013 IEEE international symposium on information theory*, pages 2069–2073. IEEE, 2013.