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Robust Numerical Path Tracking for Polynomial Homotopies

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Polynomial homotopies define solution paths of polynomial systems. By random complex constants in the homotopy, the paths are free of singular solutions and do not diverge, except perhaps at the end when the target system has singular solutions or solutions at infinity. Numerical continuation methods apply adaptive steplength control algorithms [7] to track the solution paths. Path jumping (identified in [9] as a reliability issue) occurs when the numerical approximations of one path jump onto another path. Although the paths themselves are free from singularities, path jumping occurs most likely when paths come near singularities.

We developed a new adaptive steplength control algorithm to track the solution paths using estimates of the distance to the nearest value of the continuation parameter where singularities or solutions at infinity occur. Our point of departure is Fabry's ratio theorem [6]. With this theorem we can detect singular points based on the coefficients of the Taylor series [5]. In particular, if for the series

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + \cdots \,, \text{ we have } \lim_{n \to \infty} c_n / c_{n+1} = z,$$

then z is a singular point of the series and it lies on the boundary of the circle of convergence of the series. The radius of this circle is then less than $|z|$.

Padé approximants [1] are *locally the best* rational approximants of power series [8]. Constructed directly from the coefficients of the power series, Padé approximants realize an efficient analytic continuation of the series beyond its circle of convergence. The relationship with Fabry's theorem is immediate if we consider Padé approximants where the denominator has degree one. If the numerator of the Padé approximant is degree n, then the numerator is $1 - c_{n+1}/c_n t$, and therefore its pole is c_n/c_{n+1}, in agreement with Fabry's theorem.

In addition to taking the nearest pole of the Padé approximants into account, we estimate the distance to the nearest (other) solution path using the information of the Jacobian and the Hessians of the polynomials in the system. Our steplength control algorithm decreases the step size as the numerical conditioning of the Jacobian matrix worsens and the curvature increases.

The Taylor series of the solution paths are computed by Newton's method [3]. Our new path tracking algorithm is 'robust' as it prevents path jumping and works well in double precision arithmetic. We implemented our algorithm in Julia and PHCpack (available on github). In our numerical experiments, we compare with [2] and [4].

Keywords: adaptive steplength control, continuation, homotopy, Newton's method, Padé approximant, path tracking, polynomial system, power series.

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