Integrability in Planar AdS/CFT and Yangian Symmetry

Niklas Beisert

ITP, ETH zürich

Workshop “Space, Time and Matrices”
Institut des Hautes Études Scientifiques
Bures-sur-Yvette, 26 February 2019

Introduction and Overview

This talk is about **Integrability in the planar AdS/CFT duality**:

- Proposed duality between string/gauge (gravity/QFT) models.
- Integrable structures observed in the planar limit in some models.
- Integrability provides toolkit for efficient computations.
- Integrability applied to confirm central aspects of AdS/CFT.
- Yangian invariance as a hidden symmetry related to integrability.

Goals: Improve understanding of QFT and gravity in terms of:

- algebraic and analytical methods
- non-perturbative results
- unconventional symmetries

Overview:

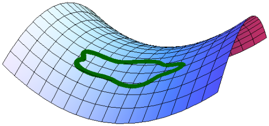
- AdS/CFT correspondence (introduction)
- integrability and achievements (review)
- Yangian symmetry (current work)

I. Cast of Characters

AdS/CFT Correspondence

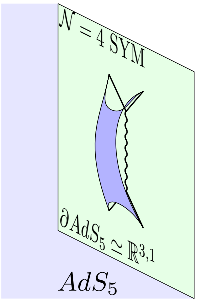
Need to understand strings on curved target spaces:

- non-linear equations
- spectrum difficult
- scattering?! how to get started?



Major achievement: conjectured exact **AdS/CFT duality** [Maldacena hep-th/9711200]

- string (gravitational) theory on AdS target space,
- conformal field theory (CFT) on boundary of AdS.



Prototype duality:

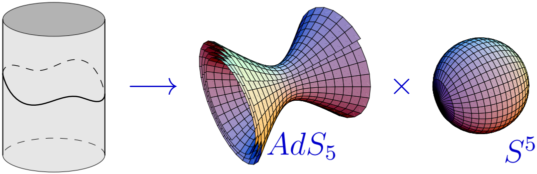
- IIB strings on $AdS_5 \times S^5$ target space
- $\mathcal{N}=4$ supersymmetric Yang–Mills (4D CFT)

Features:

- highly symmetric, highly accessible;
- but: non-linear models, strong/weak duality.

Strings on $AdS_5 \times S^5$

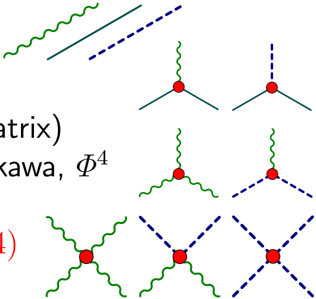
IIB superstrings on curved $AdS_5 \times S^5$ space:



- 2D non-linear sigma model (QFT),
- worldsheet coupling λ,
- string coupling: g_{s},
- weakly coupled for large λ,
- symmetry: background isometries $\widetilde{\mathrm{PSU}}(2,2|4)$.

$\mathcal{N}=4$ Super Yang–Mills Theory

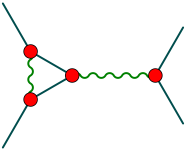
4D Quantum Field Theory Model: [Brink Schwarz Scherk]



- gauge field A_μ, 4 fermions Ψ, 6 scalars Φ.
- gauge group typically $\mathrm{SU}(N_\mathrm{c})$
- all fields massless and adjoint ($N_\mathrm{c} \times N_\mathrm{c}$ matrix)
- standard couplings: non-abelian gauge, Yukawa, Φ^4
- coupling constant g_YM, topological angle θ
- exact superconformal symmetry $\widetilde{\mathrm{PSU}}(2,2|4)$

Supersymmetry helps:

- protects some quantities, e.g. $\beta = 0$,
- but still model far from trivial!



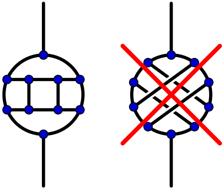
Weakly coupled for small g_YM
compute by Feynman graphs (hard!)

Planar Limit

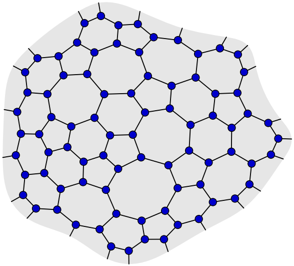
Planar limit in gauge theory:

['t Hooft Nucl. Phys. B72, 461]

- large-N_c limit: $N_\mathrm{c} = \infty$, $g_\mathrm{YM} = 0$,
  't Hooft coupling $g_\mathrm{YM}^2 N_\mathrm{c} =: \lambda$ remains,
- only planar Feynman graphs,
  no crossing propagators,
- drastic combinatorial simplification.

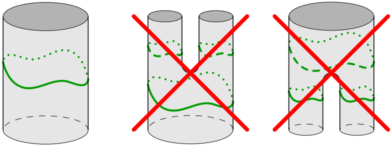


Surface of Feynman graphs becomes 2D string worldsheet:



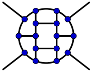
Planar limit in string theory:

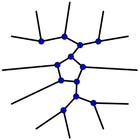
- no string coupling $g_\mathrm{s} = 0$,
  no string splitting or joining.
- worldsheet coupling λ remains.



Integrability

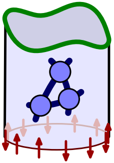
Standard QFT approach: **Feynman graphs**





- enormously difficult at higher loops …
- … but also at lower loops and many legs.

Planar $\mathcal{N}=4$ SYM is **integrable** … see review collection [NB et al. 1012.3982]



- …so is the AdS/CFT dual string theory.
- integrability vastly simplifies calculations.
- spectrum of local operators now largely understood.
- can compute observables at finite coupling λ. [NB, Eden Staudacher]
- simple integral equation for cusp dimension $D_{\text{cusp}}(\lambda)$.

Local, gauge-invariant operators, e.g. / dual to string states:

$$\mathcal{O} = \operatorname{Tr} \mathcal{D}^{n_1}\Phi\, \mathcal{D}^{n_2}\Phi \cdots \mathcal{D}^{n_L}\Phi \quad \longleftrightarrow$$

Observable: scaling dimension $D_{\mathcal{O}}$ / dual to energy of string state

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle \sim |x-y|^{-2D_{\mathcal{O}}}.$$

II. Planar AdS/CFT Spectrum using Integrability

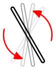
Cusp Dimension from Bethe Equations

Asymptotic Bethe equations for scaling dimensions [Minahan Zarembo][NB Staudacher][Serban Staudacher]

$$\exp(ip_k L) = \prod_{j=1, j\neq k}^{M} S(p_k, p_j; \lambda), \qquad D_{\mathcal{O}} = D_0 + \sum_{j=1}^{M} D(p_k; \lambda).$$

Coupling constant λ enters analytically. [NB, Dippel Staudacher][NB Staudacher]

Useful object: Twist-two operators with spin S / dual spinning string:

 : $\mathcal{O}_S \simeq \operatorname{Tr} \Phi \overleftrightarrow{\mathcal{D}}^S \Phi \qquad \longleftrightarrow$ 

A lot is known about their anomalous dimensions δD_S:

- QCD: δD_S responsible for scale violations in DIS.
- DGLAP, BFKL evolution equations.
- Large-S behaviour: cusp dimension D_{cusp}

$$D_S = D_{\text{cusp}} \log S + \ldots .$$

Cusp Dimension

Cusp dimension determined by AdS/CFT planar integrable system!

Compute cusp dimension using Bethe equations. Integral eq.: [Eden Staudacher]

$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{dy\,y}{e^{2\pi y/\sqrt{\lambda}}-1}\,\psi(y).$$

Kernel $K = K_0 + K_1 + K_\mathrm{d}$ made from Bessel $\mathrm{J}_{0,1}$ with [NB, Eden Staudacher]

$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_\mathrm{d}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{dz\,z}{e^{2\pi z/\sqrt{\lambda}}-1}\,K_0(z,y).$$

Cusp anomalous dimension: $D_\mathrm{cusp} = (\lambda/\pi^2)\psi(0)$.

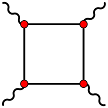
Weak/Strong Expansion

Weak-coupling solution of integral equation [NB, Eden Staudacher]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\,\frac{\lambda}{\pi^2} - \frac{1}{96}\,\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\,\frac{\lambda^3}{\pi^2} - \Big(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\Big)\,\frac{\lambda^4}{\pi^2} \pm \ldots.$$

Confirmed by gluon scattering amplitudes [Bern Dixon Smirnov][Bern, Czakon, Dixon Kosower, Smirnov]

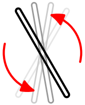
$$A(p,\lambda) \simeq A^{(0)}(p)\exp\big(2D_{\text{cusp}}(\lambda)M^{(1)}(p)\big).$$



Connection between integrability & scattering amplitudes? later...

Strong-coupling asymptotic solution of integral equation [Casteill Kristjansen][Basso Korchemsky Kotański]

$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots.$$



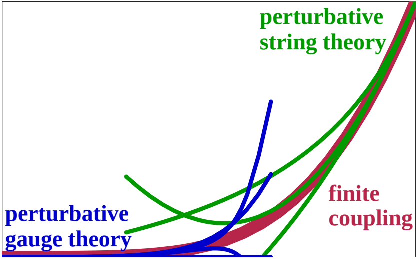
Agreement with energy of spinning string. [Gubser Klebanov Polyakov][Frolov Tseytlin][Roiban Tirziu Tseytlin]

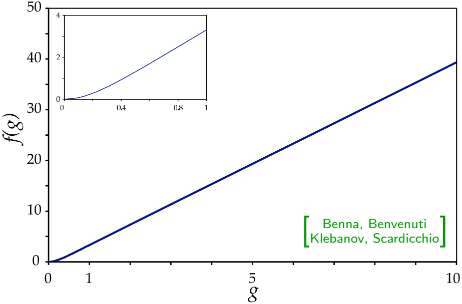
Finite-Coupling Interpolation

Cusp dimension can be computed numerically at finite coupling λ.

Smooth interpolation between perturbative gauge and string theory

- in the Bethe equations (left), [NB, Eden Staudacher]
- in the cusp dimension (right). [Benna, Benvenuti Klebanov, Scardicchio]



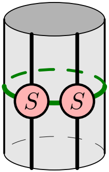


An exact result in a (planar) 4D gauge theory at finite coupling.

Thermodynamic Bethe Ansatz

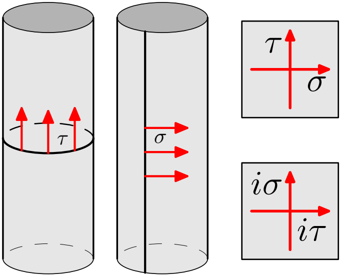
Bethe equations not exact for finite size?

- scattering assumes infinite worldsheet,
- actual string states defined on finite cylinder,
- Lüscher terms: virtual particles around cylinder.



Thermodynamic Bethe Ansatz:

- idea: space has finite extent,
  but time is infinite.
- consider evolution in space,
  scattering problem on infinite line.
- in 2D: double Wick rotation.
  Same S-matrix.

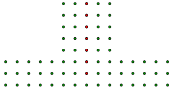


Obtain infinite set of coupled integral equations.

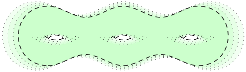
Techniques and Applications

Arsenal of improved integrable techniques:

- T/Y-System
- Hirota equations



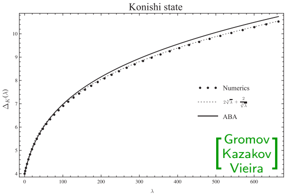
- Baxter equations
- quantum curves



- finite non-linear integral equations

Consider now particular state ("Konishi"), e.g.

$$\mathcal{O} = \operatorname{Tr} \Phi_m \Phi_m .$$



[Gromov Kazakov Vieira]

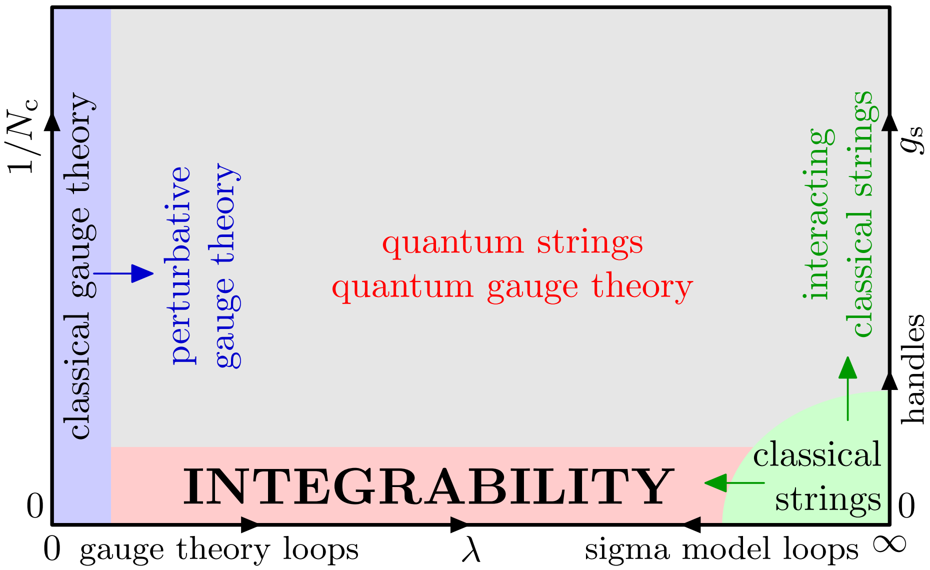
Can now compute the dimension or energy:

- interpolation from weak to strong coupling,
- 8 loops: sum of (multiple) zeta values

[Bajnok Janik] [Leurent Volin]

$$D = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} - \frac{\big(78 - 18\zeta(3) + 45\zeta(5)\big)\lambda^4}{2048\pi^8} + \ldots .$$

Charted Territory

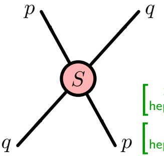


III. Selected Achievements using Integrability

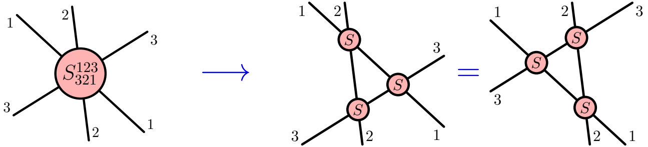
Worldsheet Scattering

Integrability methods rely on scattering picture for 2D worldsheet.

- 8 bosonic + 8 fermionic excitations,
- $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ residual symmetry,
- 2-particle scattering matrix S,
- $S(p,q;\lambda)$ at finite λ determined by symmetry,
- integrability: factorised multi-particle scattering, YBE.



[Staudacher hep-th/0412188]
[NB hep-th/0511082]



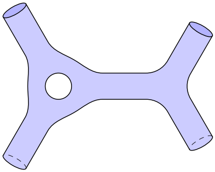
Unusual non-local **symmetry**:

- infinite-dimensional quantum algebra: Yangian
- novel deformation of Yangian $\mathrm{Y}[\mathfrak{u}(2|2)]$ (maths)
- investigations of algebra ongoing

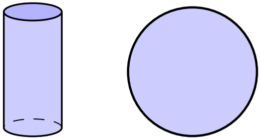
[Gomez Hernández][Plefka Spill Torrielli]
[NB 0704.0400]
[Matsumoto Moriyama Torrielli][NB Spill]
[...]

Correlation Functions

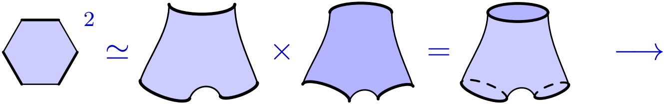
Integrable methods to compute correlation functions efficiently?

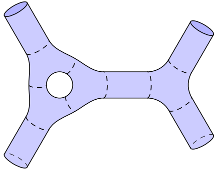


But: Integrability applies to annulus and disc topology only!



Stitch together two hexagons to a pair of pants, then glue arbitrary correlator (as in string theory): [Escobedo, Gromov Sever, Vieira][Basso Komatsu Vieira][...]

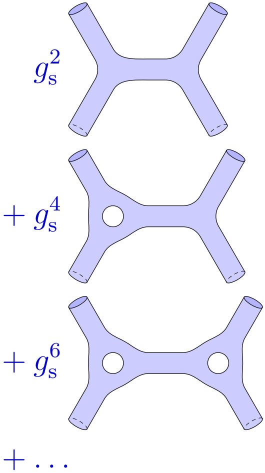




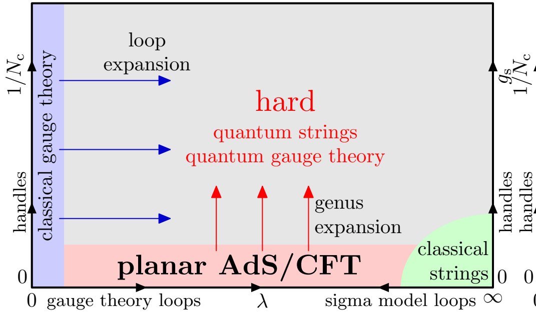
Excitations propagate on/around 2D worldsheet: scattering!

Genus Expansion

Outlook: Genus expansion can now be done by gluing (in principle)

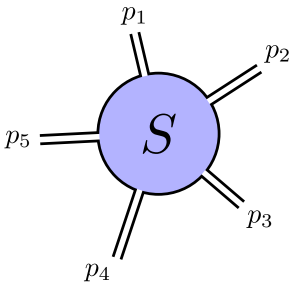


Alternative expansion scheme for $\mathcal{N} = 4$ SYM theory:



Planar Scattering in Gauge Theory

Another observable: colour-ordered **planar scattering**
Much progress in past 15 years (on-shell, geometric & integrable) [. . .]



Generic infrared factorisation for $S_n(\lambda, p)$:

$$S_n^{(0)}(p) \exp\bigl(D_{\text{cusp}}(\lambda) M_n^{(1)}(p) + R_n(\lambda, p)\bigr)$$

- tree level scattering $S_n^{(0)}(p)$
- one loop factor $M_n^{(1)}(p)$ (IR-divergent)
- cusp anomalous dimension $D_{\text{cusp}}(\lambda)$
- remainder function $R_n(p, \lambda)$ (finite)

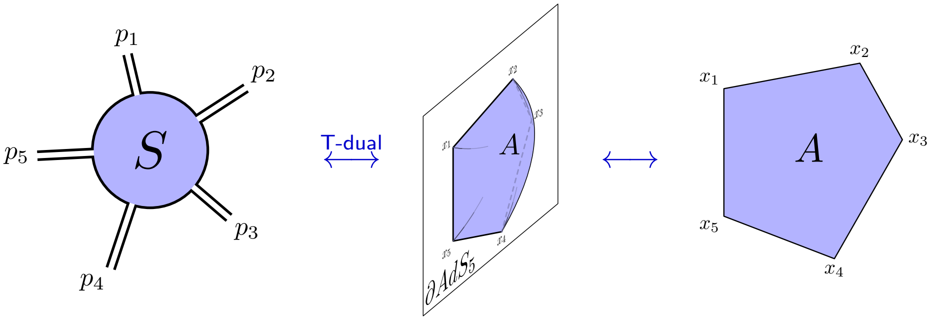
Intriguing observation for $n = 4, 5$ legs: $R_n = 0$! [Anastasiou, Bern Dixon, Kosower][Bern Dixon Smirnov]

- Computed/confirmed at 4 loops using unitarity. [Bern Dixon Smirnov][Bern, Czakon, Dixon Kosower, Smirnov]
- Exact result for scattering at finite λ! Why simple?
- Generalise to $n \geq 6$ legs! Compute exact R_n?!

Planar Scattering and Wilson Loops

AdS/CFT provides a **string analog** for planar scattering. [Alday Maldacena]

Area of a minimal surface in AdS_5 ending on a null polygon on ∂AdS_5.



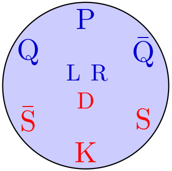
AdS/CFT backwards: [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]

- Minimal surfaces correspond to Wilson loops in gauge theory.
- Amplitudes "T-dual" to null polygonal Wilson loops

Dual Conformal and Yangian Symmetries

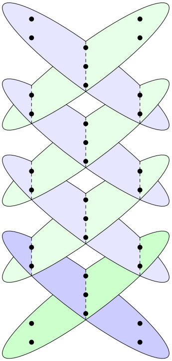
$\mathcal{N} = 4$ SYM is superconformal: $\mathrm{PSU}(2,2|4)$ symmetry.

- Amplitudes are conformally invariant.
- Wilson loops are conformally invariant.



Two conformal symmetries: [Drummond, Henn Smirnov, Sokatchev][Drummond Korchemsky Sokatchev]

- different action on amplitudes and Wilson loops
- ordinary conformal symmetry $\updownarrow$ T-duality dual conformal symmetry [Alday Maldacena]
- together: generate infinite-dimensional … [NB, Ricci Tseytlin, Wolf] … Yangian algebra $\mathrm{Y}(\mathrm{PSU}(2,2|4))$. [Drummond Henn Plefka]



Dual conformal symmetry **explains simplicity**:

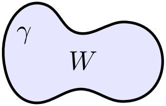
- No dual conformal cross ratios for $n = 4, 5$.
- Remainder function must be trivial: $R_n = 0$.

Wilson Loops

Invariance of null polygonal Wilson loops spoiled by divergences.

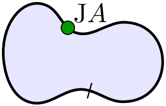
Better: can define finite Wilson loops in $\mathcal{N}=4$ SYM: [Maldacena hep-th/9803002]

$$W = \mathrm{P}\,\mathrm{Tr}\left[\exp \oint_\gamma \left(A_\mu dx^\mu + \Phi_m q^m d\tau\right)\right], \quad |dx| = |q| d\tau.$$



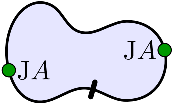
Conformal action (level-zero Yangian) equivalent to single insertion

$$\mathrm{J}^C W = \mathrm{P}\,\mathrm{Tr}\left[\oint \mathrm{J}^C A(\tau) \exp \oint A\right].$$



Level-one Yangian action: bi-local insertion

$$\widehat{\mathrm{J}}^C W = \mathrm{P}\,\mathrm{Tr}\left[f^C_{AB} \iint_{\tau_2>\tau_1} \mathrm{J}^A A(\tau_1)\, \mathrm{J}^B A(\tau_2) \exp \oint A\right].$$



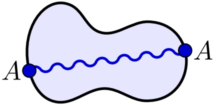
Wilson Loop Expectation Value

Yangian invariance of Wilson loops requires [Müller, Münkler Plefka, Pollok, Zarembo][NB, Müller Plefka, Vergu]

$$\langle \mathrm{J}^C W \rangle = 0, \qquad \langle \widehat{\mathrm{J}}^C W \rangle = 0.$$

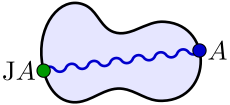
Wilson loop expectation value at $\mathcal{O}(\lambda)$:

$$\langle W \rangle \sim \iint \langle A_1\, A_2 \rangle.$$



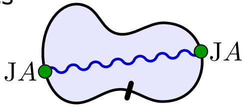
Conformal action vanishes upon integration

$$\langle \mathrm{J}^C W \rangle = \iint \langle \mathrm{J}^C A_1\, A_2 + A_1\, \mathrm{J}^C A_2 \rangle \simeq 0.$$



Vanishing of level-one action uses additional features

$$\langle \widehat{\mathrm{J}}^C W \rangle = f^C_{AB} \iint \langle \mathrm{J}^A A_1\, \mathrm{J}^B A_2 \rangle \simeq 0.$$



Yangian invariance at $\mathcal{O}(\lambda)$! **Outlook:** Higher perturbative orders?

Status of AdS/CFT Integrability

implications understood for several observables
applied to compute at finite coupling
well-defined math concepts at leading weak coupling
some perturbative corrections under control
understood well in classical string theory on $AdS_5 \times S^5$

Open Questions:

What is integrability?
How to define it? How to prove it?
... at higher loops and finite coupling?

IV. Yangian Symmetry of $\mathcal{N} = 4$ Super Yang–Mills Theory

Symmetry of the Model?

Aim: Show planar Yangian invariance of the action [NB, Garus, Rosso 1701.09162]

$$\widehat{\mathrm{J}}\,\mathcal{S} = 0.$$

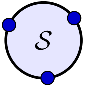
How to apply $\widehat{\mathrm{J}}$ to the action $\mathcal{S}$?

- distinction of planar and non-planar parts not evident
- which representation: free, non-linear, quantum?

Hints:

- Wilson loop displays Yangian symmetry
- OPE of Wilson loops contains Lagrangian $\mathcal{L}$

Essential features of the action:



- action is single-trace (disc topology)
- action is conformal (required for cyclicity)
- action is not renormalised (no anomalies?)

Equations of Motion

Application of $\widehat{\mathrm{J}}$ on the action needs extra care. Consider e.o.m.:

$$\widehat{\mathrm{J}}(\text{e.o.m.}) \overset{?}{\sim} \text{e.o.m.}$$

Need for consistent of quantum formalism of symmetry!

Dirac equation is easiest:

$$D{\cdot}\Psi + [\Phi, \bar{\Psi}] = \partial{\cdot}\Psi + i[A, \Psi] + [\Phi, \bar{\Psi}] = 0.$$

Bi-local level-one action on Dirac equation:

$$\begin{aligned} &D{\cdot}\widehat{\mathrm{J}}^C\Psi + i[\widehat{\mathrm{J}}^C A, \Psi] + [\Phi, \widehat{\mathrm{J}}^C\bar{\Psi}] \\ &\qquad + f^C_{AB}\big[i\{\mathrm{J}^A A, \mathrm{J}^B\Psi\} + \{\mathrm{J}^A\Phi, \mathrm{J}^B\bar{\Psi}\}\big] = 0. \end{aligned}$$

All terms cancel for proper choice of single-field action $\widehat{\mathrm{J}}^C\Psi, \widehat{\mathrm{J}}^C A$

Equations of Motion Yangian-invariant!

Classical Invariance of the Action

Exact symmetries ensure all-order Ward–Takahashi identities

- … if action is invariant (invariance of e.o.m. not sufficient)
- … and if there are no quantum anomalies.

Would like to show **invariance of action** $\widehat{\mathrm{J}}\mathcal{S} = 0$. Difficulties:

- cyclicity: where to cut open trace? (modulo level zero)
- non-linearity: how to deal with terms of different length (2,3,4)?

Using invariance of e.o.m. construct "$\widehat{\mathrm{J}}\mathcal{S}$" such that [NB, Garus, Rosso 1803.06310]

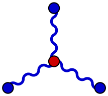
$$\widehat{\mathrm{J}}\mathcal{S} = \ldots = 0.$$

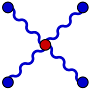
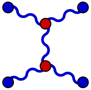
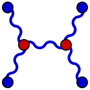
very unusual features: • coefficients depend on number of fields,
• overlapping bi-local terms, • gauge invariance not manifest.

Proper definition of integrability!

Correlators of Fields

Derive concrete implication of Yangian symmetry.
Consider correlators of fields: [NB, Garus 1804.09110]

$\big\langle A_1\, A_2\, A_3 \big\rangle =$ 

$\big\langle A_1\, A_2\, A_3\, A_4 \big\rangle =$  $+$  $+$ 

Yangian symmetry implies Ward–Takahashi identities, e.g.

$$\mathrm{J}\big\langle A_1\, A_2\, A_3 \big\rangle = \big\langle \mathrm{J}A_1\, A_2\, A_3 \big\rangle + \big\langle A_1\, \mathrm{J}A_2\, A_3 \big\rangle + \big\langle A_1\, A_2\, \mathrm{J}A_3 \big\rangle = 0,$$

$$\widehat{\mathrm{J}}\big\langle A_1\, A_2\, A_3 \big\rangle = \big\langle \mathrm{J}A_1\, \mathrm{J}A_2\, A_3 \big\rangle + \ldots + \big\langle \widehat{\mathrm{J}}A_1\, A_2\, A_3 \big\rangle + \ldots = 0.$$

Verified for several correlators (formal transformations).

However: need to fix a gauge in quantised gauge theory!

Gauge Fixing

Fix gauge by Faddeev–Popov method; new BRST symmetry Q.

Impact of gauge fixing on Yangian symmetry?

- introduction of ghost fields, extra terms $\mathcal{S}_{\text{g.f.}}$ in action;
- must consider BRST cohomology ($\mathrm{Q}^2 = 0$);
- how to represent symmetry on unphysical fields and ghosts?

Conformal symmetry: Residual terms must be BRST exact

$$\mathrm{J}\,\mathcal{S} = \mathrm{J}\,\mathcal{S}_{\text{g.f.}} = \mathrm{Q}\,\mathcal{K}[\mathrm{J}].$$

Level-one symmetry: extra bi-local compensating terms required

$$\widehat{\mathrm{J}}\,\mathcal{S} = \mathrm{Q}\,\mathcal{K}[\widehat{\mathrm{J}}] + \mathrm{J}\,\mathcal{K}[\mathrm{Q}\wedge\mathrm{J}] + (\mathrm{Q}\wedge\mathrm{J})\,\mathcal{K}[\mathrm{J}].$$

Action satisfies gauge-fixed invariance condition!
Slavnov–Taylor identities for correlation functions tested.

Anomalies?

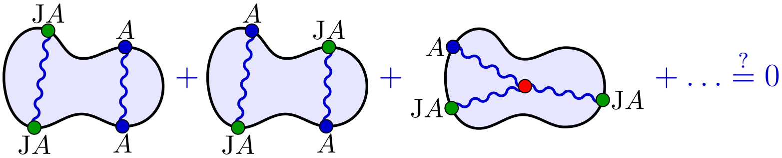
Classical symmetries may suffer from quantum anomalies:

- Not clear how to deal with anomalies for non-local action (in colour-space not necessarily in spacetime).
- Violation of (non-local) current? Cohomological origin?

Clues:

- Integrability appears to work well at finite λ: expect no anomalies?
- Not an issue for Wilson loop expectation value at $\mathcal{O}(\lambda)$.

Outlook: Consider Wilson loop at $\mathcal{O}(\lambda^2)$ [NB, Hansen, Münkler (in progress)]



Regularise carefully!

V. Conclusions

Conclusions

Review of AdS/CFT Integrability:

- Motivation and constituent models.
- Planar spectrum at finite coupling.
- Progress: correlators, scattering amplitudes, Wilson loops.

Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM:

- Equations of motion & action variation invariant.
- Classical planar $\mathcal{N} = 4$ SYM integrable.
- Ward–Takahashi identities due to Yangian symmetry.
- Symmetry compatible with gauge fixing.
- No quantum anomalies to be expected?!