Index Theorems for Gauge Theories, and Holonomy Saddles

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K. Hori, H. Kim, P.Y. 2014
S-J. Lee, P.Y. 2016
S-J. Lee, P.Y. 2017
C. Hwang, P.Y. 2017
C. Hwang, S. Lee, P.Y. 2018

an old problem and an old puzzle

index theorems via localization, or not

rational invariants and the old problem revisited

how "H-saddles" resolve the old puzzle
and glue gauge theories across dimensions

back in 1997

$$\frac{5}{4} \quad = \quad 1 \quad + \quad \frac{1}{4}$$

back in 1997

$$\mathcal{I}^{SU(2)}_{\mathcal{N}=16:\text{bulk}} \; = \; \mathcal{I}^{SU(2)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(2)}_{\mathcal{N}=16}$$

$$\frac{5}{4} \quad = \quad 1 \quad + \quad \frac{1}{4}$$

back in 1997

$$\mathcal{I} = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} = \mathrm{Tr}_{H=0}(-1)^{\mathcal{F}}$$

$$\mathcal{I}^{SU(2)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(2)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(2)}_{\mathcal{N}=16}$$

$$\frac{5}{4} = 1 + \frac{1}{4}$$

in an effort to confirm the M-theory hypothesis, of course

Witten 1995

M on $\mathbf{S}^1 \times \mathcal{R}^{9+1}$ = IIA on $\mathcal{R}^{9+1}$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
moving freely on $\mathcal{M}_{9+1}$

$$\mathcal{I}_{\mathcal{N}=\mathbf{16}}^{SU(N)} = 1$$

Hoppe 1982
Claudson, Halpern 1985
De Witt, Hoppe, Nicolai 1988
Hoppe 1988
Froehlich, Hoppe 1988
……

Banks, Fischler, Shenker, Susskind 1996

which is, perhaps, one of the most convoluted ways to obtain '1'

$$\mathcal{I} = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} = \mathrm{Tr}_{H=0}(-1)^{\mathcal{F}}$$

$$\uparrow$$

$$\mathcal{I}^{SU(2)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(2)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(2)}_{\mathcal{N}=16}$$

$$\frac{5}{4} = 1 + \frac{1}{4}$$

$$\uparrow$$

$$\lim_{\beta\to 0} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} \quad\to\quad \mathcal{Z}^{SU(2)}_{\mathcal{N}=16} = \int_{SU(2)/Z_2} dX\, d\Psi\;\; e^{-[X,X]^2/4 + X_\mu \Psi \Gamma_\mu \Psi/2}$$

P.Y. / Sethi, Stern 1997

which is, perhaps, one of the most convoluted ways to obtain '1'

$$\mathcal{I} = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} = \mathrm{Tr}_{H=0}(-1)^{\mathcal{F}}$$

$$\uparrow$$

$$\mathcal{I}^{SU(2)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(2)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(2)}_{\mathcal{N}=16}$$

$$\frac{5}{4} = 1 + \frac{1}{4} \leftarrow \mathcal{I}^{U(1)/Z_2}_{\mathcal{N}=16;\mathrm{bulk}}$$

P.Y. 1997

$$\uparrow$$

$$\lim_{\beta\to 0} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} \to \mathcal{Z}^{SU(2)}_{\mathcal{N}=16} = \int_{SU(2)/Z_2} dX\, d\Psi\;\, e^{-[X,X]^2/4 + X_\mu \Psi \Gamma_\mu \Psi/2}$$

P.Y. / Sethi, Stern 1997

the defect term arises entirely from the boundary
→ asymptotic dynamics suffices
→ computable by the asymptotic Coulomb branch

$$-\delta\mathcal{I}_{\mathcal{N}}^{SU(2)}$$

$$= -\delta\mathcal{I}_{\mathcal{N}}^{U(1)/Z_2}$$

$$= \quad \mathcal{I}_{\mathcal{N};\mathrm{bulk}}^{U(1)/Z_2}$$

P.Y. 1997

arbitrary high rank cases followed, soon

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(N)}_{\mathcal{N}=16}$$

$$\boxed{\mathcal{Z}^{SU(N)}_{\mathcal{N}=16} = \sum_{p|N;p\geq 1} \frac{1}{p^2}} = 1 + \sum_{p|N;p>1} \frac{1}{p^2}$$

Nekrasov, Moore, Shatashvili 1998

Green, Gutperle 1997
Kac, Smilga 1999

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16} = 1$$

similar problems with smaller supersymmetry address
Seiberg-Witten vs. IIA theory on local Calabi-Yau conifold

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{I}^{SU(N)}_{\mathcal{N}=4,8} - \delta\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8}$$

$$\boxed{\mathcal{Z}^{SU(N)}_{\mathcal{N}=4,8} = \frac{1}{N^2}} = 0 + \boxed{\frac{1}{N^2} = \mathcal{I}^{U(1)^{N-1}/S_N}_{\mathcal{N}=4,8:\text{bulk}}}$$

$$\Downarrow$$

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8} = 0$$

P.Y. 1997
Sethi, Stern 1997
Gutperle, Green 1997
Moore, Nekrasov, Shatashvili 1998

one would have naturally expected,
for other simple gauge groups…

$$\mathcal{I}^{G}_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{I}^{G}_{\mathcal{N}=4,8} - \delta\mathcal{I}^{G}_{\mathcal{N}=4,8}$$

$$\mathcal{Z}^{G}_{\mathcal{N}=4,8} \quad = \quad 0 \quad + \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8:\text{bulk}}$$

yet, …

$$\Uparrow$$

$$\mathcal{I}^{G}_{\mathcal{N}=4,8} = 0$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

P.Y. / Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000 / Pestun 2002

$\mathcal{N}=4$	$\mathcal{I}^G_{\rm bulk} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\rm bulk} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{51}{512}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{1275}{16384}$
$Sp(5)$	$\frac{663}{8192}$	$\frac{8415}{131072}$
$Sp(6)$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
$SO(15)$	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

?

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$
$$= -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$
$$= \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8;\text{bulk}}$$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$
$$= \mathcal{Z}^G_{\mathcal{N}=4,8}$$

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{51}{512}$
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$SO(7)$	$\frac{15}{128}$	$\frac{25}{256}$

?

IIA with an orienti-point

M on $\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$ = IIA on $\mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$

anomaly cancelation requires a single chiral fermion supported on $\mathbf{S}^1 \times \mathcal{R}^{0+1}$

➡

Kaluza-Klein reduction generates two towers of fermionic harmonic oscillators, resulting in four Hilbert spaces whose partition functions constitute the two generating functions above

Dasgupt, Mukhi 1995

Kol, Hanany, Rajaraman 1999

which requires

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

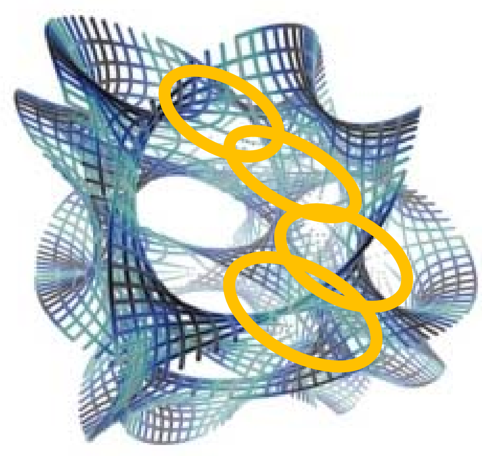
$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1 + t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1 + t^{2n})$$

but the program for proving the latter two was stuck, well before we come to this maximal supersymmetry then … jumping forward some 15 years

wall-crossing, rational invariants, quiver invariants, partition functions, localization, black hole microstates, …

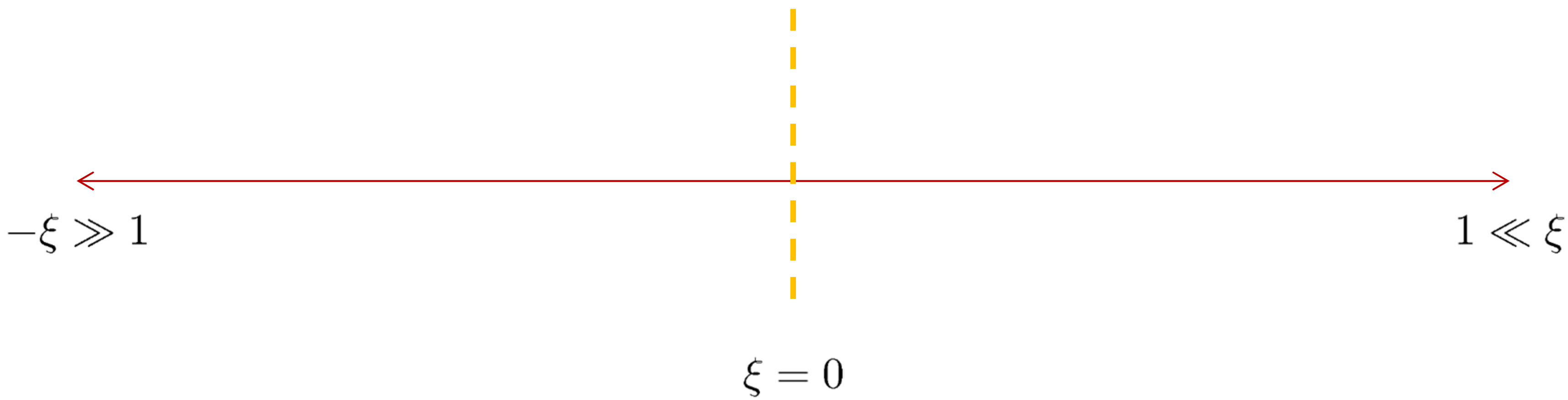
$$\underline{R^1} \times R^3 \times$$



1d/2d Gauged Linear Sigma Models with 4 Supercharges

$SU(2)_R \times U(1)_R$ gauge fields $(A_0, \lambda_\alpha, X_i, D)^a$ FI constants ξ^i for U(1)'s

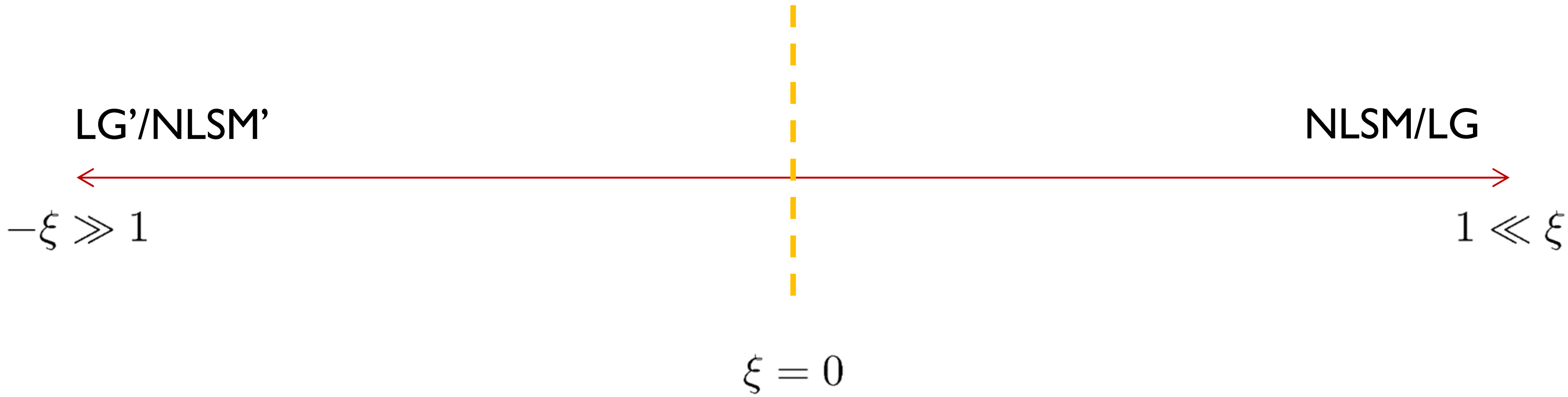
$J_{1,2,3}$ R chirals $(X, \psi_\alpha, F)^I$



1d/2d Gauged Linear Sigma Models with 2 Supercharges

$U(1)_R$

gauge fields	$(A_\mu, \lambda_-, D)^a$	FI constants ξ^i for U(1)'s
chirals	$(X, \psi_+)^I$	
fermi	$(\psi_-, F)^I$	



$\mathcal{I}$ as Ω

$$\mathcal{I}(\mathbf{y};x) \quad \equiv \quad \mathrm{Tr}_{\mathrm{kernel}(Q)}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}\right]$$

$$\Omega(\mathbf{y};x) \quad \equiv \quad \lim_{e^2\to 0}\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

a sweeping generalization of geometric index theorem
via path-integral by Alvarez-Gaume, ~1983, to gauged systems

Hori, Kim, P.Y. 2014

with the naïve invariance of index under continuous deformation, or under the banner of "localization"

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\mathrm{Re}\left(\int d\theta^2\ \mathrm{tr}\, W_\alpha W^\alpha\right)$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\int d\theta^2 d\bar{\theta}^2\ \mathrm{tr}\,\bar{\Phi} e^V \Phi$$

$$\mathcal{L}_{\text{usperpotential}} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi\int d\theta^2 d\bar{\theta}^2 \mathrm{tr}\, V$$

Benini, Eager, Hori, Tachikawa 2013
Hori, Kim, P.Y. 2014

scale up FI to send $e\xi$ to infinite for a reason to be explained,

then, after a long, long, long song and dance, ……

a Jeffrey-Kirwan contour integral (for χ_y genus if compact and geometric)

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$= \sum \text{JK-Res}_{\eta:\{Q_i\}}\, g(u,\bar{u};0)$$

$$g(u,\bar{u};D=0) = \left(\frac{1}{\mathbf{y}-\mathbf{y}^{-1}}\right)^{\mathrm{rank}} \prod_\alpha \frac{t^{-\alpha/2}-t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1}-t^{-\alpha/2}\mathbf{y}}$$

$$\times \prod_i \frac{t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-(R_i/2-1)}-t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2-1}}{t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2}-t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-R_i/2}}$$

Hori, Kim, P.Y. 2014

Szenes, Vergne 2004
Brion, M.Vergne 1999
Jeffrey, Kirwan 1993

2d GLSM Elliptic Genera

Benini + Eager + Hori + Tachikawa 2013

$\xi < 0$ $\xi = 0$ $0 < \xi$

1d GLSM Equivariant Index

Hori + Kim + P.Y. 2014

N=4 CP(N -1)

chirals	$U(1)$	$U(N)_F$
X	1	N

null

$\xi < 0$

$0 < \xi$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

$N = 6$

quintic CY3 hypersurface in CP4

	P	$X_{1,2,3,4,5}$
$U(1)$	-5	1

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

$\xi < 0$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$0 < \xi$

N=4 rank 2 GLSM Q.M. for CY3 in WCP$_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

hybrid

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

geometric

Landau-Ginsburg

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

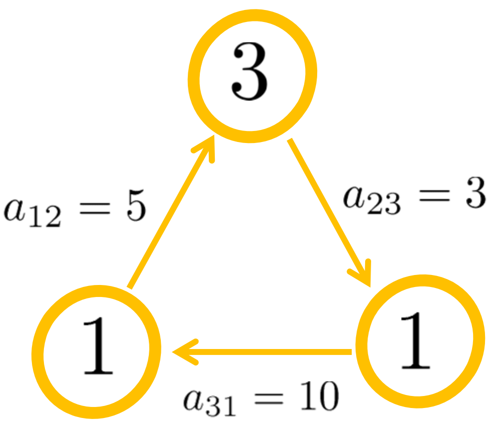
orbifold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

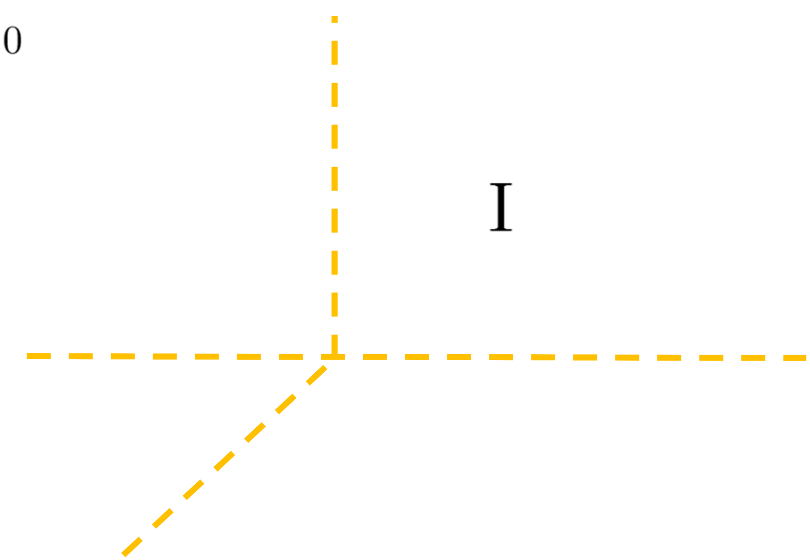
for the entire class of N=4 quiver quantum mechanics,
the entire Hodge diamonds can be recursively read off
from such χ_y genus
in each and every wall-crossing chambers!!!

J.Manschot, B.Pioline, A.Sen 2010~2013
S.J. Lee, Z.L. Wang, P.Y. 2012~2014

N = 4 (3,1,1) triangle quiver

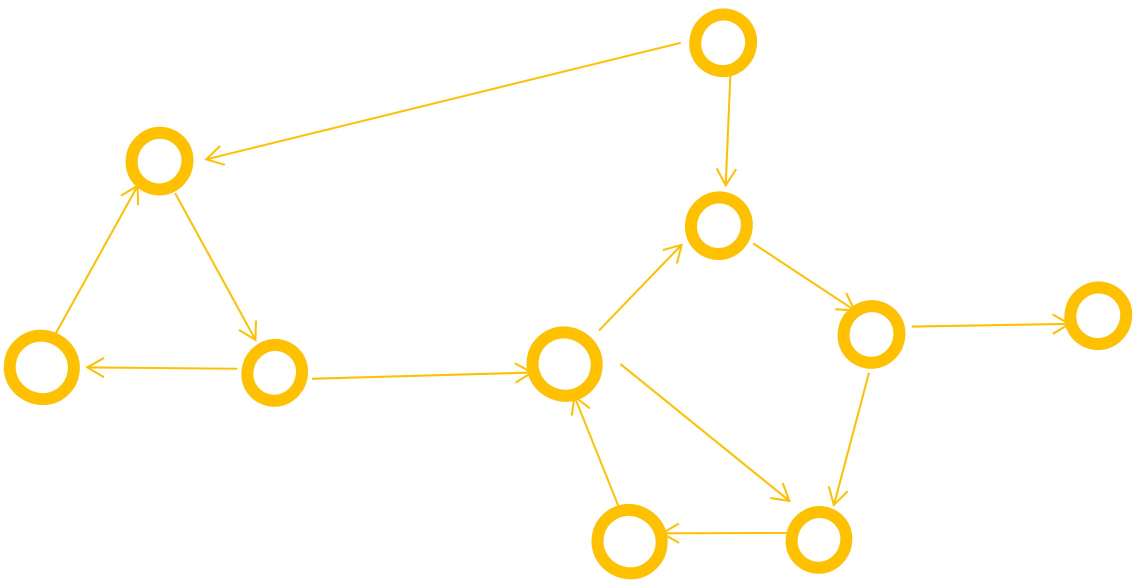


1
0 0
0 2 0
0 0 0 0
0 0 4 0 0
0 0 6 6 0 0
0 0 0 5 0 0 0
0 0 6 6 0 0
0 0 4 0 0
0 0 0 0
0 2 0
0 0
1



elliptic genus & witten index

cohomology

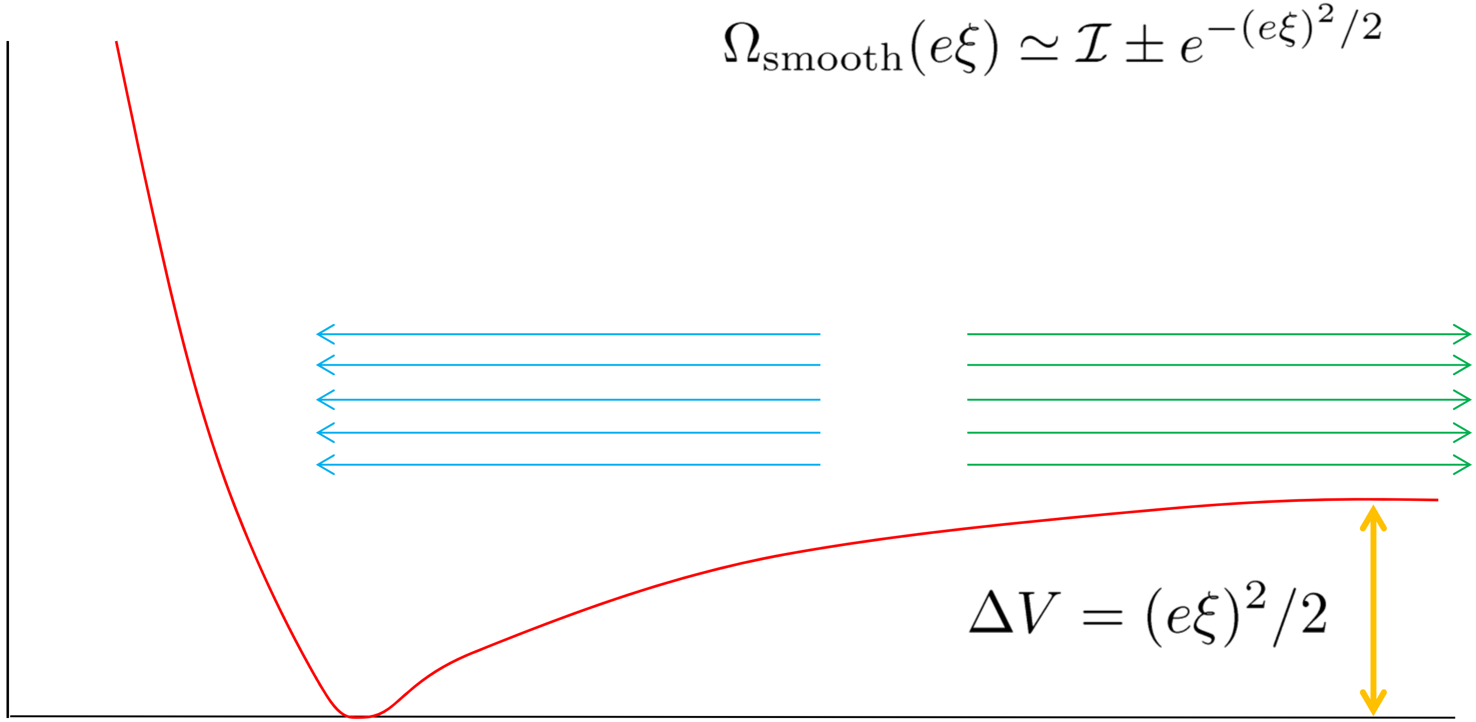


wall-crossing

quiver invariants

4d N=2 black hole microstates

wall-crossing happens at $\xi = 0$ because
a continuum direction touches the ground state



Hori, Kim, P.Y. 2014

what if such asymptotic flat directions
cannot be lifted by a parameter tuning?
can we still count the relevant Witten index reliably via path integral?

the only generic answer has to be "NO"
and higher supersymmetry does not help either

$\mathcal{I}$ from Ω

rational invariants to the rescue

when the asymptotic flatness arises from the Coulomb side

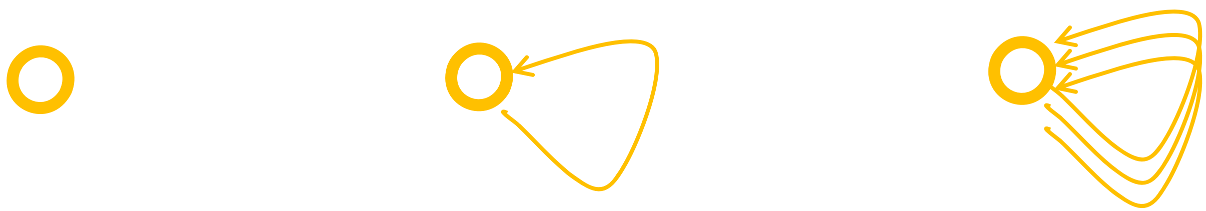
&

how this fixed a road to the M-theory hypothesis

S.J. Lee + P.Y., 2016

back to supersymmetric pure Yang-Mills quantum mechanics

$$\mathcal{N} = 4, 8, 16$$



S.J. Lee + P.Y., 2016

after rigorous applications of HKY procedure,

$$\Omega^{SU(2)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{2}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})} \qquad \rightarrow \qquad \frac{1}{2^2}$$

$$\Omega^{SU(3)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{3}\frac{1}{(\mathbf{y}^{-2}+1+\mathbf{y}^2)} \qquad \rightarrow \qquad \frac{1}{3^2}$$

$$\Omega^{SU(4)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^3)} \qquad \rightarrow \qquad \frac{1}{4^2}$$

$$\Omega^{SU(N)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{N}\frac{\mathbf{y}^{-1}-\mathbf{y}}{\mathbf{y}^{-N}-\mathbf{y}^{N}} \qquad \rightarrow \qquad \frac{1}{N^2}$$

S.J. Lee + P.Y., 2016

other rank 2 examples

$$\Omega^{SO(4)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{8}\left[\frac{2}{\mathbf{y}^{-2}+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

$$\Omega^{G_2}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{12}\left[\frac{2}{\mathbf{y}^{-2}-1+\mathbf{y}^2}+\frac{2}{\mathbf{y}^{-2}+1+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

S.J. Lee + P.Y., 2016

higher rank examples

$$\Omega^{SU(4)/SO(6)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})}$$

$$\Omega^{SO(7)/Sp(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{48}\left[\frac{8}{\mathbf{y}^{-3}+\mathbf{y}^{3}}+\frac{6}{(\mathbf{y}^{-2}+\mathbf{y}^{2})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{3}}\right]$$

$$\Omega^{SO(8)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{192}\left[\frac{32}{(\mathbf{y}^{-3}+\mathbf{y}^{3})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{12}{(\mathbf{y}^{-2}+\mathbf{y}^{2})^{2}}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{4}}\right]$$

S.J. Lee, P.Y. 2016

which can be organized into

elliptic Weyl elements only

$$0 \neq \mathrm{Det}\,(1 - w)$$

$$\Omega^G_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{|W_G|} \sum_{w \in W_G}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)}$$

Weyl group

analog of Kontsevich-Soibelman's rational invariants,
or analog of Gopakumar-Vafa's multicover formulae,
meaning that these captures multi-particle plane-wave sectors

elliptic Weyl elements for some classical groups

G	W	Elliptic Weyl Elements
$SU(N)$	S_N	$(123\cdots N)$
$SO(4)$	$Z_2 \times S_2$	$(\dot{1})(\dot{2})$
$SO(5)/Sp(2)$	$(Z_2)^2 \times S_2$	$(1\dot{2}), \quad (\dot{1})(\dot{2})$
$SO(6)$	$(Z_2)^2 \times S_3$	$(1\dot{2})(\dot{3})$
$SO(7)/Sp(3)$	$(Z_2)^3 \times S_3$	$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$
$SO(8)$	$(Z_2)^3 \times S_4$	$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$

why? because the localization implicitly computes the bulk part, and the index $\mathcal{I}$ is zero for these theories

$$\mathcal{I} \quad = \quad \mathcal{I}_{\text{bulk}} \quad + \quad \delta\mathcal{I}$$

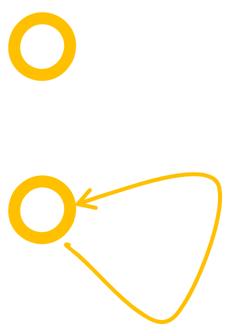
$$\Big\Downarrow$$

$$\lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$e^{2/3}\beta \to 0 \qquad \Big\Uparrow$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

these equal those of the Cartan orbifolds, for $\mathcal{N}=4,8$ cases



$$0 = \mathcal{I}^G_{\mathcal{N}=4,8} = \mathcal{I}^G_{\mathcal{N}=4,8;\mathrm{bulk}} + \delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$\Omega^G_{\mathcal{N}=4,8} = \mathcal{I}^G_{\mathcal{N}=4,8;\mathrm{bulk}} = -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$= -\delta\mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8}$$

$$= \mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8;\mathrm{bulk}}$$

P.Y. 1997

Green+Gutperle 1997
Kac+Smilga 1999

S.J. Lee + P.Y., 2016

the left hand side now agrees with the right hand side

with the smallest example being

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{8}\left[\frac{2}{\mathbf{y}^{-2}+\mathbf{y}^{2}} + \frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{2}}\right] \qquad \rightarrow \qquad \frac{5}{32}$$

and now demonstrate the vanishing index
for all simple groups, for lower supersymmetries

$$\mathcal{I}^{G}_{\mathcal{N}=4,8:\mathrm{bulk}} = \mathcal{I}^{G}_{\mathcal{N}=4,8} - \delta\mathcal{I}^{G}_{\mathcal{N}=4,8}$$

$$\Omega^{G}_{\mathcal{N}=4,8} = 0 + \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8:\mathrm{bulk}}$$

$$\Uparrow$$

$$\mathcal{I}^{G}_{\mathcal{N}=4,8} = 0$$

S.J. Lee, P.Y. 2016

S.J. Lee, P.Y. 2016

$\mathcal{N}=16$ with general simple Lie groups

$$\Omega^{G}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \quad \mathcal{I}^{G}_{\mathcal{N}=16} + \sum_{G'\subset G} \# \cdot \Delta^{G'}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\left(\mathbf{y}^{R_a-1}x^{F_a/2}-\mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a}x^{F_a/2}-\mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

elliptic Weyl elements only

$0 \neq \mathrm{Det}\,(1-w)$

S.J. Lee, P.Y. 2016

$$\Omega^{SU(N)}_{\mathcal{N}=16} = 1 + \sum_{p|N;p>1} 1 \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} = 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16}$$

$$\Omega^{G_2}_{\mathcal{N}=16} = 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16}$$

$$\Omega^{SO(7)}_{\mathcal{N}=16} = 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(3)}_{\mathcal{N}=16} = 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16}$$

$$\Omega^{SO(8)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16}$$

$$\Omega^{SO(9)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16} \cdot \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(4)}_{\mathcal{N}=16} = 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16} \cdot \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y}, x) = \frac{1}{|W|} \sum_{w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1} - \mathbf{y}\cdot w\right)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\left(\mathbf{y}^{R_a-1} x^{F_a/2} - \mathbf{y}^{1-R_a} x^{-F_a/2} \cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a} x^{F_a/2} - \mathbf{y}^{-R_a} x^{-F_a/2} \cdot w\right)}$$

the results suffice for reading off the Witten index $\mathcal{I}^{G}_{\mathcal{N}=16}$ from the unique integral part

$$\begin{aligned}
\mathcal{I}^{SU(N)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{SO(5)/Sp(2)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{G_2}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(7)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(8)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(9)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} &= 2 \\
&\vdots
\end{aligned}$$

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and similarly

$$
\begin{aligned}
\mathcal{I}^{SU(N)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{O(5)}_{\mathcal{N}=16} = \mathcal{I}^{Sp(2)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{G_2}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{O(7)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{O(8)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{O(9)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} &= 2 \\
&\vdots
\end{aligned}
$$

S.J. Lee, P.Y. 2017

which can be organized into the generating functions

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1+t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1+t^{2n})$$

S.J. Lee + P.Y., 2017

reproducing the predicted numbers from IIA with an orienti-point

M on $\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$ = IIA on $\mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
along fixed points of the orienti-point

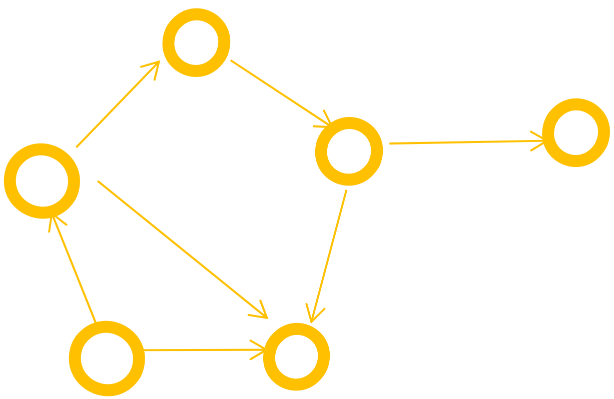
Dasgupta, Mukhi 1995
Kol, Hanany, Rajaraman 1999
Kac, Smilga 1999

⋮

S.J.Lee, P.Y. 2016 & 2017

this same story leads us back to multi-cover formulae for quivers

such rational Ω with integral $\mathcal{I}$ cleverly embedded
is not limited to pure Yang-Mills quantum mechanics,
as can be also seen in the refined version of
KS wall-crossing formulae, or in solutions thereof, for quivers



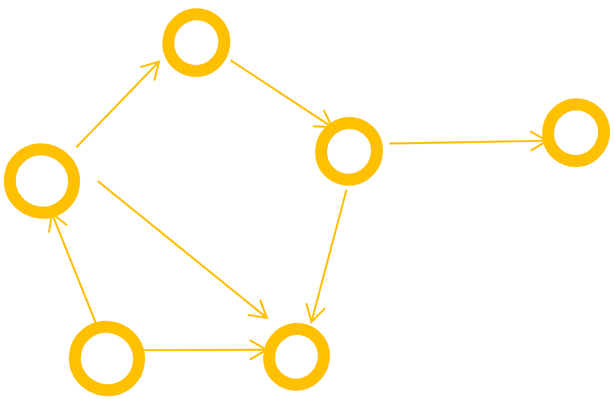
$$G = S\left[\prod_i U(N_i)\right] \qquad\qquad \omega(\Gamma;\mathbf{y}) \equiv \sum_{p|\Gamma} \mathcal{I}(\Gamma/p;\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$

proposal : the twisted partition functions compute, directly, these rational invariants for quiver theories

S.J. Lee + P.Y., 2016

$$\Omega(\Gamma; \mathbf{y}) = \mathcal{I}_{\text{bulk}} = \omega(\Gamma; \mathbf{y})$$

with generic superpotential



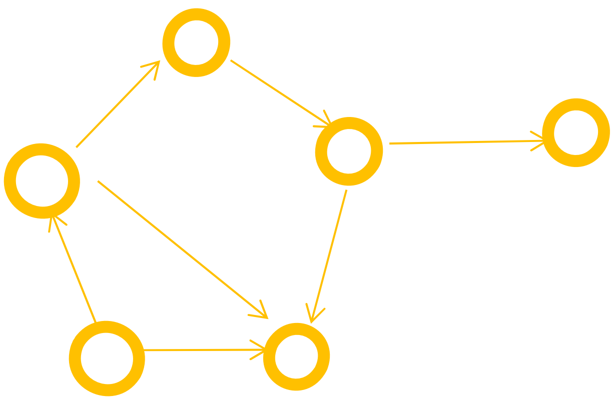
$$G = S\left[\prod_i U(N_i)\right]$$

proposal : the twisted partition functions of quivers compute these rational invariants for quiver theories

S.J. Lee + P.Y., 2016

$$\Omega(\Gamma;\mathbf{y}) = \mathcal{I}_{\text{bulk}} = \omega(\Gamma;\mathbf{y})$$

with generic superpotential

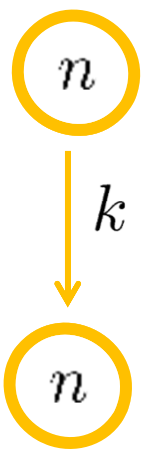


$$\mathcal{I}(\Gamma;\mathbf{y}) = \sum_{p|\Gamma} \mu(p)\cdot\Omega(\Gamma/p;\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$

$$G = S\left[\prod_i U(N_i)\right]$$

example : Kronecker quiver

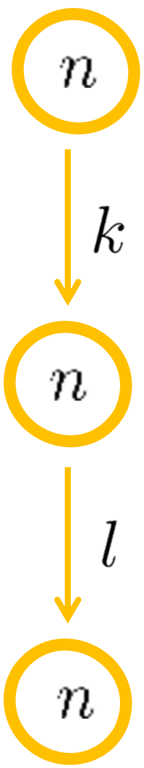
$$\mathcal{I}(\mathcal{Q}^k_{n,n};\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}^k_{n/p,n/p};\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$



$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^1_{2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^2_{2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^3_{2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^4_{2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^5_{2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2)-2\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)
\end{aligned}$$

example : 3-node quiver

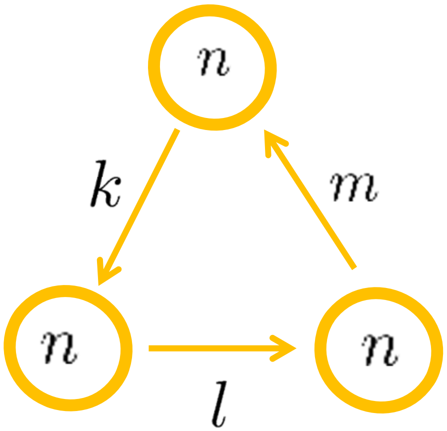
$$\mathcal{I}(\mathcal{Q}^{k,l}_{n,n,n};\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}^{k,l}_{n/p,n/p,n/p};\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$



$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^{1,1}_{2,2,2};\mathbf{y}) &= 0 \\
\mathcal{I}(\mathcal{Q}^{1,2}_{2,2,2};\mathbf{y}) &= 0 \\
\mathcal{I}(\mathcal{Q}^{1,3}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^{1,4}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^{1,5}_{2,2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2)-2\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^{2,2}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2) \\
\mathcal{I}(\mathcal{Q}^{2,3}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)
\end{aligned}$$

example : triangle quiver

$$\mathcal{I}(\mathcal{Q}^{k,l,m}_{n,n,n};\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}^{k,l,m}_{n/p,n/p,n/p};\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$



$$\begin{aligned}
\mathcal{I}(\mathcal{Q}^{1,1,-1}_{2,2,2};\mathbf{y}) &= 0\\
\mathcal{I}(\mathcal{Q}^{2,1,-1}_{2,2,2},;\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,1,-2,}_{2,2,2};\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,2,-1}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,1,-2}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-2\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{3,1,-1}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{7/2}(\mathbf{y}^2)-3\chi_{5/2}(\mathbf{y}^2)-2\chi_{3/2}(\mathbf{y}^2)-2\chi_{1/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{1,1,-3}_{2,2,2};\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)\\
\mathcal{I}(\mathcal{Q}^{2,2,-2}_{2,2,2};\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2)-\chi_{11/2}(\mathbf{y}^2)-4\chi_{9/2}(\mathbf{y}^2)\\
&\quad -3\chi_{7/2}(\mathbf{y}^2)-4\chi_{5/2}(\mathbf{y}^2)-\chi_{3/2}(\mathbf{y}^2)-\chi_{1/2}(\mathbf{y}^2)
\end{aligned}$$

1d challenges, conceptual and practical

partition functions and indices for quivers at vanishing FI ?

(quiver invariants → 4d N=2 black hole microstate counting!)

high rank asymptotics: a better contour prescription ?

rational invariants for general orientifolded quivers ?

then, what went wrong 15 years ago?

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

P.Y. / Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000 / Pestun 2002

$\mathcal{N}=4$	$\mathcal{I}^G_{\rm bulk} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\rm bulk} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{51}{512}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{1275}{16384}$
$Sp(5)$	$\frac{663}{8192}$	$\frac{8415}{131072}$
$Sp(6)$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
$SO(15)$	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\rm bulk}$$
$$= -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$
$$= \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8;\rm bulk}$$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\rm bulk}$$
$$= \mathcal{Z}^G_{\mathcal{N}=4,8:\rm matrix}$$

recall that the localization implicitly computes the bulk part

$$\mathcal{I} \quad = \quad \mathcal{I}_{\text{bulk}} \quad + \quad \delta\mathcal{I}$$

$$\lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$e^{2/3}\beta \to 0$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = \Omega^G$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{9}{64}$
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$Sp(6)$	$\frac{4641}{65536}$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
$SO(15)$	$\frac{16575}{262144}$	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

$$\mathcal{I}^G_{\text{bulk}} = \Omega^G$$

Lee, P.Y. 2016

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{Z}^G_{\mathcal{N}=4,8}$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

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Lee, P.Y. 2016

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$SO(12)$	$\frac{1589}{32768}$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
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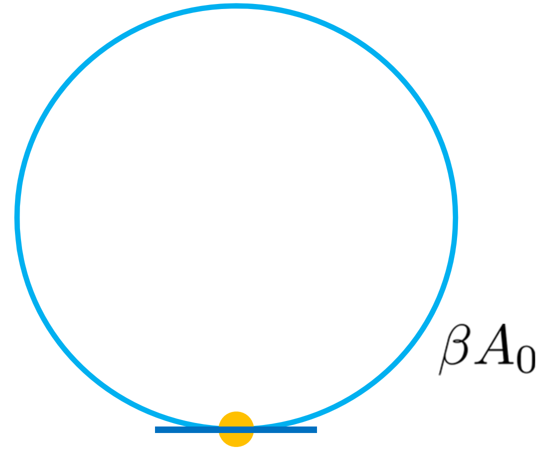
P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

?

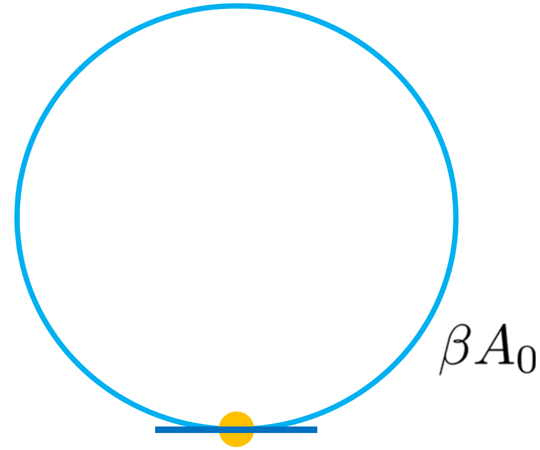
$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{Z}^G_{\mathcal{N}=4,8}$$



$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} = \mathcal{Z}^G_{\text{matrix integral}}$$

this particular matrix integral is from
the 1d path integral reduced to 0d
in the region near trivial Wilson line

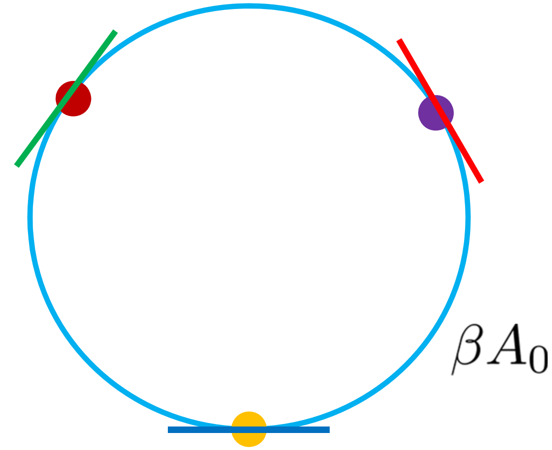
Hwang, P.Y. 2017



$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} \neq \mathcal{Z}^G_{\text{matrix integral}}$$

this particular matrix integral is from
the 1d path integral reduced to 0d
in the region near trivial Wilson line

Hwang, P.Y. 2017

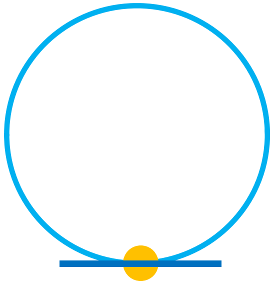


$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} = \sum_{H\subset G}\int dZ\, d\Phi\; \frac{O(\beta^0)}{Z^{2(g-h)}}\; e^{-[Z,Z]^2/4+Z_\mu K_\mu(\Phi)/2}$$

other special Wilson line values,
separated away at distance of order β^{-1},
do contribute generally;
at such *H* saddles the effective 0d theory
must have no decoupled free fermions

Hwang, P.Y. 2017

a trivial example
SU(N)

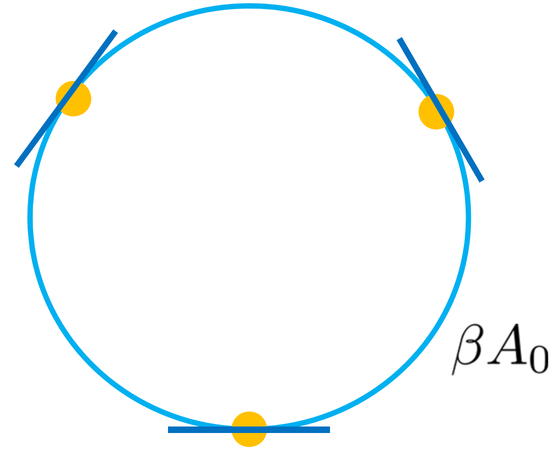


βA_0

$$\mathcal{I}^{G}_{\text{bulk}}\Big|_{\beta\to 0} = \mathcal{Z}^{SU(N)/Z_N}$$

Hwang, P.Y. 2017

a trivial example
SU(N)

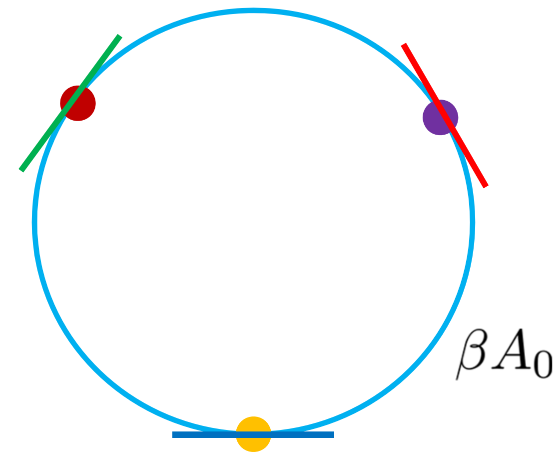


$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} = \sum_{u_{SU(N)}} \int dZ\, d\Phi\; e^{-[Z,Z]^2/4+Z_\mu K_\mu(\Phi)/2}$$

$$= N \times \int dZ\, d\Phi\; e^{-[Z,Z]^2/4+Z_\mu K_\mu(\Phi)/2}$$

$$= \mathcal{Z}^{SU(N)/Z_N}$$

Hwang, P.Y. 2017



$$\mathcal{I}^G_{\rm bulk}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^G(z') + \sum_{H<G} d_{G:H}\, \frac{|\det(Q^G)|/|W_G|}{|\det(Q^H)|/|W_H|}\, \mathcal{Z}^H(z')$$

H saddles: maximal non-Abelian subgroups left unbroken by a Wilson line

Hwang, P.Y. 2017

$$\mathcal{I}^{SU(N)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SU(N)}(z')$$

$$\mathcal{I}^{Sp(K)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{Sp(K)}(z') + \sum_{m=1}^{K-1} \frac{1}{4}\,\mathcal{Z}^{Sp(m)\times Sp(K-m)}(z')$$

$$\mathcal{I}^{SO(2N)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SO(2N)}(z') + \sum_{m=2}^{N-2} \frac{1}{8}\,\mathcal{Z}^{SO(2m)\times SO(2N-2m)}(z')$$

$$\mathcal{I}^{SO(2N+1)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SO(2N+1)}(z') + \sum_{m=2}^{N} \frac{1}{4}\,\mathcal{Z}^{SO(2m)\times SO(2N+1-2m)}(z')$$

Hwang, P.Y. 2017

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

$$\mathcal{I}^G_{\text{bulk}} = \Omega^G$$

Lee, P.Y. 2016

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = \Omega^G$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\cancel{\mathcal{I}^G_{\text{bulk}}} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{15}{128}$	$\frac{51}{512}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{195}{2048}$	$\frac{1275}{16384}$
$Sp(5)$	$\frac{663}{8192}$	$\frac{663}{8192}$	$\frac{8415}{131072}$
$Sp(6)$	$\frac{4641}{65536}$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
$SO(15)$	$\frac{16575}{262144}$	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

$$\mathcal{I}^G_{\text{bulk}} \neq \mathcal{Z}^G_{\text{matrix model}}$$

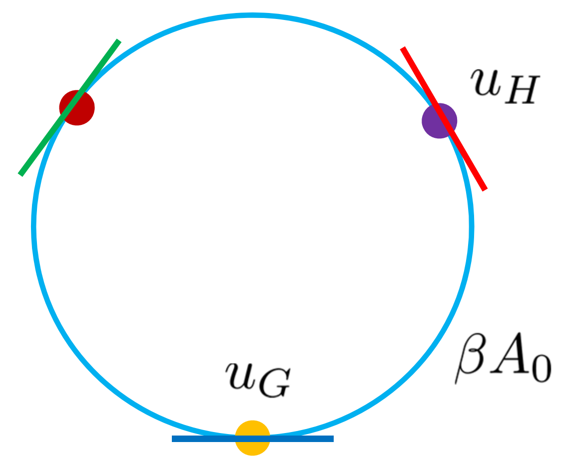
such *H*-saddles appears due to the integration
over the gauge holonomy, and thus are potentially relevant
for all susy partition functions on a vanishing circle,
regardless of space-time dimensions;

they explain many of subtleties out there,
in relating partition functions of susy gauge theories
in two adjacent dimensions

dimensional reduction is multi-branched for susy gauge theories

$S^1 \times \mathcal{M}_{d-1}$ $\qquad\qquad\qquad$ $\mathcal{M}_{d-1}$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \quad \to \quad \sum_{u_H} \; c_{G:H}(\beta)\; \mathcal{Z}_{d-1}^H(\tilde{z})$$

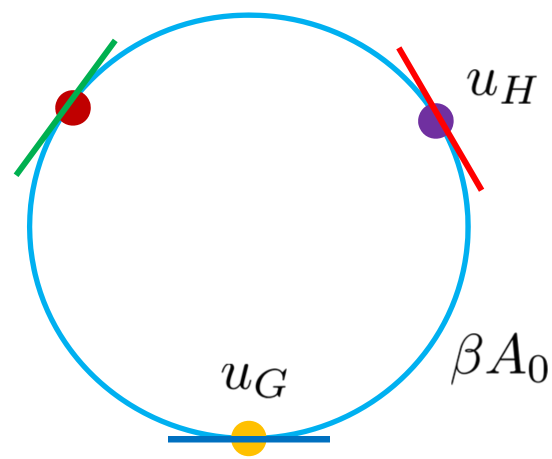


→ (1) equivariant Witten indices of different gauge theories can now be related across dimensions systematically

$S^1 \times T^{d-1}$ $\qquad\qquad$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})$$

purely algebraic factors



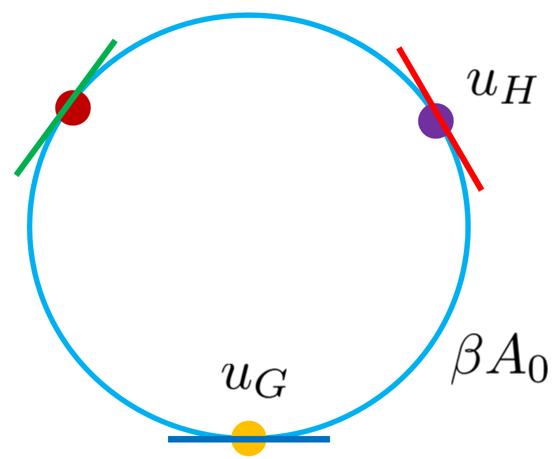
→ (1′) or, more precisely, twisted partition functions can now be related across dimensions systematically

$S^1 \times T^{d-1}$ $\qquad\qquad\qquad\qquad$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\mathrm{bulk};\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})\Big|_{\mathrm{bulk}}$$

↓

purely algebraic factors

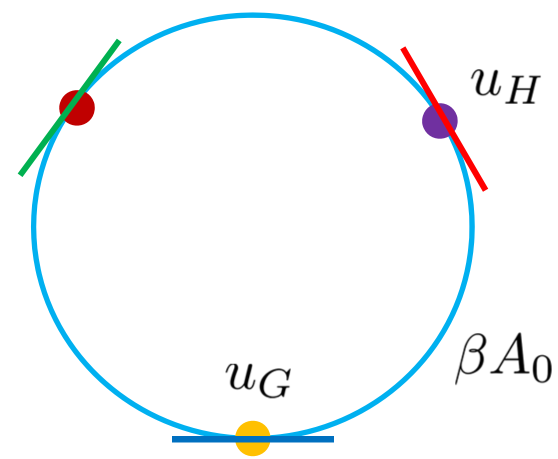


which really characterizes H-saddles and identifies their discrete locations in the space of holonomies

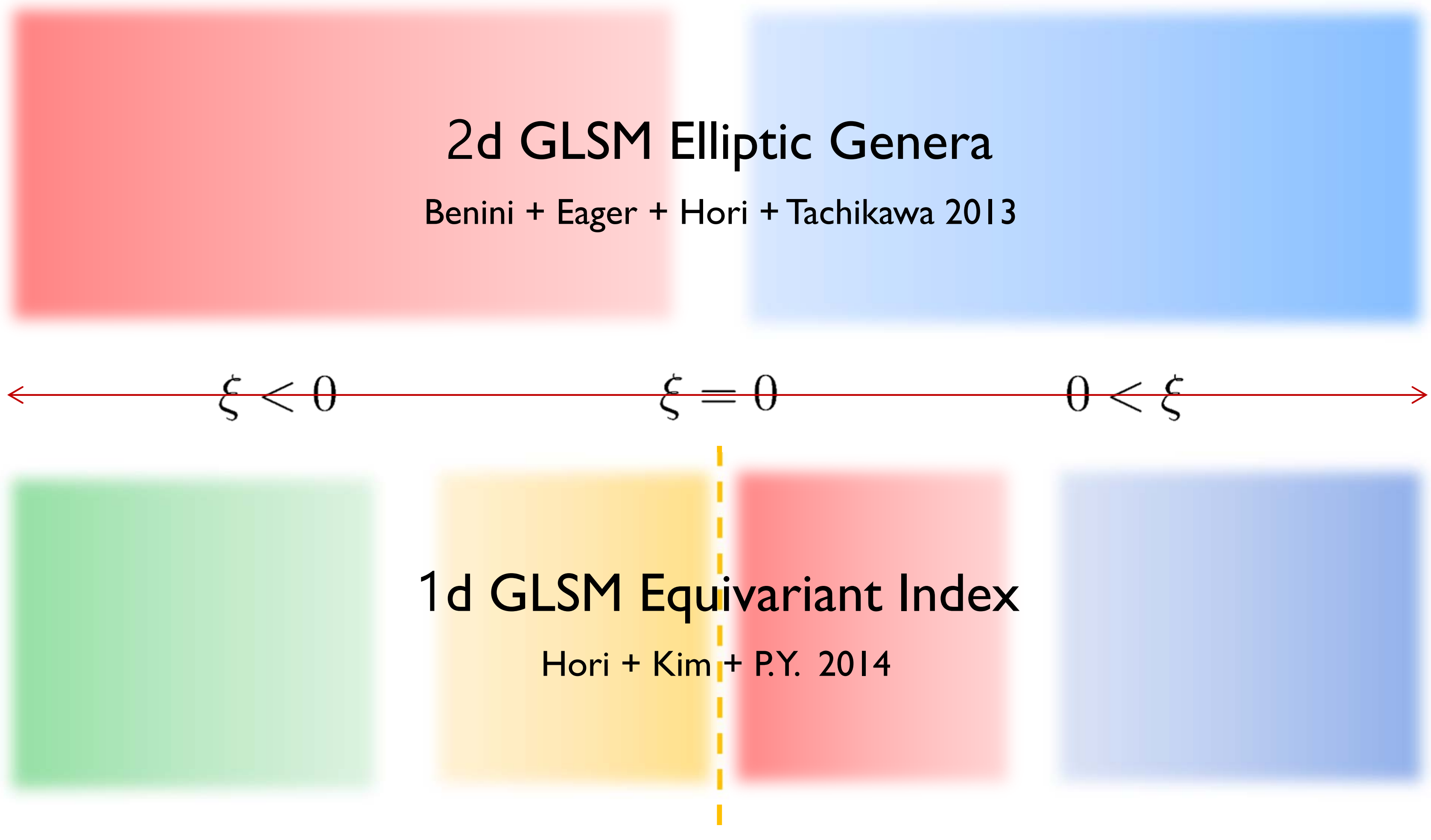
$S^1 \times T^{d-1}$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\mathrm{bulk};\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})\Big|_{\mathrm{bulk}}$$

purely algebraic factors



this phenomenon also underlies why the 2d elliptic genera fail to capture the 1d wall-crossing phenomena



→ (2) dimensional reduction of a dual pair, on a circle, produces many such dual pairs in 1d less, at best

$$S^1 \times \mathcal{M}_{d-1} \qquad\qquad \mathcal{M}_{d-1}$$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_H} c_{G:H}(\beta)\ \mathcal{Z}_{d-1}^H(\tilde{z})$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$= \ \Omega_d^{G'}(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_{H'}} c_{G':H'}(\beta)\ \mathcal{Z}_{d-1}^{H'}(\tilde{z})$$

→ (2′) such saddle-by-saddle dualities could fail for d<3 where the holonomy cannot have a vev even in the non-compact limit

$S^1 \times \mathcal{M}_{d-1}$ $\qquad\qquad$ $\mathcal{M}_{d-1}$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \;\to\; \sum_{u_H} c_{G:H}(\beta)\; \mathcal{Z}_{d-1}^H(\tilde{z})$$

$$\Updownarrow$$

$$=\;\Omega_d^{G'}(\beta\tilde{z})\Big|_{\beta\to 0} \;\to\; \sum_{u_{H'}} c_{G':H'}(\beta)\; \mathcal{Z}_{d-1}^{H'}(\tilde{z})$$

→ (3) there may be multiple Cardy exponents and the Dominant one does not generically equal the naïve one

$S^1 \times \mathcal{M}_{d-1}$

$\mathcal{M}_{d-1}$

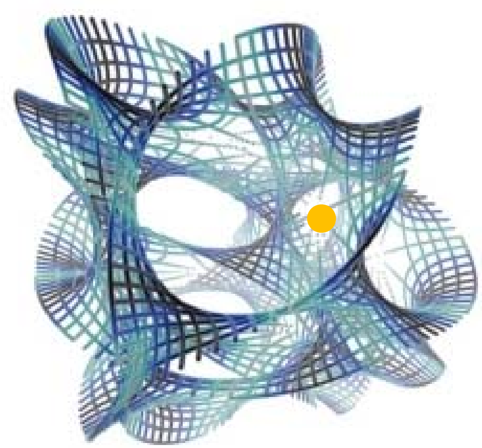
$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \;\to\; \sum_{u_H} \underbrace{c_{G:H}(\beta)}_{\sim\, e^{S^{G:H}_{\mathrm{Cardy}}/\beta}}\; \mathcal{Z}^H_{d-1}(\tilde{z})$$

how *H*-saddles manifest in the Bethe-Ansatz-driven partition functions of massive 4d N=1 theories

Hwang, Lee, P.Y. 2018

more generally, an entire class of susy partition functions was proposed for Riemannian surfaces with circle bundles

$$\text{A-twisted}\quad \left| \begin{array}{c} T^2 \\ \downarrow \\ \Sigma_g \end{array} \right. \qquad \times$$



Closset, Kim, Willett 2017

partition functions as a sum over BAE vacua on A-twisted geometry

$$\Omega_4^G = \sum_{u_* \in \mathcal{S}_{\mathrm{BE}}} \mathcal{F}_1(u_*, \nu; \tau)^{p_1} \, \mathcal{F}_2(u_*, \nu; \tau)^{p_2} \, \mathcal{H}(u_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\mathrm{BE}} = \{ u_* \mid \Phi_a(u_*, \nu; \tau) = 1, \forall a, \quad w \cdot u_* \neq u_*, \forall w \in W_G \} / W_G$$

complex structure of the fibre torus $\tau \sim i \dfrac{\beta_2}{\beta_1}$

complexified flavor holonomies

complexified gauge holonomies

Closset, Kim, Willett 2017

partition functions as a sum over BAE vacua on A-twisted geometry

$$\Omega_4^G = \sum_{u_* \in \mathcal{S}_{\mathrm{BE}}} \mathcal{F}_1(u_*, \nu; \tau)^{p_1}\, \mathcal{F}_2(u_*, \nu; \tau)^{p_2}\, \mathcal{H}(u_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\mathrm{BE}} = \{ u_* \mid \Phi_a(u_*, \nu; \tau) = 1, \forall a, \quad w \cdot u_* \neq u_*, \forall w \in W_G \} / W_G$$

$$\Phi_a(u, \nu; \tau) \equiv \exp(2\pi i \partial_a \mathcal{W})$$

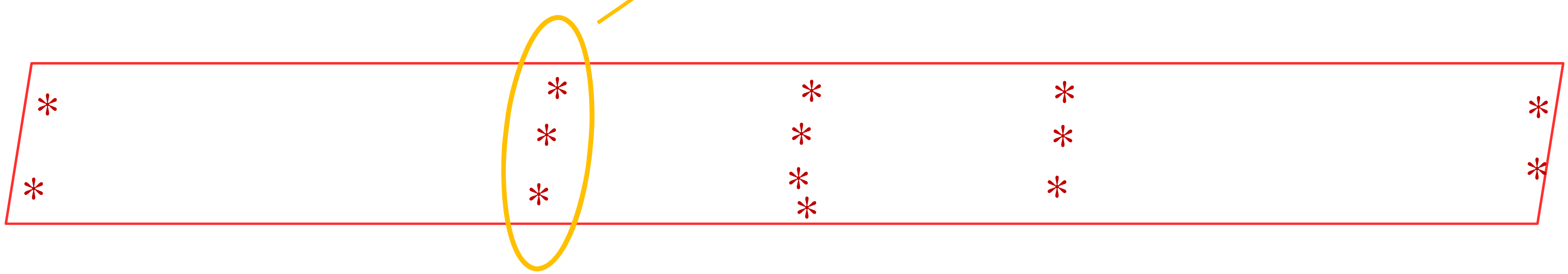
twisted superpotential in the Coulomb phase
on Σ_g due to the infinite towers
of 2d chiral fields from T^2 compactification

Closset, Kim, Willett 2017

in the Cardy limit or in the Casimir limit, related by SL(2,Z)

$$\Omega_4^G = \sum_{u_H} \sum_{\sigma_* \in \mathcal{S}_{\mathrm{BE}}^H} \mathcal{F}_1^H(\sigma_*, \nu; \tau)^{p_1}\, \mathcal{F}_2^H(\sigma_*, \nu; \tau)^{p_2}\, \mathcal{H}^H(\sigma_*, \nu; \tau)^{g-1}$$

$$\rightarrow \sum_{u_H} c_{u_H}(\tau)\; \mathcal{Z}_3^H$$

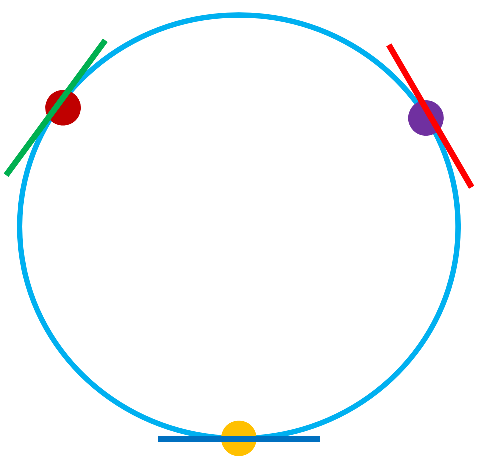


$$u = u_1 \tau + u_2$$
$$\tilde{u} = u_1 + u_2/\tau$$

→ (4) there may be multiple Cardy exponents/Casimir energies and, generically, the dominant ones need not equal the naïve ones

$$\frac{1}{2\pi i\tau}\log\left(c_{u_H}(\tau)\right) \qquad \frac{1}{2\pi i\tilde{\tau}}\log\left(c_{\tilde{u}_H}(\tilde{\tau})\right)$$

$$= \ (g-1)\times\left[-\frac{1}{12}(\mathrm{tr}_f R)+\frac{1}{2}\sum_\alpha \epsilon_\alpha(1-\epsilon_\alpha)+\frac{1}{2}\sum_i (r_i-1)\sum_{\rho_i}\epsilon_{\rho_i}(1-\epsilon_{\rho_i})\right]+\cdots$$



$$\epsilon_Q = \{Q\cdot u_H/\tau\} = \{Q\cdot \tilde{u}_H/\tilde{\tau}\}$$

Hwang, Lee, P.Y. 2018

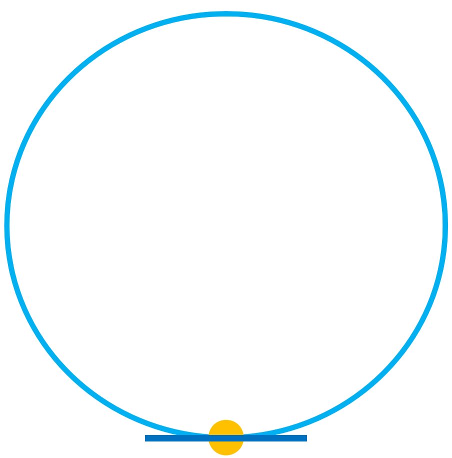
as opposed to more familiar but naïve exponents at $\begin{matrix} u_H = 0 \\ \tilde{u}_H = 0 \end{matrix}$:

$$\frac{1}{2\pi i \tau} \log\left(c_{u_H=0}(\tau)\right) \qquad \frac{1}{2\pi i \tilde{\tau}} \log\left(c_{\tilde{u}_H=0}(\tilde{\tau})\right)$$

$$= \quad (g-1) \times \left[-\frac{1}{12}(\mathrm{tr}_f R)\right] + \cdots$$

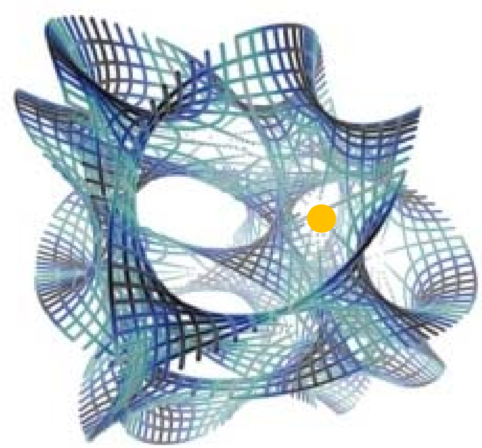
$$\propto \quad [(a-c)]$$

Di Pietro + Komargodski 2014



something similar can be done for superconformal indices:
the naïve Cardy exponents are generically modified
due to the presence of *H*-saddles

$$\underline{S^1 \times S^3} \times$$

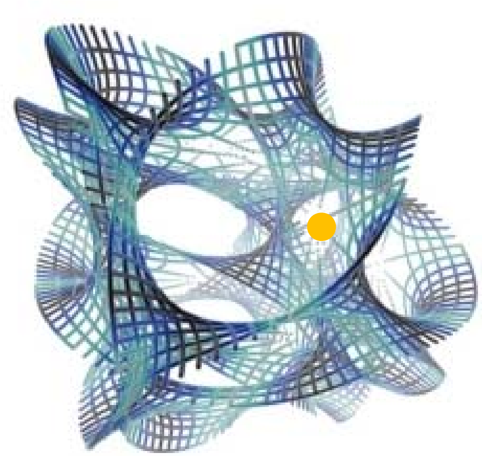


Hwang, Lee, P.Y. 2018
cf) Ardehali 2015

something similar can be done for superconformal indices:
the naïve Casimir exponents hold, however,
despite the presence of *H*-saddles

$$\cancel{SL(2,Z)}$$

$$\underline{S^1 \times S^3} \times$$



Hwang, Lee, P.Y. 2018
Assel, Cassani, Di Pietro, Komrgodski, Lorenzen, Martelli 2015

a chapter closed, in the M-theory hypothesis,
via localization

ubiquitous "H-saddles"

gluing supersymmetric gauge theories
across dimensions

"H-saddles" for (2,0) theories on T^2 ?