Spectrum and Ground States of Membrane Matrix Models

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Space Time Matrices
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Outline of Talk

1. Introduction to the quantum membrane
2. Spectrum and ground state conjecture
3. Approaches to the study of ground states
4. Outlook

Extremal bosonic membrane in $\mathbb{R}^{1,1+d}$

World-volume topology: $\mathbb{R} \times \Sigma$,
Σ fixed 2D compact manifold (Riemann surface)

Embedding coordinate functions: $\mathbf{x} = (x_{j=1,\ldots,d}) \colon \mathbb{R} \times \Sigma \to \mathbb{R}^d$
(*light-front coordinates*)

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Hamiltonian: $H[\mathbf{x}, \mathbf{p}] = \int_\Sigma \left(\sum_{j=1}^{d} p_j^2 + \sum_{1 \le j < k \le d} \{x_j, x_k\}_\Sigma^2 \right)$

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Canonical Poisson bracket on Σ: $\{f, g\}_\Sigma \sim \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$
Dynamical Poisson bracket: $\{x_j(\varphi), p_k(\varphi')\}_{\mathrm{PB}} = \delta_{jk} \delta(\varphi, \varphi')$

Hoppe et. al., previous talks,
J. de Woul, J. Hoppe, D.L., *Partial Hamiltonian reduction of relativistic extended objects in light-cone gauge*, JHEP, 2011

Matrix regularization (or "1st quantization")

Infinite-dimensional Poisson algebra of zero-mean real-valued functions x_j	$\rightarrow$	(N^2-1)-dimensional algebra of traceless hermitian $N \times N$ matrices X_j
$\{x_j, x_k\}_\Sigma$	$\rightarrow$	$\frac{1}{i}[X_j, X_k]$
$\int_\Sigma$	$\rightarrow$	Tr

(with convergence of structure constants $f_{ABC}^{(N)}$ in a basis $\{T_A\}$)

Respects symmetries (constraints):

Diffeomorphism invariance	$\rightarrow$	$\mathrm{SU}(N)$ invariance

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$$= \sum_{j,A} p_{jA}^2 + \frac{1}{2} \sum_{j,k,A} (f_{ABC} x_{jB} x_{kC})^2, \quad \{x_{jA}, p_{kB}\}_{\mathrm{PB}} = \delta_{jk}\delta_{AB}$$

Quantization (or "2nd quantization")

Schrödinger representation on $\mathcal{H}_B = L^2(\mathbb{R}^d \otimes \mathbb{R}^{N^2-1})$:

$$X_j \quad \rightarrow \quad \hat{X}_j = x_{jA} T_A, \qquad x_{jA} \text{ coordinate multiplication operators}$$

$$P_j \quad \rightarrow \quad \hat{P}_j = p_{jA} T_A, \qquad p_{jA} = -i\partial_{x_{jA}}$$

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Hamiltonian: $H_B = -\Delta_{\mathbb{R}^d \otimes \mathbb{R}^{N^2-1}} + \frac{1}{2} \sum_{j,k,A} (f_{ABC} x_{jB} x_{kC})^2$

Symmetry: $\mathrm{SO}(d) \times \mathrm{SU}(N) \to \mathrm{SO}(d) \times \mathrm{SO}(N^2 - 1)$

Physical Hilbert space $\mathcal{H}_{B,\mathsf{phys}}$: $\mathrm{SU}(N)$-invariant states Ψ

$$f_{ABC} x_{jB} p_{jC} \Psi = 0$$

Standard Dirac (constraint) quantization.

J. Goldstone, unpublished; J. Hoppe, MIT Ph.D. thesis, 1982

Supersymmetry

Add spin degrees of freedom $\rightarrow$ supermembrane $\rightarrow$ SUSY QM

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Supersymmetric quantum mechanics $(\mathcal{H}, K, H, \mathcal{Q}_j)$

- Hilbert space $\mathcal{H}$
- Grading operator $K^2 = 1 \;\Rightarrow\; \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$
- Hamiltonian operator H even, self-adj.
- Supercharge operators $\mathcal{Q}_{j=1,\dots,\mathcal{N}}$ odd s.t. $\{\mathcal{Q}_j, \mathcal{Q}_k\} = 2\delta_{jk} H$

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Symmetries in the spectrum:

$$H = \mathcal{Q}_j^2 \geq 0 \quad \Rightarrow \; \operatorname{spec} H \subseteq [0, \infty)$$

$$\begin{aligned} H\Psi &= E\Psi, \;\; \Psi \in \mathcal{H}_\pm \setminus \{0\}, \;\; E > 0 \\ \Rightarrow \;\; H\Phi &= E\Phi, \;\; \Phi \in \mathcal{H}_\mp \setminus \{0\}, \;\; \Phi := \mathcal{Q}_j \Psi \end{aligned}$$

Supermembrane matrix model

Spin representations $\mathrm{Spin}(d) \times \mathrm{Spin}(N^2 - 1) \to \mathcal{L}(\mathcal{F})$

Clifford algebras:

Over $\mathbb{R}^d$: $\{\gamma^j, \gamma^k\} = 2\delta^{j,k}$ real irrep: $\mathbb{R}^{\mathcal{N}_d}$

Over $\mathbb{R}^{\mathcal{N}_d} \otimes \mathbb{R}^{N^2-1}$: $\{\boldsymbol{\theta}_{\alpha A}, \boldsymbol{\theta}_{\beta B}\} = 2\delta_{\alpha,\beta}\delta_{A,B}$ irrep: $\mathcal{F} = \mathbb{C}^{2^{\frac{1}{2}\mathcal{N}_d(N^2-1)}}$

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Hamiltonian:

$$H = p_{jA}p_{jA} + \frac{1}{2}\sum_{A,j,k}(f_{ABC}x_{jB}x_{kC})^2 + \frac{i}{2}x_{jC}f_{CAB}\gamma^j_{\alpha\beta}\,\boldsymbol{\theta}_{\alpha A}\,\boldsymbol{\theta}_{\beta B}$$

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Supercharges: $\mathcal{Q}_{\alpha=1,\ldots,\mathcal{N}_d} = \left(p_{jA}\gamma^j_{\alpha\beta} + \frac{1}{2}f_{ABC}x_{jB}x_{kC}\gamma^{jk}_{\alpha\beta}\right)\boldsymbol{\theta}_{\beta A}$

s.t. $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 2\delta_{\alpha\beta}H + 4\gamma^j_{\alpha\beta}x_{jA}J_A$

Requirement: $d = 2, 3, 5$, or 9 $\Rightarrow \mathcal{N}_d = 2(d-1) = 2, 4, 8$, or 16

Supermembrane matrix model (cont.)

d	$\mathcal{C}l(\mathbb{R}^d)$	$\mathcal{N}_d$	$\mathcal{S}_d$	$\circlearrowleft \mathrm{Spin}(d)$	
1	$\mathbb{R} \oplus \mathbb{R}$	1	$\mathbb{R}$	$\mathbb{R}$	⇠
2	$\mathbb{R}^{2\times 2}$	2	$\mathbb{R}^2$	$\mathbb{C}$	$\leftarrow$
3	$\mathbb{C}^{2\times 2}$	4	$\mathbb{C}^2$	$\mathbb{H}$	$\leftarrow$
4	$\mathbb{H}^{2\times 2}$	8	$\mathbb{H}^2$	$\mathbb{H}_+ \oplus \mathbb{H}_-$	
5	$\mathbb{H}^{2\times 2} \oplus \mathbb{H}^{2\times 2}$	8	$\mathbb{H}^2$	$\mathbb{H}^2$	$\leftarrow$
6	$\mathbb{H}^{4\times 4}$	16	$\mathbb{H}^4$	$\mathbb{C}^4 \oplus \mathbb{C}^4$	
7	$\mathbb{C}^{8\times 8}$	16	$\mathbb{C}^8$	$\mathbb{R}^8 \oplus \mathbb{R}^8$	
8	$\mathbb{R}^{16\times 16}$	16	$\mathbb{R}^{16}$	$\mathbb{R}^8_+ \oplus \mathbb{R}^8_-$	
9	$\mathbb{R}^{16\times 16} \oplus \mathbb{R}^{16\times 16}$	16	$\mathbb{R}^{16}$	$\mathbb{R}^{16}$	$\leftarrow$

D.L., L. Svensson, *Clifford algebra, geometric algebra, and applications*, 2009 (2016)

Supermembrane matrix model (cont.)

Full Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^d \otimes \mathbb{R}^{N^2-1}) \otimes \mathcal{F}$

Physical Hilbert space $\mathcal{H}_{\mathsf{phys}}$: $J_A \Psi = 0$, where $\mathrm{SU}(N) \to \mathrm{Spin}(N^2-1) \to \mathcal{L}(\mathcal{H})$ by

$$J_A = f_{ABC}\Big(x_{jB} p_{jC} - \frac{i}{4} \boldsymbol{\theta}_{\alpha B} \boldsymbol{\theta}_{\alpha C} \Big)$$

$\mathrm{Spin}(d)$-symmetry:

$$J_{jk} = x_{jA} p_{kA} - x_{kA} p_{jA} - \frac{i}{8} \gamma^{jk}_{\alpha\beta} \boldsymbol{\theta}_{\alpha A} \boldsymbol{\theta}_{\beta A}$$

M. Baake, P. Reinicke, V. Rittenberg, *Fierz identities for real Clifford algebras and the number of supercharges*, J. Math. Phys., 1985; Claudson, Halpern, 1985; Flume, 1985

B. de Wit, J. Hoppe, H. Nicolai, *On the quantum mechanics of supermembranes*, Nucl. Phys. B, 1988

Supermembrane matrix model (cont.)

A **Fock space** representation: $\mathcal{F} = \mathrm{Span}_{\mathbb{C}}\{\prod_{\hat{\alpha},A} \lambda^{\dagger}_{\hat{\alpha}A}|0\rangle\}$

$$\lambda^{(\dagger)}_{\hat{\alpha}A} := \frac{1}{2}\left(\boldsymbol{\theta}_{\hat{\alpha}A} \overset{+}{\scriptstyle(-)} i\,\boldsymbol{\theta}_{\frac{\mathcal{N}_d}{2}+\hat{\alpha},A}\right), \quad \hat{\alpha} = 1,\ldots,\mathcal{N}_d/2 = d-1$$

$$\{\lambda_{\hat{\alpha}A}, \lambda^{\dagger}_{\hat{\beta}B}\} = \delta_{\hat{\alpha}\hat{\beta}}\delta_{AB}, \qquad \{\lambda_{\hat{\alpha}A}, \lambda_{\hat{\beta}B}\} = 0, \qquad \{\lambda^{\dagger}_{\hat{\alpha}A}, \lambda^{\dagger}_{\hat{\beta}B}\} = 0.$$

With $\mathbb{R}^d \rightsquigarrow \mathbb{R}^{d-2} \times \mathbb{C}$, $\gamma \rightsquigarrow \Gamma$ and corresponding split of coordinates

$$\mathbf{x} = (\mathbf{x}', \mathrm{Re}\, z, \mathrm{Im}\, z), \quad \mathbf{x}' := (x_{\hat{\jmath}})_{\hat{\jmath}=1,\ldots,d-2}, \quad z := x_{d-1} + i x_d,$$

$$H = H_B - 2i x_{\hat{\jmath}C} f_{CAB} \Gamma^{\hat{\jmath}}_{\hat{\alpha}\hat{\beta}} \lambda^{\dagger}_{\hat{\alpha}A}\lambda_{\hat{\beta}B} + z_C f_{CAB}\lambda^{\dagger}_{\hat{\alpha}A}\lambda^{\dagger}_{\hat{\alpha}B} + \bar{z}_C f_{CAB}\lambda_{\hat{\alpha}A}\lambda_{\hat{\alpha}B}$$

B. de Wit, J. Hoppe, H. Nicolai, *On the quantum mechanics of supermembranes*, Nucl. Phys. B, 1988

Supermembrane matrix model (cont.)

$d = 3, 5$ alternative: Use complex structure $J^2 = -1$ ($\mathbb{C}$ or $\mathbb{C} \subseteq \mathbb{H}$)

$$\hat{\lambda}_{\hat{\alpha}A} := \frac{1}{2}(\boldsymbol{\theta}_{\beta A} + iJ_{\gamma\beta}\,\boldsymbol{\theta}_{\gamma A})e^{\hat{\alpha}}_{\beta}, \qquad \{e^{\hat{\alpha}}\}_{\hat{\alpha}=1,\ldots,\mathcal{N}_d/2=d-1}$$

$$H = H_B - 2ix_{sC}f_{CAB}\gamma^{s}_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\dagger}_{\hat{\alpha}A}\hat{\lambda}_{\hat{\beta}B},$$

$$J_A = L_A - if_{ABC}\hat{\lambda}^{\dagger}_{\hat{\alpha}B}\hat{\lambda}_{\hat{\alpha}C},$$

$$J_{st} = L_{st} - \frac{i}{2}\gamma^{st}_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\dagger}_{\hat{\alpha}A}\hat{\lambda}_{\hat{\beta}A},$$

Claudson, Halpern, 1985
D.L., *Zero-energy states in supersymmetric matrix models*, Ph.D. thesis, KTH, 2010

Supermembrane matrix model (cont.)

$d = 1$ (degenerate) model:

$$H = -\Delta_{\mathbb{R}^{N^2-1}} - 2x_A J_A \qquad (V = 0)$$

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with

$$\mathcal{Q} := -i\,\boldsymbol{\theta}_A \frac{\partial}{\partial x_A} \sim \nabla \quad \Rightarrow \quad H = \mathcal{Q}^2 \text{ on } \mathcal{H}_{\mathsf{phys}},$$

$$J_A = f_{ABC}\Big(x_B p_C - \frac{i}{4}\,\boldsymbol{\theta}_B\,\boldsymbol{\theta}_C\Big)$$

or

$$Q := -i\hat{\lambda}^\dagger_A \frac{\partial}{\partial x_A} \sim \mathrm{d}, \quad \{Q, Q^*\} = H + 2x_A J_A, \quad Q^2 = 0,\ (Q^*)^2 = 0,$$

$$J_A := f_{ABC}\left(x_B p_C - i\hat{\lambda}^\dagger_B \hat{\lambda}_C\right).$$

Claudson, Halpern, 1985; Samuel 1997; Trzetrzelewski 2007; D.L., Ph.D. thesis, 2010

Spectrum

Surprising differences between the energy spectra of:

- Classical (regularized) membrane
- Quantum regularized membrane
- Quantum regularized supermembrane

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Illustrated conveniently using **toy models**.

Spectrum: Classical model

Hamiltonian: $H = \sum_{j=1}^{d} \mathrm{Tr}\, P_j^2 + V$

Potential: $V = \sum_{1 \leq j < k \leq d} \mathrm{Tr}\, (i[X_j, X_k])^2 \quad \geq 0$

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Potential: $V = \sum_{1 \leq j < k \leq d} \operatorname{Tr} (i[X_j, X_k])^2 \quad \geq 0$

Toy model in $\mathbb{R}^2$: $V_{\mathsf{toy}} = x^2 y^2$

Flat directions $\Rightarrow$ unconfined

Spectrum: Quantum mechanical model

Scalar Schrödinger operator: $H_B = -\Delta + V(x) \;\; \geq 0$

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Toy model: $H_{B,\text{toy}} = -\partial_x^2 - \partial_y^2 + x^2 y^2$

Purely discrete spectrum:

$$\begin{aligned} H_{B,\text{toy}} &= \frac{1}{2}(-\partial_x^2 - \partial_y^2) + \frac{1}{2}\underbrace{(-\partial_x^2 + y^2 x^2)}_{\geq |y|} + \frac{1}{2}\underbrace{(-\partial_y^2 + x^2 y^2)}_{\geq |x|} \\ &\geq \frac{1}{2}(-\Delta + |x| + |y|) \ > 0 \end{aligned}$$

M. Lüscher, NPB 1983; B. Simon, Ann. Phys. 1983

Garcia del Moral et. al., NPB, 2007; 2010 (BLG/ABJM type)

Spectrum: Supersymmetric quantum mechanical model

Matrix Schrödinger operator: $H = (-\Delta + V(x))1 + x_{jA} M_{jA}$
s.t. $H = \mathcal{Q}_\alpha^2 \geq 0$ on $\mathcal{H}_{\mathsf{phys}}$

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Theorem (dW-L-N)

For any $\lambda \geq 0$ there exists a sequence Ψ_t of rapidly decaying smooth $SU(N)$-invariant functions s.t. $\|\Psi_t\| = 1$ and $\|(H - \lambda)\Psi_t\| \to 0$ as $t \to \infty$. Hence, $\operatorname{spec} H = [0, \infty)$.

For toy model: $\Psi_t(x, y) := \chi_t(x)\phi_x(y)\xi$

B. de Wit, M. Lüscher, H. Nicolai, *The supermembrane is unstable*, NPB, 1989

Ground state conjecture

BFSS Conjecture

$d = 9$: *Unique normalizable zero-energy ground state for all* N

$d = 2, 3, 5$: *No normalizable zero-energy state for any* N

T. Banks, W. Fischler, S. Shenker, L. Susskind, *M Theory As A Matrix Model: A Conjecture*, Phys. Rev. D, 1997

Ground state conjecture (cont.)

Conjecture supported by:

- Rigorous proof (by contradiction) for $d = 2, N = 2$

  J. Fröhlich, J. Hoppe, *On Zero-Mass Ground States in Super-Membrane Matrix Models*, CMP, 1998

- Asymptotics (necessary decay known for $N = 2$)

  M. B. Halpern, C. Schwartz, *Asymptotic Search for Ground States of SU(2) Matrix Theory*, Int. J. Mod. Phys. A, 1998
  J. Fröhlich, G. M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, *Asymptotic form of zero energy wave functions in supersymmetric matrix models*, NPB, 2000

- Witten index calculations

  P. Yi, *Witten Index and Threshold Bound States of D-Branes*, NPB, 1997
  S. Sethi, M. Stern, *D-Brane Bound States Redux*, CMP, 1998

  Green, Gutperle, 1998; Krauth, Nicolai, Staudacher, 1998; Kac, Smilga, 2000; Moore, Nekrasov, Shatashvili, 2000

  Caution! Imbimbo, Mukhi, 1984; Staudacher, 2000; Jaffe, 2000

Ground state conjecture (cont.)

$d = 3, 5$ embedded eigenvalues: $H|_{\hat{\mathcal{F}}_0} = H_B|_{\hat{\mathcal{F}}_0} > 0$

Ground state conjecture (cont.)

$d = 3, 5$ embedded eigenvalues: $H|_{\hat{\mathcal{F}}_0} = H_B|_{\hat{\mathcal{F}}_0} > 0$

$d = 3$ non-normalizable states:

$$Q_{\hat{\alpha}} \sim e^{-W} \hat{\lambda}^\dagger \cdot \partial \, e^W, \quad W(x) := \frac{1}{6} \epsilon_{jkl} f_{ABC} x_{jA} x_{kB} x_{lC}$$

$$\Psi_{0,-} = e^{-W(x)} |0\rangle, \quad \Psi_{0,+} = e^{+W(x)} \hat{\lambda}^\dagger_{1,1} \dots \hat{\lambda}^\dagger_{2,N^2-1} |0\rangle$$

Ground state conjecture (cont.)

$d = 3, 5$ embedded eigenvalues: $H|_{\hat{\mathcal{F}}_0} = H_B|_{\hat{\mathcal{F}}_0} > 0$

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$d = 1$ degenerate model ($V = 0$):

$$Q \sim \hat{\lambda}^\dagger \cdot \partial$$

$$\Psi_{0,-} = |0\rangle, \quad \Psi_{0,+} = \hat{\lambda}^\dagger_1 \dots \hat{\lambda}^\dagger_{N^2-1}|0\rangle$$

Plane-wave (non)normalizable zero-energy states for any N.

Claudson, Halpern, 1985; D.L., 2010; Hynek, 2016

Some approaches to the study of ground states

I. Construction by recursive methods

J. Hoppe, D.L., M. Trzetrzelewski, *Construction of the Zero-Energy State of SU(2)-Matrix Theory: Near the Origin*, Nucl. Phys. B, 2009

Hynek, Trzetrzelewski, 2010; Michishita, 2010; 2011

II. Deformation

J. Hoppe, D.L., M. Trzetrzelewski, *Octonionic twists for supermembrane matrix models*, Ann. Henri Poincaré, 2009

III. Averaging w.r.t. symmetries

J. Hoppe, D.L., M. Trzetrzelewski, *Spin(9) Average of SU(N) Matrix Models I. Hamiltonian*, J. Math. Phys., 2009

IV. Weighted spaces and index theory

D.L., *Weighted Supermembrane Toy Model*, Lett. Math. Phys., 2010; Ph.D. thesis, 2010

D.L., *Geometric extensions of many-particle Hardy inequalities*, J. Phys. A: Math. Theor., 2015

I. Construction by recursive methods

Consider the structure of a possible ground state $\Psi(x)$ around $x = 0$:

$$\Psi(x) = \psi^{(0)} + x_{jA}\psi^{(1)}_{jA} + \frac{1}{2}x_{jA}x_{kB}\psi^{(2)}_{jA,kB} + \dots,$$

where $\quad \gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(1)}_{jA} = 0, \quad \gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(2)}_{jA,kB} = 0,$

$$\gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(3)}_{jA,kB,lC} + if_{ABC}\gamma^{kl}_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(0)} = 0, \quad \text{etc.}$$

Theorem (JH-DL-MT)

For $d = 9$, $N = 2$ we have
(where $\mathcal{F} = \otimes^3\mathcal{F}_{256}$ and $\mathcal{F}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ under $\mathrm{Spin}(9)$)

$$\psi^{(0)} \propto (44 \otimes 44 \otimes 44)_{\mathsf{sym}} + \frac{13}{36}(44 \otimes 84 \otimes 84)_{\mathsf{sym}}$$

I. Construction by recursive methods (cont.)

$$(44 \otimes 44 \otimes 44)_{\mathsf{sym}} := |jl\rangle_1 |kl\rangle_2 |jk\rangle_3$$

$$\begin{aligned}(44 \otimes 84 \otimes 84)_{\mathsf{sym}} &:= |jk\rangle_1 |jlm\rangle_2 |klm\rangle_3 \\ &\quad + |klm\rangle_1 |jk\rangle_2 |jlm\rangle_3 \\ &\quad + |jlm\rangle_1 |klm\rangle_2 |jk\rangle_3\end{aligned}$$

Michishita & Trzetrzelewski studied also $\psi^{(1)}$, $\psi^{(2)}$ (for $N = 2$)

II. Deformation

A conjugation of a combination of supercharges:

$$Q(\mu) := e^{\mu g(x)} \frac{1}{\sqrt{2}} (\mathcal{Q}_8 + i\mathcal{Q}_{16}) e^{-\mu g(x)},$$

with

$$g(x) = \frac{1}{6} f_{ABC} x_{jA} x_{kB} x_{lC} \gamma^{jkl}_{8,16},$$

leads to a family of new models $H(\mu) := \{Q(\mu)^\dagger, Q(\mu)\} \geq 0$ with $G_2 \times \mathrm{U}(1) \times \mathrm{SU}(N)$ symmetry:

$$H(\mu) = -\Delta_{1\ldots7} + (\mu-1)^2 V_{1\ldots7} + H_D + (\mu-1) x_{1\ldots7} \cdot M_1 + x_{89} \cdot M_2$$

cp. M. Porrati, A. Rozenberg, NPB, 1998

II. Deformation (cont.)

Consider $\tilde{H} := H(\mu = 1)$, which is a truncation of H

Theorem (JH-DL-MT)

$$\operatorname{spec} \tilde{H} = \operatorname{spec} H = [0, \infty)$$

Deformation approach has been successful for simpler models

L. Erdős, D. Hasler, J. P. Solovej, *Existence of the D0 - D4 bound state: A Detailed proof*, Ann. Henri Poincaré, 2005

III. Averaging w.r.t. Spin(9)

Coordinate split: $\mathbb{R}^9 = \mathbb{R}^7 \times \mathbb{R}^2$

Truncated Hamiltonian
$H_D = -\Delta_{89} + x_{89} \cdot S(x_{1\ldots7})x_{89} + x_{1\ldots7} \cdot M$

Interpretation: 2D SUSY $\mathrm{SU}(N)$ matrix model with 7D space of parameters

Simple spectrum: set of $2(N^2 - 1)$ SUSY harmonic oscillators

III. Averaging w.r.t. Spin(9) (cont.)

Slightly modified operator:

$$H'_D := -\frac{9}{2}\Delta_{89} + \frac{18}{7} x_{89} \cdot S(x_{1\ldots7}) x_{89} + \frac{36}{7} x_{1\ldots7} \cdot M$$

still simple spectrum, rescaled frequencies

Theorem (JH-DL-MT)

The average of the operator H'_D w.r.t. $\mathrm{Spin}(9)$ is equal to the full Hamiltonian H.

IV. Weighted spaces and index theory

Asymptotic analysis suggests to allow for more slowly decaying ground states (cp. also $d = 1$ model).

Weighted Hilbert space: $\mathcal{H}_\alpha = L^2(\mathbb{R}^{d(N^2-1)}, \rho_\alpha(x)dx) \otimes \mathcal{F}$,
with $\rho_\alpha(x) = (1 + |x|^2)^{-\alpha/2}$, $\alpha \geq 0$ weight.
$\Rightarrow \quad \langle \Phi, \Psi \rangle_\alpha = \langle \Phi, \rho_\alpha \Psi \rangle$

Self-adjoint Hamiltonian H_α defined by Friedrichs extension of:
$\langle \Psi, H_\alpha \Psi \rangle_\alpha := \langle \Psi, H\Psi \rangle = \|Q\Psi\|^2 \geq 0, \;\; \Psi \in C_c^\infty$.

Ground state correspondence:

$$\Psi \in \ker_{\mathcal{H}} H \quad \Rightarrow \quad \Psi \in \ker_{\mathcal{H}_\alpha} H_\alpha \quad \Rightarrow \quad \Psi \in C^\infty \text{ and } Q\Psi = 0$$

IV. Weighted spaces and index theory (cont.)

Spectral relation:

$$\langle \Psi, (H_\alpha - \lambda)\Psi \rangle_\alpha = \langle \Psi, (H - \lambda\rho_\alpha)\Psi \rangle \quad \Rightarrow \quad N(H_\alpha - \lambda)_\alpha = N(H - \lambda\rho_\alpha)$$

Hence, if H_α has a discrete spectrum in $\mathcal{H}_\alpha$
($\Leftrightarrow H - \lambda\rho_\alpha$ in $\mathcal{H}$ has finitely many negative eigenvalues $\forall\lambda$),
then

$$\ker_{\mathcal{H}_\alpha} H_\alpha \neq 0 \quad \Leftrightarrow \quad H - \lambda\rho_\alpha \text{ has a negative eigenvalue } \forall\lambda > 0$$

IV. Weighted spaces and index theory (cont.)

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Theorem (DL)

For the supermembrane toy model we have for $\alpha > 2$

$$N(H_{\mathsf{toy}} - \lambda\rho_\alpha) \leq C + o(\lambda^{\frac{3}{2}}),$$

and hence discrete spectrum of $H_{\mathsf{toy},\alpha}$.

D.L., *Weighted Supermembrane Toy Model*, Lett. Math. Phys., 2010

IV. Weighted spaces and index theory (cont.)

Sketch of proof:
Simpler to consider the domain $x > 1$ with Dirichlet boundary conditions, where

$$\begin{aligned} H_{\mathsf{toy}} - \lambda\rho_\alpha &\geq -\partial_x^2 - \partial_y^2 + x^2\left(y + \frac{1}{2x^2}\sigma_2\right)^2 - \frac{1}{4x^2} - x - \frac{\lambda}{x^\alpha} \\ &= -\partial_x^2 - \frac{1}{4x^2} \underbrace{-\partial_{\tilde{y}}^2 + x^2\tilde{y}^2 - x}_{=\sum_{k=0}^\infty 2kxP_k} - \frac{\lambda}{x^\alpha} \end{aligned}$$

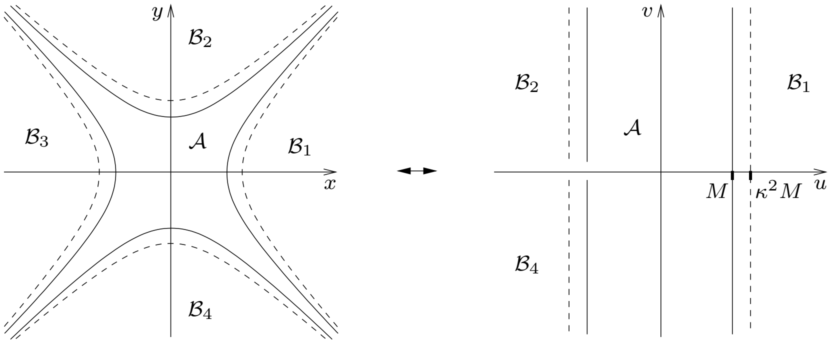
and use that for an operator-valued potential V on $(1,\infty)$, $V(x)$ acting on fibers $\hbar = L^2(\mathbb{R}, d\tilde{y})$,

$$N\left(\left(-\partial_x^2 - \frac{1}{4x^2}\right)\otimes 1_\hbar + V(x)\right) \leq C\int_1^\infty \mathrm{Tr}_{\,\hbar}\,|V(x)_-|^{\frac{3}{2}}\,x^2(\ln x)^2\,dx.$$

D. Hundertmark, *On the number of bound states for Schrödinger operators with operator-valued potentials*, Ark. Mat., 2002

IV. Weighted spaces and index theory (cont.)

For the full domain $\mathbb{R}^2$, use a partition of unity and a conformal coordinate transformation $z \mapsto z^2$ to map into regions of this form:



Partition of $\mathbb{R}^2$ into regions $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.

IV. Weighted spaces and index theory (cont.)

We have $H_\alpha = Q_\alpha^* Q_\alpha, \;\; Q_\alpha = \rho_\alpha^{-1/2} Q, \;\; Q_\alpha^* = \rho_\alpha^{-1} Q \rho_\alpha^{1/2}$
Consider $H'_\alpha := Q_\alpha Q_\alpha^*$

Weighted index:

$$I_\alpha := \mathrm{Tr}_{\mathcal{H}_\alpha} e^{-\beta H_\alpha} - \mathrm{Tr}_{\mathcal{H}_\alpha} e^{-\beta H'_\alpha} = \dim \ker_{\mathcal{H}_\alpha} H_\alpha - \dim \ker_{\mathcal{H}} H,$$

independent of $\beta > 0$ whenever H_α, H'_α have discrete spectra.

IV. Weighted spaces and index theory (cont.)

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independent of $\beta > 0$ whenever H_α, H'_α have discrete spectra.

Works fine for free line model and $d = 1$ model for sufficient α.

Toy model? Calculations suggest $I_\alpha = 0$...

D.L., *Zero-energy states in supersymmetric matrix models*, Ph.D. thesis, KTH, 2010

Outlook

I. Continued construction at $x \sim 0$ and $x \to \infty$

II. Zero-energy states for the deformed operator $\tilde{H}$?

III. Averaging of eigenstates of H_D resp. H'_D?

IV. Discreteness of H'_α, and weighted index for toy model?
$d = 2, 3, 5, 9$ SMM? Physical relevance of weighted states?

V. Embedded eigenvalues for $d = 3, 5$ SMM. Other d?

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Thank you!