Emergence of space-time from matrices

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Outline of the talk

Large-N reduction

IIB matrix model

Topology change of space-time

Large-N reduction

1. What is Large-N reduction?

Large-N reduction is a typical example of emergence of space-time from matrices.

The basic statement is

"The large-N gauge theory with periodic boundary condition *does not depend* on the volume of the space-time."

In particular, the theory in the infinite space-time is equivalent to that on one point. The space-time emerges from the internal degrees of freedom of the reduced model.

Lattice version

Consider U(N) or SU(N) lattice gauge theory

$$S_{\text{Wilson}} = -\frac{N}{\lambda}\sum_{n,\mu\neq\nu} Tr\left(U_{n,\mu}U_{n+\hat{\mu},\nu}U^{\dagger}_{n+\hat{\nu},\mu}U^{\dagger}_{n,\nu}\right)$$

in a d-dimensional periodic box of size

$$\boldsymbol{L_1 \times L_2 \times \cdots \times L_d\ .}$$

In the large-N limit $\boldsymbol{N \to \infty, \lambda}$: fixed,

***physics* does not depend on the size of the box $\boldsymbol{L_i}$ if the center invariance**

$$\boldsymbol{U_{n,\mu} \to e^{i\theta_\mu} U_{n,\mu}}$$

is not broken spontaneously.

Here "*physics*" means

(1) Free energy per unit volume $f = \dfrac{F}{V}$

$$F = -\log Z,\ Z = \int [dU] \exp\left(-S_{\text{Wilson}}\right),$$

$$V = L_1 \times L_2 \times \cdots L_d \ ,$$

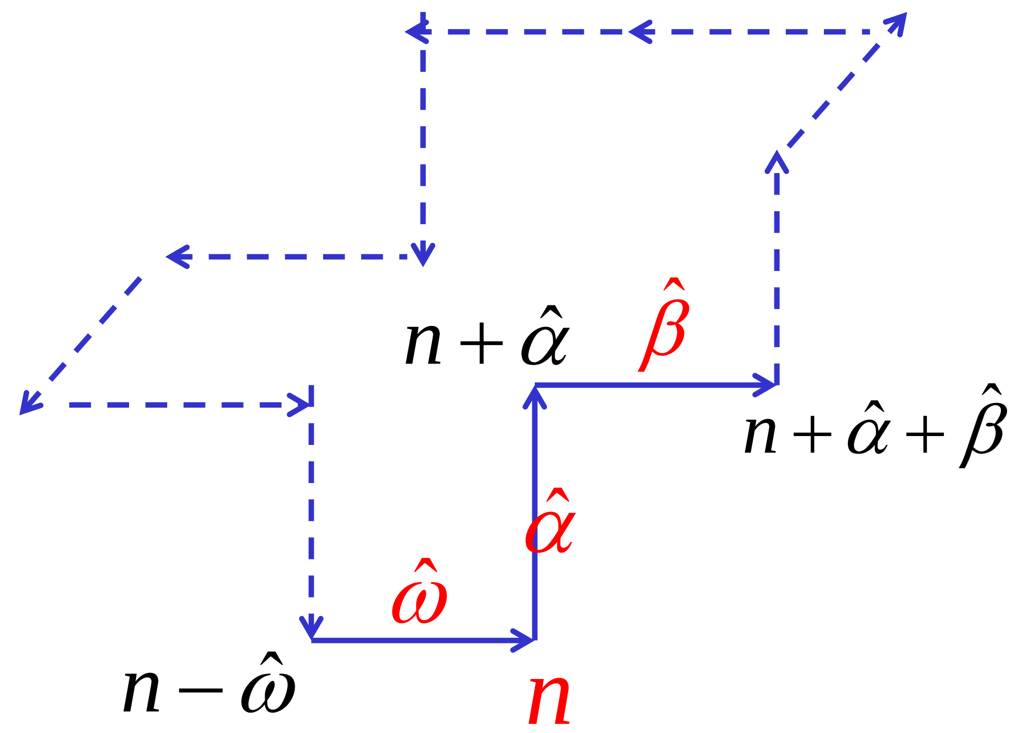
(2) Wilson loop

Wilson loop in a periodic box is defined as the next two slides:

First of all a closed loop *C* in the infinite lattice space is specified by a starting point *n* and the sequence of directions $\alpha, \beta, \cdots$.

$$C = \left(n, n+\hat{\alpha}, n+\hat{\alpha}+\hat{\beta}, \cdots, n-\hat{\omega}\right),$$

$$\alpha, \beta, \cdots = \pm 1, \pm 2, \cdots, \pm d \, .$$



Therefore we can define the corresponding loop *C'* that is folded in the periodic box by the same expression once the starting point *n'* is specified:

$$C' = \left(n', n'+\hat{\alpha}, n'+\hat{\alpha}+\hat{\beta}, \cdots, n'-\hat{\omega}\right).$$

Then the Wilson loop in the periodic box is defined as usual:

$$W(C) = \frac{\int \prod_{n,\mu} dU_{n,\mu} \exp\left(-S_{\text{Wilson}}\right) w(C)}{\int \prod_{n,\mu} dU_{n,\mu} \exp\left(-S_{\text{Wilson}}\right)}$$

$$w(C) = \frac{1}{N} Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right).$$

Lattice version

Consider U(N) or SU(N) lattice gauge theory

$$S_{\mathrm{Wilson}} = -\frac{N}{\lambda} \sum_{n,\mu\neq\nu} Tr\left(U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}\right)$$

in a d-dimensional periodic box of size

$$L_1 \times L_2 \times \cdots \times L_d \,.$$

In the large-N limit $N \to \infty, \lambda$: fixed,

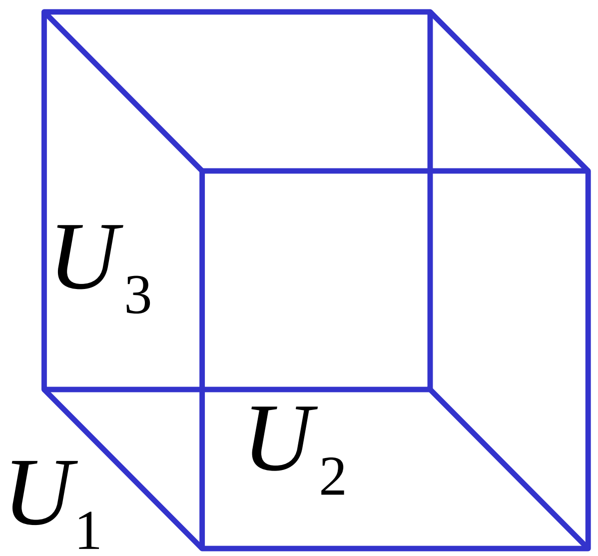
***physics* does not depend on the size of the box L_i if the center invariance**

$$U_{n,\mu} \to e^{i\theta_\mu} U_{n,\mu}$$

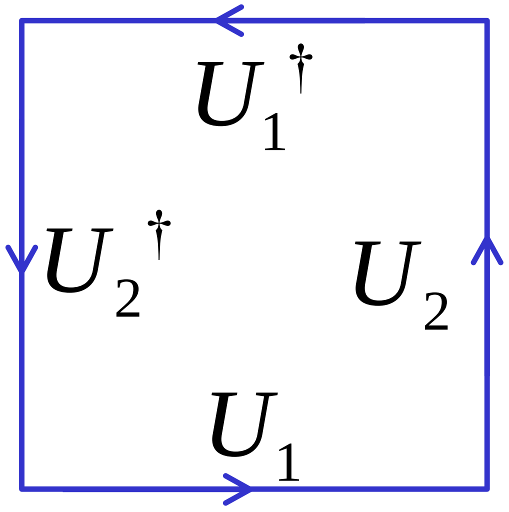
is not broken spontaneously.

In particular,

if we consider the minimum size of the box $\mathbf{1} \times \mathbf{1} \times \cdots \times \mathbf{1}$, we have a model with *d* unitary matrices:



$$S_{\text{reduced}} = -\frac{N}{\lambda} \sum_{\mu \neq \nu = 1}^{d} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right).$$



“the large-N reduced model”

Continuum version

The large-N reduced model

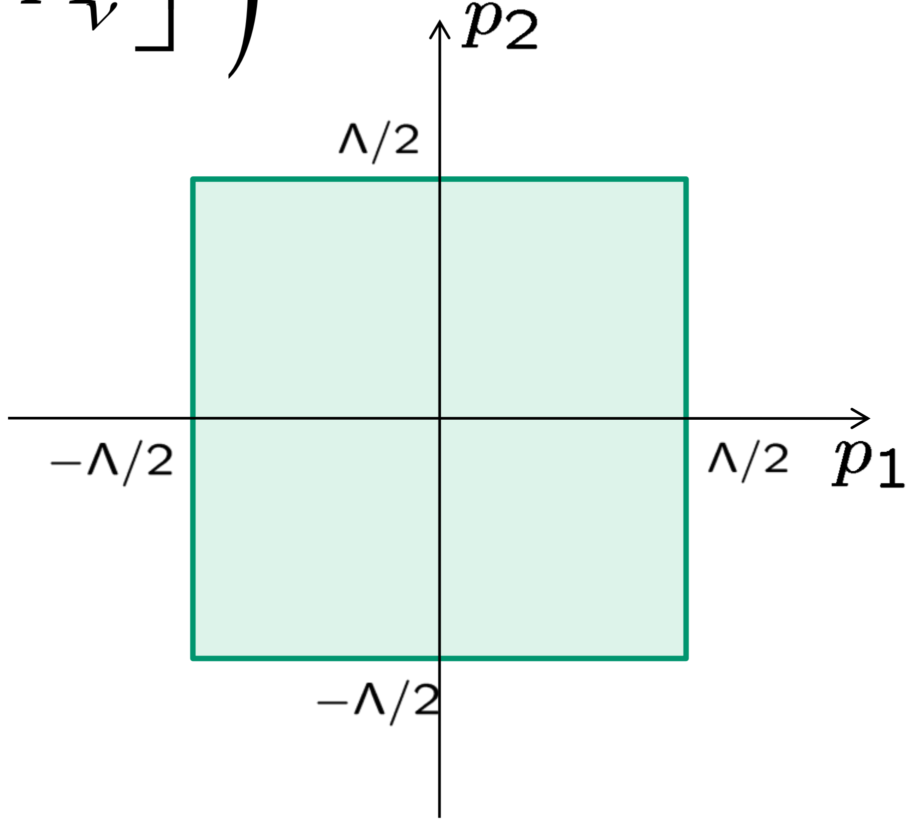
$A_\mu : N \times N$

Hermitian

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[A_\mu, A_\nu\right]^2\right)$$

is equivalent to the d-dimensional Yang-Mills

***if* the eigenvalues of A_μ are uniformly distributed.**



⇔ center invariance

However, it is not automatically realized. It is known that the eigenvalues collapse to one point unless we do something.

2. How is the center invariance broken spontaneously?

To be concrete we consider the continuum version.

In order to examine the center invariance, let's consider the one-loop effective action for the diagonal elements of A_μ obtained after integrating out the off-diagonal elements:

$$A_\mu = \begin{pmatrix} p_\mu^{(1)} & & * \\ & \ddots & \\ * & & p_\mu^{(N)} \end{pmatrix}$$

The quadratic part of the action

$$S = -Tr\left(\left[A_\mu, A_\nu\right]^2\right).$$

$A_\mu = P_\mu + A_\mu$

$$P_\mu = \begin{pmatrix} p_\mu^{(1)} & & 0 \\ & \ddots & \\ 0 & & p_\mu^{(N)} \end{pmatrix}$$

A_μ : off diagonal

is given by

$$S_2 = Tr\left(\left[P_\mu, \tilde{A}_\nu\right]^2 + \left[P_\mu, b\right]\left[P_\mu, c\right]\right)$$

$$= \sum_{i<j}\left(p_\mu^{(i)} - p_\mu^{(j)}\right)^2 \left|\left(\tilde{A}_\nu\right)_{i,j}\right|^2$$

$$+ \sum_{i<j}\left(p_\mu^{(i)} - p_\mu^{(j)}\right)^2 \left(\left(b\right)_{i,j}\left(c\right)_{j,i} + \left(b\right)^*_{i,j}\left(c\right)^*_{j,i}\right),$$

and the one-loop effective action becomes

$$S_{\mathrm{eff}}^{(1\text{-loop})} = N^2(d-2)\sum_{i<j}\log\left(\left(p_\mu^{(i)} - p_\mu^{(j)}\right)^2\right).$$

$S_{\text{eff}}^{(1\text{-loop})}$ **is of order N^2 for N variables. It is minimized in the large-N limit.**

If $d > 2$, the eigenvalues of A_μ are attractive, and collapse to a point.

This indicates the spontaneous breaking of the translational invariance of the eigenvalues: $A_\mu \to A_\mu + c$

$$U_\mu = \exp\left(iaA_\mu\right)$$

This is the continuum version of the center invariance: $U_\mu \to e^{i\theta_\mu} U_\mu$

3. How can we recover the center invariance?

In order for space-time to emerge from matrices, the center invariance should be recovered.

Actually there are several ways to recover the center invariance.

Strong coupling

If the coupling is sufficiently strong, quantum fluctuation might overwhelm the attractive force.
It actually happens at least for the lattice version of the reduced model.

$$S_{\text{reduced}} = -\frac{N}{\lambda}\sum_{\mu\neq\nu=1}^{d} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right),\ \lambda > \lambda_c \sim 4.$$

quenching

We constrain the diagonal elements of A_μ to a uniform distribution by hand

$$(A_\mu)_{ii} = p_\mu^{(i)} \ .$$

Then the perturbation series reproduce that of the *d*-dimensional gauge theory. However, this is rather formal, and the gauge invariance is no longer manifest.

A lattice version of quenching that keeps manifest gauge invariance was proposed, but now it is known that it does not work.

Bhanot–Heller–Neuberger,
Gross–Kitazawa

twisting

Gonzalez-Arroyo, Korthals Altes ('83)

If we expand A_μ around the non-commutative back ground

$$A_\mu^{(0)} = \hat{p}_\mu$$

$$\left[\hat{p}_\mu, \hat{p}_\nu \right] = i\, B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

the theory is equivalent to gauge theory in a non-commutative space-time.

Because the equation of motion of the reduced model is given by

$$\left[A_\mu,\left[A_\mu,A_\nu\right]\right]=0,$$

the non-commutative back ground is a classical solution.

But it is not the absolute minimum of the action.

One way to make it stable is to modify the model to

$$\tilde{S}=-\left(\frac{2\pi}{\Lambda}\right)^d\frac{N}{4\lambda}Tr\left(\left(\left[A_\mu,A_\nu\right]-i\,B_{\mu\nu}\right)^2\right).$$

The lattice version of this is called the twisted reduced model: Gonzalez-Arroyo, Okawa

$$S_{\text{reduced}} = -\frac{N}{\lambda} \sum_{\mu \neq \nu = 1}^{d} e^{i\theta_{\mu\nu}} Tr\left(U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}\right).$$

Several MC analyses have been made on the twisted reduced model, and they found some discrepancy from the infinite volume theory, which is related to the UV-IR mixing.

Heavy adjoint fermions

Kovtun-Unsal-Yaffe (2007), Bringoltz-Sharpe (2009), Poppitz, Myers, Ogilvie, Cossu, D'Elia, Hollowood, Hietanen, Narayanan, **Azeyanagi, Hanada, Yacobi**

They have introduced additional heavy adjoint fermions.

Then the collapse of the eigenvalues can be avoided without changing the long distance physics.

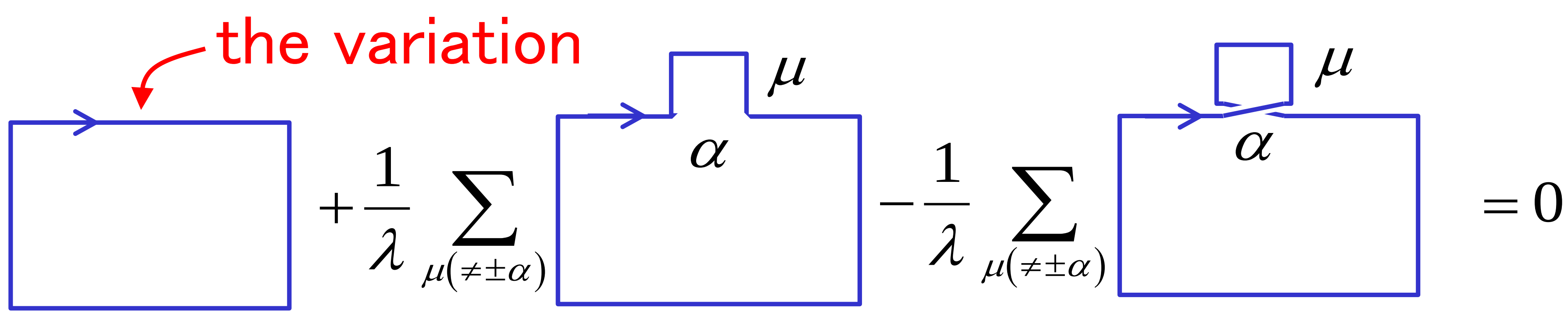
4. Proof of Large-N reduction

There are several proofs, each of which shows an aspect of the large-N reduction.

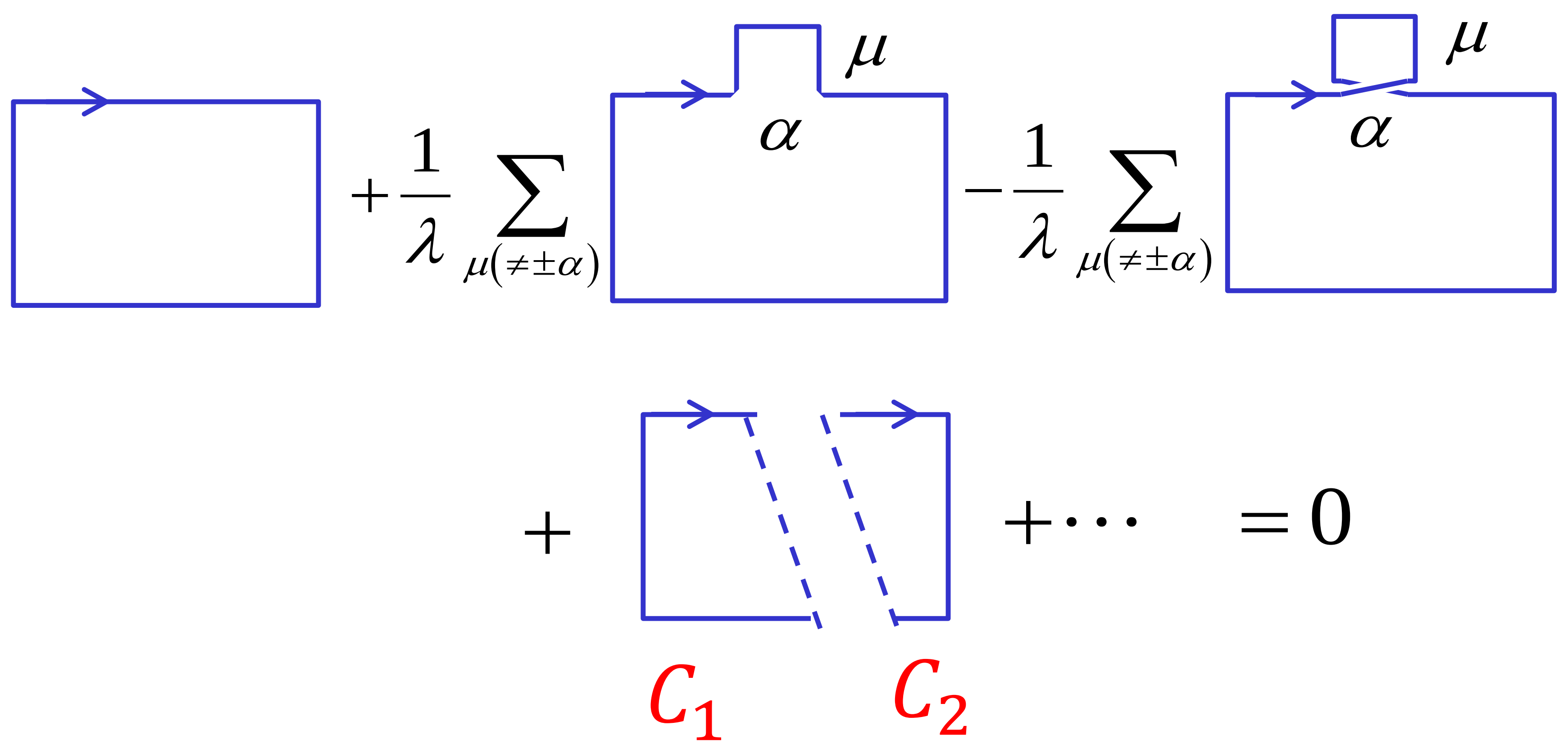
Loop equations

Loop equations are nothing but SD equations obtained from the variation of a link variable on a Wilson loop.

For example, for a non self intersecting loop it looks like



However for the corresponding folded loop in a periodic box we have additional terms:



Here each of C_1 and C_2 is closed in the periodic box but not in the infinite space.

In the large N limit, traced operators are factorized in general, and we have

$$\langle w(C_1) w(C_2) \rangle = \langle w(C_1) \rangle \langle w(C_2) \rangle \, .$$

The crucial point is that

C_1 (or C_2) contains different numbers of $U_{n,\mu}$ and $U^{\dagger}_{n,\mu}$ at least for one direction μ, because it is not closed in the infinite space.

Therefore if the center invariance

$$U_{n,\mu} \to e^{i\theta_\mu} U_{n,\mu}$$

is not broken spontaneously, $\langle w(C_1) \rangle$ is zero, and the additional terms disappear.

Strong coupling expansion

"The strong-coupling expansion series of the Wilson action and the reduced model agree in the large-N limit."

The essence is captured by the Weingarten model that is obtained from the Wilson action by replacing the unitary measure with the Gaussian measure:

$$S_{\text{Weingarten}}$$

$$= -\frac{N}{\lambda}\sum_{n,\mu\neq\nu} Tr\left(V_{n,\mu}V_{n+\hat{\mu},\nu}V^{\dagger}_{n+\hat{\nu},\mu}V^{\dagger}_{n,\nu}\right) + N\sum_{n,\mu} Tr\left(V_{n,\mu}V^{\dagger}_{n,\mu}\right).$$

$$V_{n,\mu} : N\times N \text{ complex matrix}$$

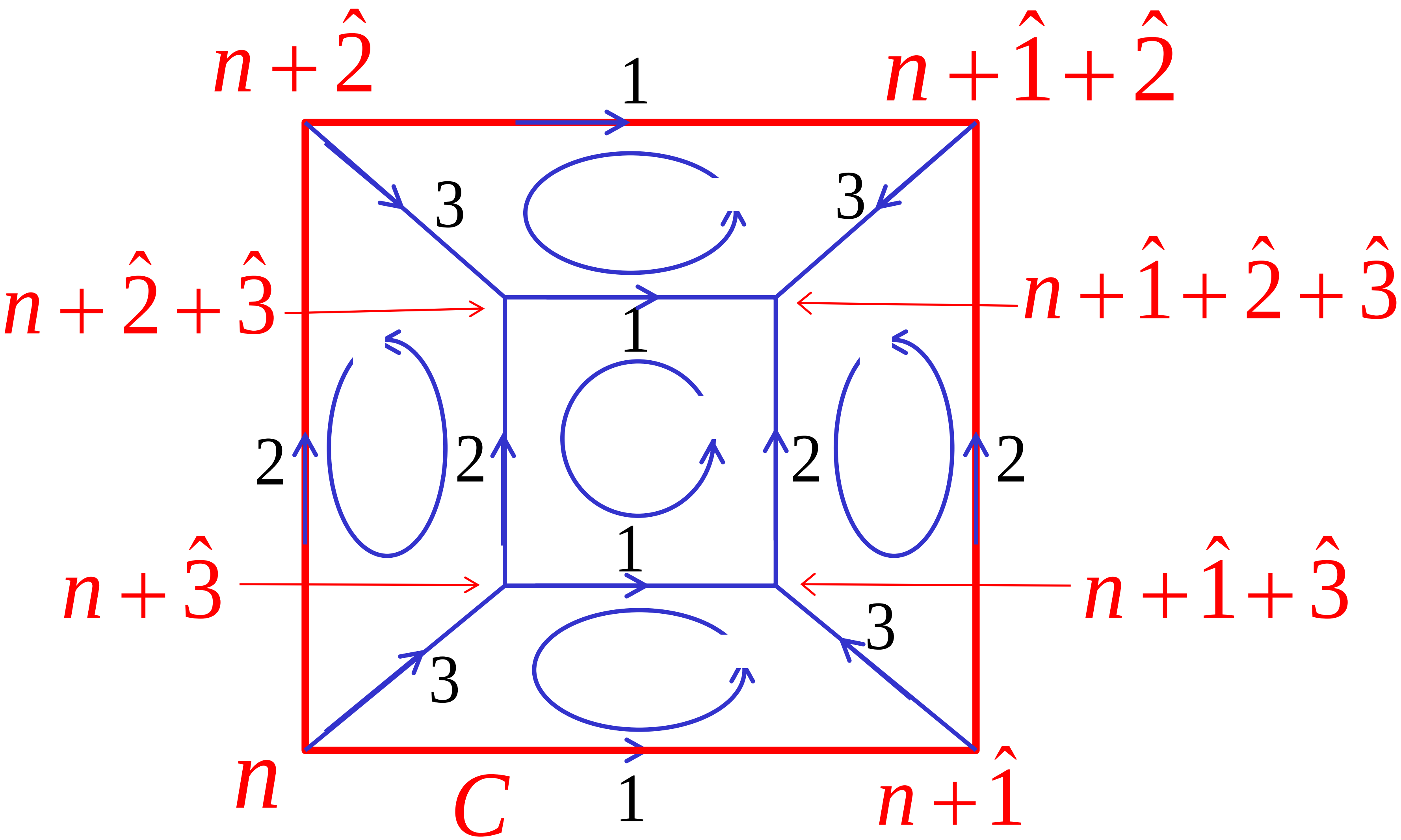
Wilson loop is simply defined by replacing $\boldsymbol{U_{n,\mu}}$ with $\boldsymbol{V_{n,\mu}}$:

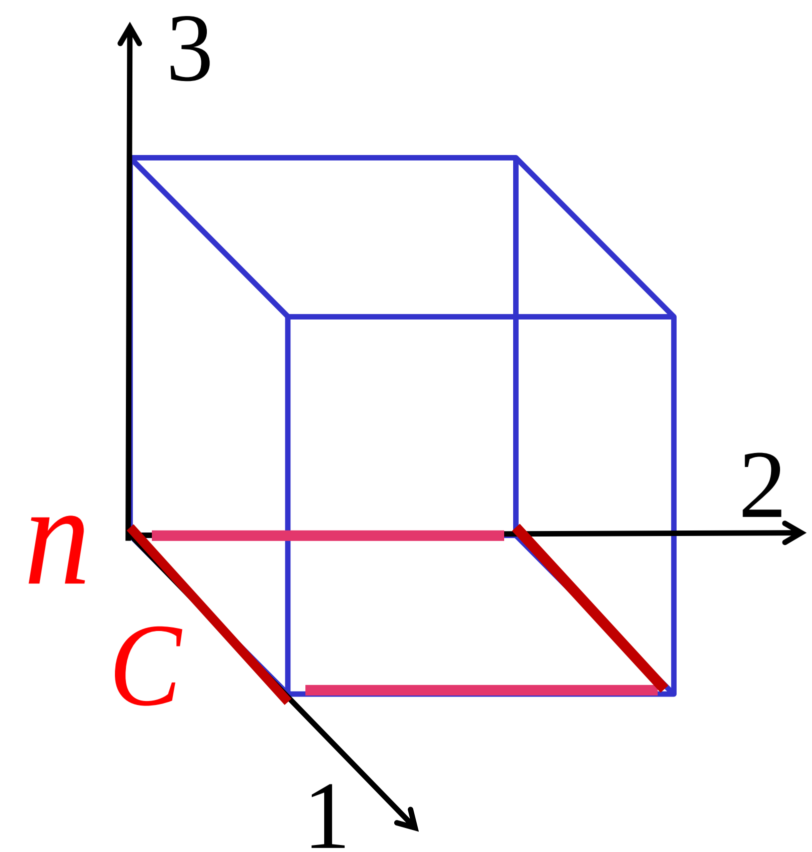
$$W(C) = \frac{\int \prod_{n,\mu} dV_{n,\mu} \exp\left(-S_{\text{Weingarten}}\right) w(C)}{\int \prod_{n,\mu} dV_{n,\mu} \exp\left(-S_{\text{Weingarten}}\right)}$$

$$w(C) = \frac{1}{N} Tr\left(V_{n,\alpha} V_{n+\hat{\alpha},\beta} \cdots V_{n-\hat{\omega},\omega}\right),$$

where $V_{n,-\mu} = V^{\dagger}_{n-\hat{\mu},\mu}$.

Then the Feynman diagrams for a Wilson loop look like





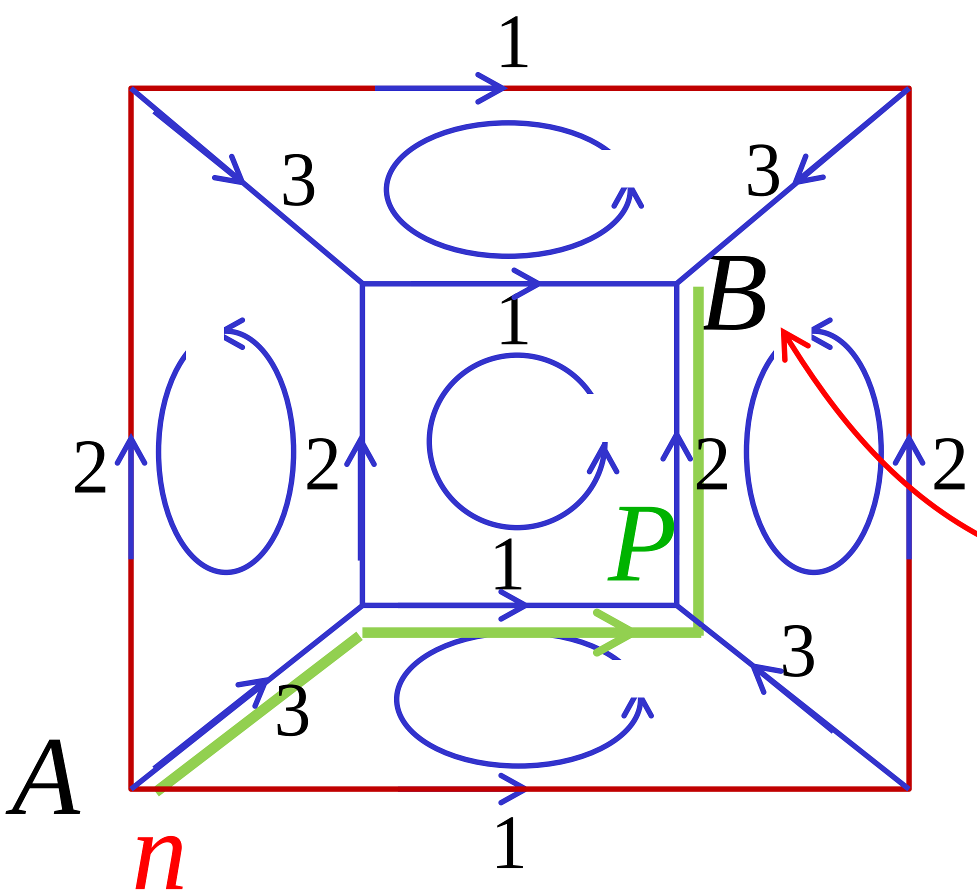
Each face corresponds to the V^4 interaction.

Each side corresponds the propagator.

The crucial point is that we do not need the precise information of the sites if the graph is planar:

Suppose a vertex *A* corresponds to the site *n* .

For any vertex *B* find a path P from *A* to *B*,



The site corresponding to *B* is obtained by summing up the displacement vectors along *P*:

$$n + \hat{3} + \hat{1} + \hat{2}.$$

This does not depend on the choice of P,

The situation is analogous to the existence of a potential for a rotation free vector field.

This means that Weingarten model and the reduced Weingarten model give the same values of Wilson loop.

A similar analysis can be applied to the Wilson action by using the standard source formula:

$$W(C)=\frac{\exp\left(-S_{\text{Wilson}}\left(\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\right)w\left(C,\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\exp\left(\sum_{n,\mu}f\left(J_{n,\mu}\right)\right)\Bigg|_{\{J=0\}}}{\exp\left(-S_{\text{Wilson}}\left(\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\right)\exp\left(\sum_{n,\mu}f\left(J_{n,\mu}\right)\right)\Bigg|_{\{J=0\}}},$$

$$\exp\left(f\left(J\right)\right)=\int dU\exp\left(i\,N\left(Tr\left(JU\right)+Tr(J^{\dagger}U^{\dagger}\right)\right).$$

We can show that
the strong-coupling expansion series of the Wilson action and the reduced model agree in the large-N limit.

Perturbative expansion around diagonal background

As the simplest example we start with the large-N ϕ^3 theory in the continuum space,

$$S=\int d^d x\, Tr\left(\frac{1}{2}\left(\partial_\mu \phi\right)^2+\frac{1}{2}m^2\phi^2+\frac{1}{3}\kappa\phi^3\right),$$

$\phi : N\times N$ hermitian matrix,

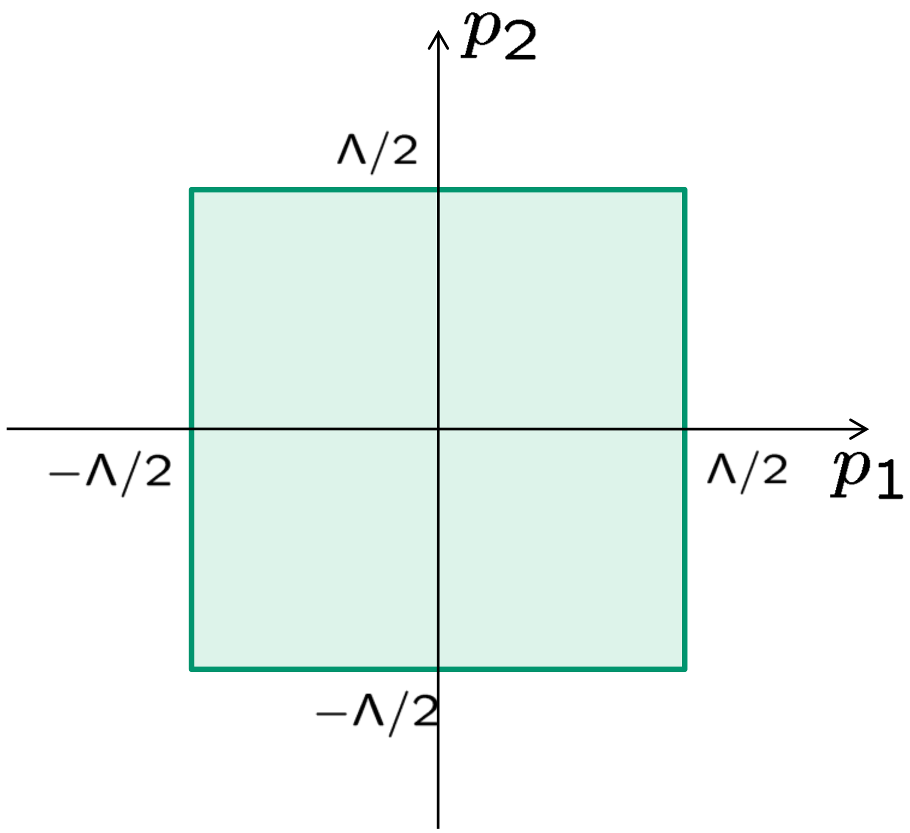
and the expectation values of single trace operators such as

$$O_n\left(x_1, x_2, \cdots, x_n\right)=Tr\left(\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right).$$

Parisi's reduced model

(1) Let $\hat{P}_\mu \ (\mu = 1, \cdots, d)$ be $N \times N$ diagonal matrices whose elements distribute uniformly in the d- dimensional space, which we regard as the momentum space.

$$\hat{P}_\mu = \begin{pmatrix} p_\mu^{(1)} & & & \\ & p_\mu^{(2)} & & \\ & & \ddots & \\ & & & p_\mu^{(N)} \end{pmatrix}$$



(2) Corresponding to the field $\phi(x)$,
introduce a $N \times N$ Hermitian matrix $\tilde{\phi}$,
and construct the corresponding action and operators by substituting

$$\phi(x) \to \tilde{\phi}(x) = \exp\left(i\,\hat{P}_\mu x^\mu\right)\tilde{\phi}\,\exp\left(-i\,\hat{P}_\mu x^\mu\right),$$

to the original expression.

For the action the space-time integral of 1 should be replaced with

$$\int d^d x 1 \to \left(\frac{2\pi}{\Lambda}\right)^d .$$

Volume of the unit cell

Λ is the cut off that appears in $\hat{P}_\mu$.

Then we have

$$\partial_\mu \phi(x)$$

$$\rightarrow \partial_\mu \tilde{\phi}(x) = \partial_\mu \left(\exp\left(i\,\hat{P}_\mu x^\mu\right) \tilde{\phi} \exp\left(-i\,\hat{P}_\mu x^\mu\right) \right)$$

$$= \exp\left(i\,\hat{P}_\mu x^\mu\right) \left[i\hat{P}_\mu, \tilde{\phi} \right] \exp\left(-i\,\hat{P}_\mu x^\mu\right),$$

$$S = \int d^d x \; Tr\left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} \kappa\, \phi^3 \right)$$

$$\rightarrow \tilde{S} = \int d^d x \boxed{Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi} \right]^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{3} \kappa \tilde{\phi}^3 \right)}$$

← **no *x* dependence**

$$\rightarrow \left(\frac{2\pi}{\Lambda} \right)^d Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi} \right]^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{3} \kappa \tilde{\phi}^3 \right),$$

Thus we obtain

action

$$\tilde{S} = \left(\frac{2\pi}{\Lambda}\right)^d Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi}\right]^2 + \frac{1}{2}m^2\tilde{\phi}^2 + \frac{1}{3}\kappa\tilde{\phi}^3\right),$$

operators

$$\tilde{O}_n\left(x_1, x_2, \cdots, x_n\right) = Tr\left(\tilde{\phi}\left(x_1\right)\tilde{\phi}\left(x_2\right)\cdots\tilde{\phi}\left(x_n\right)\right).$$

"In the large-N limit the expectation value

$$\left\langle \tilde{O} \right\rangle = \frac{\int d\tilde{\phi}\,\tilde{O}\exp\left(-\tilde{S}\right)}{\int d\tilde{\phi}\exp\left(-\tilde{S}\right)}$$

agrees with the original field theory."

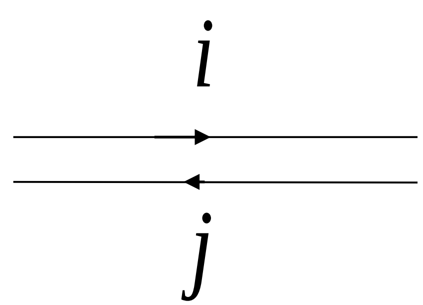
Proof of Parisi's rule

For simplicity, we consider the free energy. The generalization to the expectation values of the operators are straightforward.

From the quadratic part

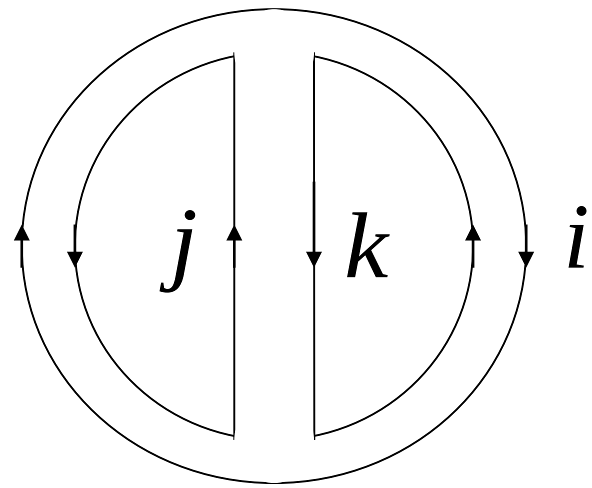
$$Tr\left(\left[\hat{P}_\mu,\tilde{\phi}\right]^2\right)=\sum_{i,j}\left(\left(p_\mu{}^{(i)}-p_\mu{}^{(j)}\right)^2+m^2\right)\left|\left(\tilde{\phi}\right)_{i,j}\right|^2,$$

the propagator for $\left(\tilde{\phi}\right)_{i,j}$ is given by



$$\left(\frac{\Lambda}{2\pi}\right)^d\frac{1}{\left(p^{(i)}-p^{(j)}\right)^2+m^2}.$$

Feynman diagrams are something like



$$\lambda^2\left(\frac{\Lambda}{2\pi}\right)^d \sum_i\sum_j\sum_k \frac{1}{p^{(i)}-p^{(j)}\,{}^{2}+m^2 \quad p^{(j)}-p^{(k)}\,{}^{2}+m^2 \quad p^{(k)}-p^{(i)}\,{}^{2}+m^2}.$$

In the large-N limit we can replace $\sum_i \to \frac{N}{\Lambda^d}\int d^d p$

$$=\lambda^2\left(\frac{\Lambda}{2\pi}\right)^d\left(\frac{N}{\Lambda^d}\right)^3\int\int\int d^D p\, d^D p' d^D p'' \frac{1}{p-p'\,{}^{2}+m^2 \quad p'-p''\,{}^{2}+m^2 \quad p''-p\,{}^{2}+m^2}$$

$$=\left(\frac{2\pi}{\Lambda}\right)^d \lambda^2 N^3 \int\int \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d}\frac{1}{k_1{}^2+m^2 \quad k_2{}^2+m^2 \quad (k_1+k_2)^2+m^2}$$

$\uparrow$

free energy per unit cell

end

We apply Parisi's rule to the continuum gauge theory.

$$A_\mu(x) \to \exp\left(i\,\hat{P}_\mu x^\mu\right)\tilde{A}_\mu \exp\left(-i\,\hat{P}_\mu x^\mu\right)$$

$$\partial_\mu A_\nu(x) \to \exp\left(i\,\hat{P}_\mu x^\mu\right)\left[i\,\hat{P}_\mu, \tilde{A}_\nu\right]\exp\left(-i\,\hat{P}_\mu x^\mu\right)$$

$$\begin{aligned}
-iF_{\mu\nu}(x) &= \left[-i\partial_\mu + A_\mu(x), -i\partial_\nu + A_\nu(x)\right] \\
&= -i\partial_\mu A_\nu(x) + i\partial_\nu A_\mu(x) + \left[A_\mu(x), A_\nu(x)\right] \\
&\to \exp\left(i\,\hat{P}_\mu x^\mu\right)\left(\left[\hat{P}_\mu, \tilde{A}_\nu\right] + \left[\tilde{A}_\mu, \hat{P}_\nu\right] + \left[\tilde{A}_\mu, \tilde{A}_\nu\right]\right)\exp\left(-i\,\hat{P}_\mu x^\mu\right) \\
&= \exp\left(i\,\hat{P}_\mu x^\mu\right)\left[\hat{P}_\mu + \tilde{A}_\mu, \hat{P}_\nu + \tilde{A}_\nu\right]\exp\left(-i\,\hat{P}_\mu x^\mu\right)
\end{aligned}$$

$$S=\int d^d x Tr\left(\frac{N}{4\lambda}F_{\mu\nu}(x)^2\right)$$

$$\rightarrow \tilde{S}=-\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda}Tr\left(\left[\hat{P}_\mu+\tilde{A}_\mu,\hat{P}_\nu+\tilde{A}_\nu\right]^2\right)$$

If we define $A_\mu=\hat{P}_\mu+\tilde{A}_\mu,$

the action becomes

$$\tilde{S}=-\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda}Tr\left(\left[A_\mu,A_\nu\right]^2\right),$$

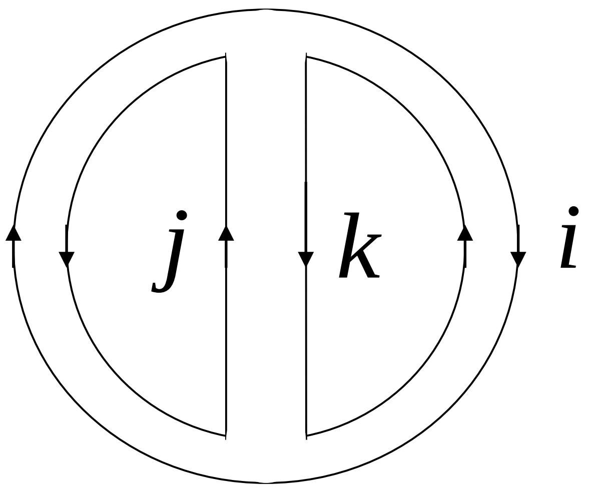
and $\hat{P}_\mu$ **'s disappear from the theory.**

One might conclude that this theory is equivalent to the gauge theory in d-dimensions.
But it is too naïve.

Actually, in the proof of Parisi's rule, we have assumed that the diagonal elements are negligible, because we have only N such variables while the action is of order N^2.

But it is not necessarily true in massless theory.

In that case, the propagators for diagonal elements become infinite.



$$\sum_i \sum_j \sum_k \frac{1}{\left[\left(p^{(i)}-p^{(j)}\right)^2+m^2\right]\left[\left(p^{(j)}-p^{(k)}\right)^2+m^2\right]\left[\left(p^{(k)}-p^{(i)}\right)^2+m^2\right]}.$$

We have to be careful, when we apply Parisi's rule to a massless theory such as gauge theory.

This is why we have to worry about the center invariance.

Expansion around non-commutative background

As I said for the twisted reduced model,

if we expand A_μ in the matrix model action

$$S = -\frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right)$$

around a non-commutative back ground

$$A_\mu = A_\mu^{(0)} + \hat{a}_\mu .$$

$$A_\mu^{(0)} = \hat{p}_\mu$$

$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = i B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

***S* becomes the action of gauge theory in a non-commutative space-time.**

This can be shown in the next two slides:

First we introduce a mapping between operators and functions

$$\hat{o} = \int \frac{d^d k}{(2\pi)^d} \tilde{o}(k) \exp(i k_\mu \hat{x}^\mu) \leftrightarrow o(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{o}(k) \exp(i k_\mu x^\mu)$$

$$\hat{x}^\mu = C^{\mu\nu} \hat{p}_\nu, \;\; B_{\mu\nu} C^{\nu\lambda} = \delta_\mu^\lambda .$$

Then we have the following correspondence

$$\left[\hat{p}_\mu, \hat{o}\right] \leftrightarrow i\partial_\mu o,$$

$$\hat{o}_1 \hat{o}_2 \qquad \leftrightarrow o_1 * o_2,$$

and identity

$$Tr(\hat{o}) = \frac{\sqrt{\det B}}{(2\pi)^{d/2}} \int d^d x \, o(x).$$

The crucial point is that the commutator $\left[A_\mu, A_\nu\right]$ is mapped to the field strength in the noncommutative space-time:

$$
\begin{aligned}
\left[A_\mu, A_\nu\right] &= \left[\hat{p}_\mu + \hat{a}_\mu, \hat{p}_\nu + \hat{a}_\nu\right] \\
&= \left[\hat{p}_\mu, \hat{p}_\nu\right] + \left[\hat{p}_\mu, \hat{a}_\nu\right] + \left[\hat{a}_\mu, \hat{p}_\nu\right] + \left[\hat{a}_\mu, \hat{a}_\nu\right] \\
&\leftrightarrow \quad iB_{\mu\nu} + i\partial_\mu a_\nu - i\partial_\nu a_\nu + a_\mu * a_\nu - a_\nu * a_\mu \\
&\qquad = iB_{\mu\nu} + i\left(F_{\mu\nu}\right)_* .
\end{aligned}
$$

Then the matrix model action becomes a field theory on the non-commutative space-time:

$$
S = \frac{\sqrt{\det \mathrm{B}}}{(2\pi)^{d/2}} \int d^d x \left(-\frac{1}{4} F_{\mu\nu}{}^2 \right)_*
$$

Naively, if UV-IR mixing is not there, this theory is equivalent to the large-N gauge theory in the low energy region. But it is not true, and the matrix theory is *not* completely equivalent to the ordinary field theory.

UV-IR mixing and SSB of the center invariance are related.

Actually the twisted reduced model

$$S_{\text{reduced}} = -\frac{N}{\lambda}\sum_{\mu\neq\nu=1}^{d} e^{i\theta_{\mu\nu}} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right)$$

is equivalent to the original reduced model in the strong coupling region, because the phases cancel out in the strong coupling expansion.

$\Rightarrow$ no UV-IR mixing in the strong coupling region

5. Large-N reduction and string theory

Large-N reduced model looks like world sheet string theory.

The basic idea

Nambu ('77)

"Worldsheet of string has a structure of phase space."

This situation becomes manifest when we express the string in terms of the Schild action.

In fact, in the Schild action, the worldsheet can be regarded as a symplectic manifold, and the action is given by the integration of a quantity that is expressed in terms of the Poisson bracket.

For simplicity, we start with bosonic string.

Schild action

Bosonic string is described by the Nambu-Goto action

$$S_{NG} = -\rho \int d^2\xi \sqrt{-\frac{1}{2}\Sigma^2}, \quad \Sigma^{\mu\nu} = \varepsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}.$$

It is nothing but the area of the worldsheet, which is expressed in terms of an anti-symmetric tensor $\Sigma^{\mu\nu}$ that is constructed from the space-time coordinate X^{μ}.

It is known that the Nambu-Goto action is equivalent to the Schild action

$$S_{\text{Schild}} = \frac{\alpha}{2\pi}\int d^2\xi\sqrt{g}\,\frac{1}{4}\left\{X^\mu, X^\nu\right\}^2 - \frac{\beta}{2\pi}\int d^2\xi\sqrt{g},$$

$\sqrt{g}$: a volume density on the world sheet

$$\left\{X, Y\right\} = \frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X\,\partial_b Y,$$

which is nothing but the Poisson bracket if we regard the worldsheet as a phase space.

The equivalence can be seen easily, by eliminating $\sqrt{g}$ from the Scild action:

$$\frac{\delta}{\delta\sqrt{g}}S_{\text{Schild}} = 0 \Rightarrow \sqrt{g} = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{-\left(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu\right)^2}$$

$$\Rightarrow S_{\text{Schild}} = -\frac{\sqrt{\alpha\beta}}{2\pi}\int d^2\xi\sqrt{-\frac{1}{2}\left(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu\right)^2}.$$

Symplectic structure of the worldsheet

The crucial point is that the Schild action has a structure of phase space.

In fact it is given by the integration over the phase space

$$\int d^2\xi\sqrt{g}$$

of a quantity that is expressed in terme the Poisson bracket

$$\frac{\alpha}{2\pi}\frac{1}{4}\left\{X^{\mu},X^{\nu}\right\}^2-\frac{\beta}{2\pi}.$$

Note that we do not need Worldsheet metric, but what we need is just the volume density $\sqrt{g}$.

Matrix regularization

Then we want to discretize the worldsheet in order to define the path integral.

A natural discretization of phase space is the "quantization".

If we quantize a phase space, it becomes the state-vector space, and we have the following correspondence:

$$\text{function} \quad \rightarrow \quad \text{matrix}$$

$$\{A, B\} \quad \rightarrow \quad \frac{1}{i}[A, B]$$

$$\frac{1}{2\pi}\int d^2\xi\sqrt{g}A \quad \rightarrow \quad TrA$$

$$W_\infty\text{-symmetry} \quad \rightarrow \quad U(N)\text{-symmetry}$$

Then the Schild action becomes

$$S_{\text{Matrix}} = -\alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \beta Tr(1),$$

and the path integral is regularized like

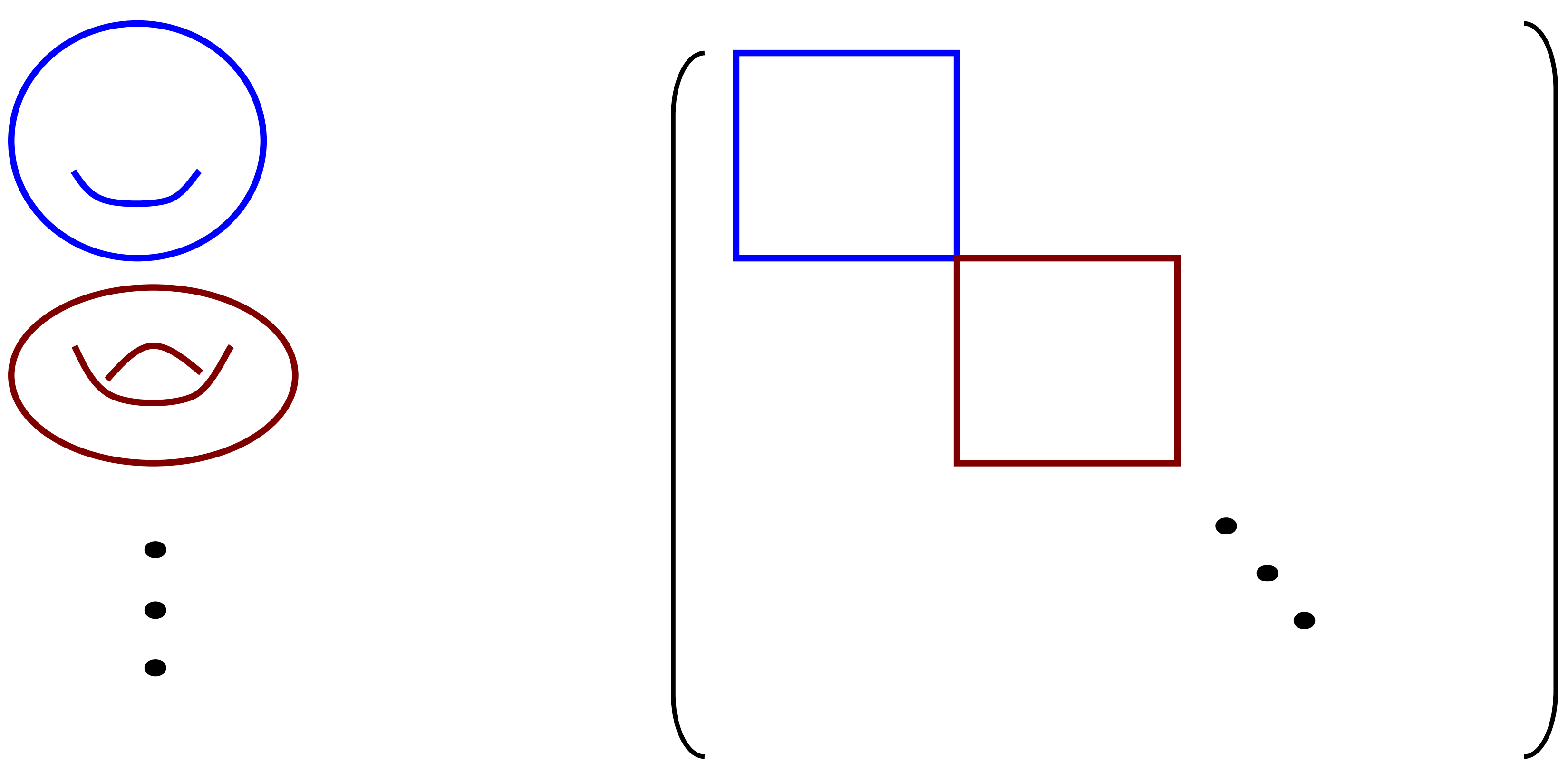
$$Z = \int \frac{[dg\, dX]}{\text{vol}(\text{Diff})} \exp\left(iS_{\text{Schild}}\right) \rightarrow \sum_{n=1}^{\infty} \int \frac{dA}{SU(n)} \exp\left(iS_{\text{Matrix}}\right).$$

Here we have used the fact that the phase space volume $\int d^2\xi \sqrt{g}$ is diff. invariant and becomes the matrix size $Tr(1) = n$ after the regularization.

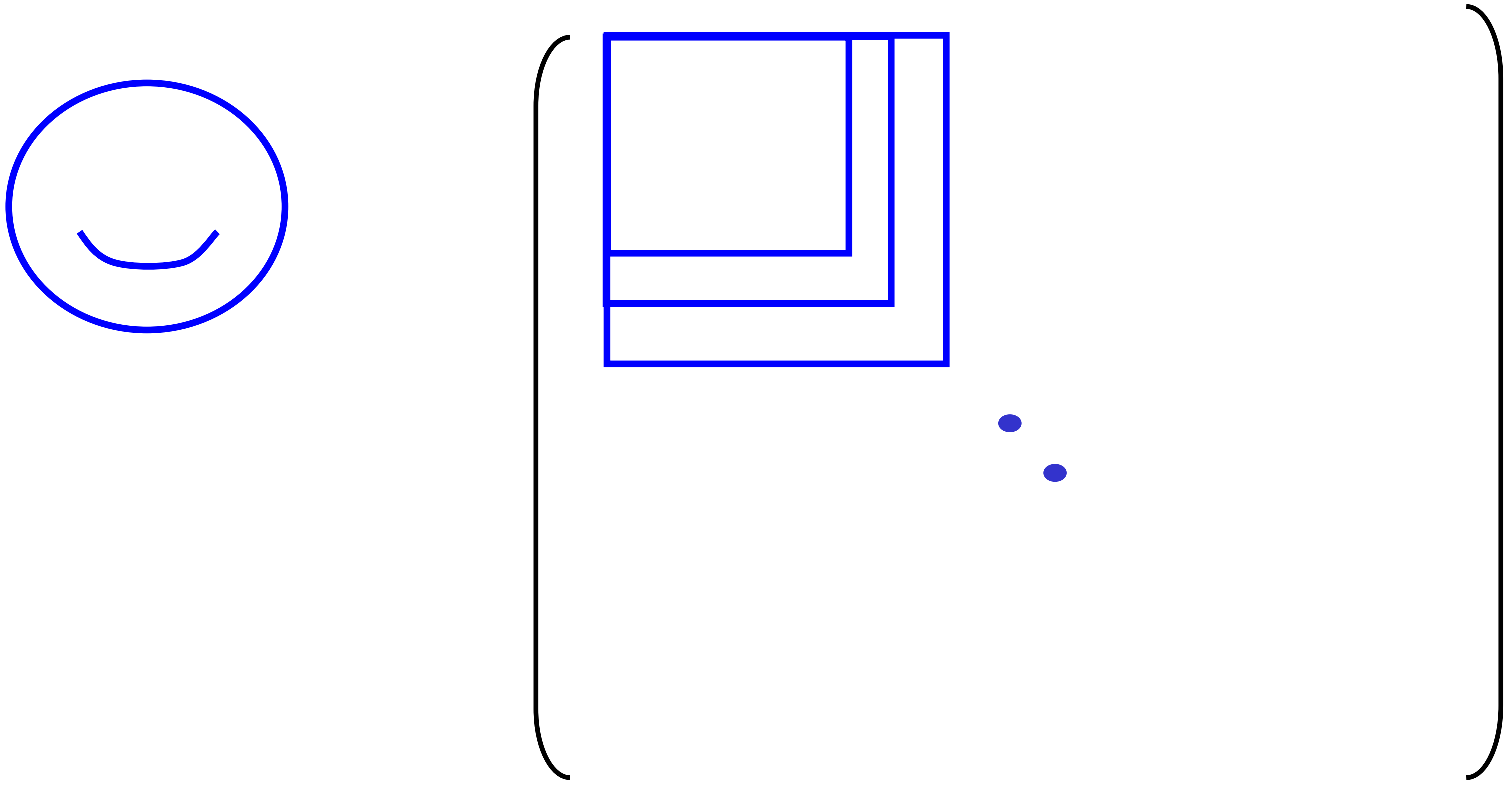
Therefor the path integral over g becomes summation over n .

Multi-string states

One good point of the matrix regularization is that all topologies of the worldsheet are automatically included in the matrix integral. Disconnected worldsheets are also included as block diagonal configurations as



Furthermore the sum over the size of the matrix is automatically included, if the worldsheet is imbedded in a larger matrix as a submatrix.



If we take this picture that all the worldsheets emerge as submatrices of a large matrix, the second term of

$$S_{\mathrm{Matrix}} = -\alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \beta\, Tr(1)$$

can be regarded as describing the chemical potential for the block size.

Thus we expect that the whole universe is described by a large matrix that obeys

$$S = -\alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right).$$

This is nothing but the large-N reduced model.

We have seen that in this model the eigenvalues collapse to one point, and it can not describe an extended space-time.
This might be related to the instability of bosonic string by tachyons.

On the other hand, if we start from type IIB superstring, we will get the reduced model for supersymmetric gauge theory. In this case eigenvalues do not collapse, and we can have non-trivial space-time.

IIB matrix model

1. Definition of IIB matrix model

Schild action of IIB string

First we constract the Schild action of type IIB superstring.

Green-Schwarz action

$X^{\mu}\ (\mu = 0 \sim 9)$

θ^1, θ^2 : 10D Mayorana-Weyl

$$S_{GS} = -\rho \int d^2\xi \Big(\sqrt{-\frac{1}{2}\Sigma^2} + i\varepsilon^{ab}\partial_a X^{\mu}\left(\bar{\theta}^1\Gamma_{\mu}\partial_b\theta^1 + \bar{\theta}^2\Gamma_{\mu}\partial_b\theta^2\right) + \varepsilon^{ab}\bar{\theta}^1\Gamma^{\mu}\partial_a\theta^1\,\bar{\theta}^2\Gamma_{\mu}\partial_b\theta^2 \Big),$$

$$\Sigma^{\mu\nu} = \varepsilon^{ab}\Pi_a^{\mu}\Pi_b^{\nu},$$

$$\Pi_a^{\mu} = \partial_a X^{\mu} - i\bar{\theta}^1\Gamma^{\mu}\partial_a\theta^1 + i\bar{\theta}^2\Gamma^{\mu}\partial_a\theta^2$$

κ-symmetry

$$\delta_\kappa \theta^1 = \alpha^1$$

$$\delta_\kappa \theta^2 = \alpha^2$$

$$\delta_\kappa X^\mu = i\bar{\theta}^1 \Gamma^\mu \alpha^1 - i\bar{\theta}^2 \Gamma^\mu \alpha^2$$

$$\alpha^1 = \left(1 + \tilde{\Gamma}\right)\kappa^1$$

$$\alpha^2 = \left(1 - \tilde{\Gamma}\right)\kappa^2,$$

$$\tilde{\Gamma} = \frac{1}{2\sqrt{-\frac{1}{2}\Sigma^2}} \Sigma^{\mu\nu}\Gamma_{\mu\nu}$$

N=2 SUSY

$$\delta_{\mathrm{SUSY}} \theta^1 = \varepsilon^1$$

$$\delta_{\mathrm{SUSY}} \theta^2 = \varepsilon^2$$

$$\delta_{\mathrm{SUSY}} X^\mu = i\bar{\varepsilon}^1 \Gamma^\mu \theta^1 - i\bar{\varepsilon}^2 \Gamma^\mu \theta^2$$

Gauge fixing for the κ-symmetry $\theta^1 = \theta^2 = \psi$

$$S_{GS} = -\rho \int d^2\xi \left(\sqrt{-\frac{1}{2}\sigma^2} + 2i\varepsilon^{ab}\partial_a X^\mu \bar{\psi}\Gamma_\mu \partial_b \psi \right),$$

$$\sigma^{\mu\nu} = \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu .$$

N=2 SUSY should be combined with *κ* symmetry so that the gauge condition is maintained

$$\delta\theta^1 = \delta_{\mathrm{SUSY}}\theta^1 + \delta_\kappa \theta^1$$

$$\delta\theta^2 = \delta_{\mathrm{SUSY}}\theta^2 + \delta_\kappa \theta^2$$

$$\delta X^\mu = \delta_{\mathrm{SUSY}} X^\mu + \delta_\kappa X^\mu \qquad \Rightarrow \delta\theta^1 = \delta\theta^2$$

$$\kappa^1 = \frac{-\varepsilon^1 + \varepsilon^2}{2}$$

$$\kappa^2 = \frac{\varepsilon^1 - \varepsilon^2}{2}$$

N=2 SUSY becomes simple if we consider a combination

$$\xi = \frac{\varepsilon^1 + \varepsilon^2}{2}$$

$$\varepsilon = \frac{\varepsilon^1 - \varepsilon^2}{2}.$$

Then we have the following simple form:

$$\delta^{(1)}\psi = -\frac{1}{2\sqrt{-\frac{1}{2}\sigma^2}}\sigma^{\mu\nu}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}X^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}X^\mu = 0.$$

Schild action

Then we convert the action to the Schild action as in the case of bosonic string:

$$S_{\text{Schild}} = \frac{\alpha}{2\pi}\int d^2\xi\sqrt{g}\left(\frac{1}{4}\left\{X^\mu, X^\nu\right\}^2 - \frac{i}{2}\bar{\psi}\Gamma_\mu\left\{X^\mu, \psi\right\}\right) - \frac{\beta}{2\pi}\int d^2\xi\sqrt{g},$$

N=2 SUSY

$$\delta^{(1)}\psi = -\frac{1}{2}\left\{X^\mu, X^\nu\right\}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}X^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}X^\mu = 0$$

Everything is written in terms of Poisson bracket.

Matrix regularization

Applying the matrix regularization, we have

$$S_{\text{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu, \psi\right]\right) - \beta\, Tr(1).$$

N=2 SUSY

$$\delta^{(1)}\psi = -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\varepsilon \quad \leftarrow F_{\mu\nu} = -i\left[A_\mu, A_\nu\right]$$

$$\delta^{(1)}A^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}A^\mu = 0$$

Everything is written in terms of commutator.

IIB matrix model

Ishibashi, HK, Kitazawa, Tsuchiya
Jevicki, Yoneya

Drop the second term, and consider large-N

$$S_{\text{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu, \psi\right]\right).$$

IIB matrix model

Formally, this is the large-N reduced model of 10D super YM theory.
A good point is that the N=2 SUSY is manifest even after the discretization.

N=2 SUSY

$$S_{\text{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu, \psi\right]\right)$$

One of the N=2 SUSY is nothing but the supersymmety of the 10D super YM theory.

$$\delta^{(1)}\psi = -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}A^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

The other one is almost trivial.

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}A^\mu = 0$$

Even so, they form non trivial N=2 SUSY:

$$\left\{Q^{(1)}, Q^{(1)}\right\} = 0,\ \left\{Q^{(2)}, Q^{(2)}\right\} = 0,\ \left\{Q^{(1)}, Q^{(2)}\right\} = P.$$

comment

The other matrix models

IIB matrix model is nothing but the dimensional reduction of the 10D super YM theory to 0D.

It is natural to think about the other possibilities. In fact, they have considered the dimensional reduction to various dimensions:

0D ⇒ IIB matrix model

1D ⇒ Matrix theory { de Witt–Hoppe–Nicolai, Banks–Fischler–Shenker–Susskind ⇒ O' Connor's

2D ⇒ Matrix string Motl, Dijkgraaf–Verlinde–Verlinde

4D ⇒ AdS/CFT

From the viewpoint of the large-N reduction, they are equivalent *if* we quench the diagonal elements of the matrices.

0D: IIB matrix model

↓ quenching diagonal elements of A_0

1D: Matrix theory

↓ quenching diagonal elements of A_1

2D:Matrix string

↓ quenching diagonal elements of A_2 and A_3

4D :AdS/CFT

However the dynamics of the diagonal elements are rather complicated.
At present the relations among them are not well-understood.

comment end

Open questions

We expect that the IIB matrix model

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\gamma^\mu [A^\mu, \Psi]\right)$$

gives a constructive definition of superstring.

However there are some fundamental open questions:

Is an infrared cutoff necessary?

How should the large-N limit be taken?

How does the space-time emerge?

How does the diff. invariance appear?

2. Ambiguities in the definition

- Euclidean or Lorentzian
- Necessity of the IR cutoff
- How to take the (double) scaling limit

Euclidean or Lorentzian?

In general, systems with gravity do not allow a simple Wick rotation, because the kinetic term of the conformal mode (the size of the universe) has wrong sign.

On the other hand, the path integral of the IIB matrix model seems well defined for the Euclidean signature, because the bosonic part of the action is positive definite:

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2\right) + \frac{1}{2} Tr\left(\overline{\Psi}\gamma^\mu [A^\mu, \Psi]\right).$$

How about Lorentzian signature?
If we simply apply the analytic continuation

$$A^0 = -iA^{10},$$

the path integral becomes unbounded:

$$Z = \int dAd\Psi \exp\left(-S_M\right),$$

$$S_M = -\frac{1}{g^2} Tr\left(\frac{1}{4}\left(\eta_{\mu\alpha}\eta_{\nu\beta}[A^\mu, A^\nu][A^\alpha, A^\beta]\right) \right) + \frac{1}{2} Tr\left(\overline{\Psi}\gamma_\mu [A^\mu, \Psi]\right)$$

$$= -\frac{1}{g^2} Tr\left(-\frac{1}{2}[A^0, A^i]^2 + \frac{1}{4}[A^i, A^j]^2 \right) + \cdots.$$

From the point of view of the large-N reduction, it is natural to take

$$Z = \int dAd\Psi \exp\left(i S_M\right).$$

Is IR cutoff necessary?

Because of the supersymmetry the force between two eigenvalues cancels between bosons and fermions

$$A_\mu = \begin{pmatrix} p_\mu{}^{(1)} & & * \\ & \ddots & \\ * & & p_\mu{}^{(N)} \end{pmatrix}$$

$$S^{(1-loop)}{}_{eff} = \left(D-2-d_F\right)\sum_{i,j}\log\left(\left(p^{(i)}-p^{(j)}\right)^2\right) = 0$$

It *seems* that we have to impose an infrared cutoff by hand to prevent the eigenvalues from running away to infinity.

$$-l < \mathrm{eigen}\left(A^\mu\right) < l$$

But there is a subtlety.
Because the diagonal elements of fermions are zero modes of the quadratic part of the action, we should keep them when we consider the effective Lagrangian.
The one-loop effective Lagrangian for the diagonal elements is given by

Aoki, Iso, Kitazawa, Tada, HK

$$A_\mu = \begin{pmatrix} p_\mu{}^{(1)} & & * \\ & \ddots & \\ * & & p_\mu{}^{(N)} \end{pmatrix} \qquad \psi = \begin{pmatrix} \xi^{(1)} & & * \\ & \ddots & \\ * & & \xi^{(N)} \end{pmatrix}$$

$$S_{\text{eff}}{}^{\text{1-loop}}(x,\xi) = \sum_{i<j} tr\left(\frac{S_{(i,j)}{}^4}{4} + \frac{S_{(i,j)}{}^8}{8} \right),$$

$$\left(S_{(i,j)}\right)_{\mu,\nu} = \left(\bar{\xi}^{(i)} - \bar{\xi}^{(j)}\right)\Gamma^{\mu\alpha\nu}\left(\xi^{(i)} - \xi^{(j)}\right)\frac{p_\alpha{}^{(i)} - p_\alpha{}^{(j)}}{\left(\left(p^{(i)} - p^{(j)}\right)^2\right)^2}.$$

Because of the fermionic degrees of freedom, there appears a weak attractive force between the eigenvalues, and at least the partition function becomes finite.
However it is not clear whether all the correlation functions are finite or not.

Austing and Wheater,
Krauth, Nicolai and Staudacher,
Suyama and Tsuchiya,
Ambjorn, Anagnostopoulos, Bietenholz, Hotta and Nishimura,
Bialas, Burda, Petersson and Tabaczek,
Green and Gutperle,
Moore, Nekrasov and Shatashvili.

Let's estimate the order of this interaction.

We first integrate out the fermionic variables

$$Z^{(1\text{–loop})}(p) = \int \prod_i d^{16}\xi^{(i)} \exp\left(-S_{\text{eff}}^{(1\text{-loop})}(p,\xi)\right)$$

Since $S_{(i,j)}$ **is quadratic in** $\xi^{(i)} - \xi^{(j)}$ **,**

which has only 16 components, we have

$$S_{(i,j)}{}^{n} = 0, \quad (n > 8)$$

and

$$\exp\left(-S_{\text{eff}}^{(1\text{-loop})}(p,\xi)\right) = \exp\left(-\sum_{i<j} tr\left(\frac{S_{(i,j)}{}^{4}}{4} + \frac{S_{(i,j)}{}^{8}}{8}\right)\right)$$

$$= \prod_{i<j}\left(1 + a\ \mathrm{tr}\left(S_{(i,j)}{}^{4}\right) + b\ \mathrm{tr}\left(S_{(i,j)}{}^{8}\right)\right).$$

Therefore, for each pair of i and j we have 3 choices

$$1, \quad a\,\mathrm{tr}\left(S_{(i,j)}{}^{4}\right), \quad b\,\mathrm{tr}\left(S_{(i,j)}{}^{8}\right)$$

which carry the powers of ξ

0, 8, 16 respectively.

On the other hand, we have $16N$ dimensional fermionic integral $\int \prod_{i=1}^{N} d^{16}\xi^{(i)}$.

Therefore the number of factors other than 1 should be less than or equal to $2N$, and we can conclude that the effective action induced from the fermionic zero modes is of $O(N)$:

$$Z^{(1\text{-loop})}(p) = \sum_{\text{various terms}} \underbrace{f\left(p_\alpha{}^{(i)} - p_\alpha{}^{(j)}\right) f\left(p_\alpha{}^{(i')} - p_\alpha{}^{(j')}\right)\cdots}_{\leq 2N}$$

$$\sim \exp\left(O\left(N\right)\right).$$

This should be compared to the bosonic case

$$Z^{(1-\text{loop})}(p) = \exp\left(-(D-2)\sum_{i<j} \log\left(p^{(i)} - p^{(j)} \right)^2 \right)$$

$$\sim \exp\left(O(N^2) \right)$$

SUSY reduces the attractive force by at least a factor 1/N.
So, in the naïve large-N limit, simultaneously diagonal backgrounds are stable.
However, it is not clear what happens in the double scaling limit.

How to take the large-N limit

In the IIB matrix model , we usually regard *A* as the space-time coordinates.

$$S=-\frac{1}{g^2}Tr\left(\frac{1}{4}[A^\mu,A^\nu]^2+\frac{1}{2}\overline{\Psi}\gamma^\mu[A^\mu,\Psi]\right)$$

So, g has dimensions of length squared.

How is the Planck scale expressed?

If it does not depend on the IR cutoff *l*, as we normally guess, we should have

$$l_{\text{Planck}}=N^\alpha g^{\frac{1}{2}}. \qquad \leftarrow \alpha\,?$$

In other words, we should take the large-N limit keeping this combination finite.

At present we have no definite answer.

3. Interpretation of the matrices

What do the matrices stand for?

If we regard the IIB matrix model

$$S = -\frac{1}{g^2} Tr(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\gamma^\mu[A^\mu, \Psi])$$

as the matrix regularization of the Schild action, A^μ are space-time coordinates.

On the other hand if we regard it as the large-N reduced model, the diagonal elements of A^μ represent momenta.

Another interesting possibility is to consider a non-commutative back ground such as

$$A_\mu^{\ 0} = \begin{cases} \hat{p}_\mu \otimes 1_k \,, & \mu = 0,\cdots,3 \\ 0, & \mu = 4,\ldots,9 \end{cases} .$$

Here $\hat{p}_\mu$ satisfy the CCR's

$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = i\,B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

and 1_k is the $k \times k$ unit matrix.
Then we have a 4D noncommutative flat space with SU(*k*) gauge theory.

There are many possibilities to realize the space-time.

Actually various models that are close to the standard model can be constructed by choosing an appropriate background.

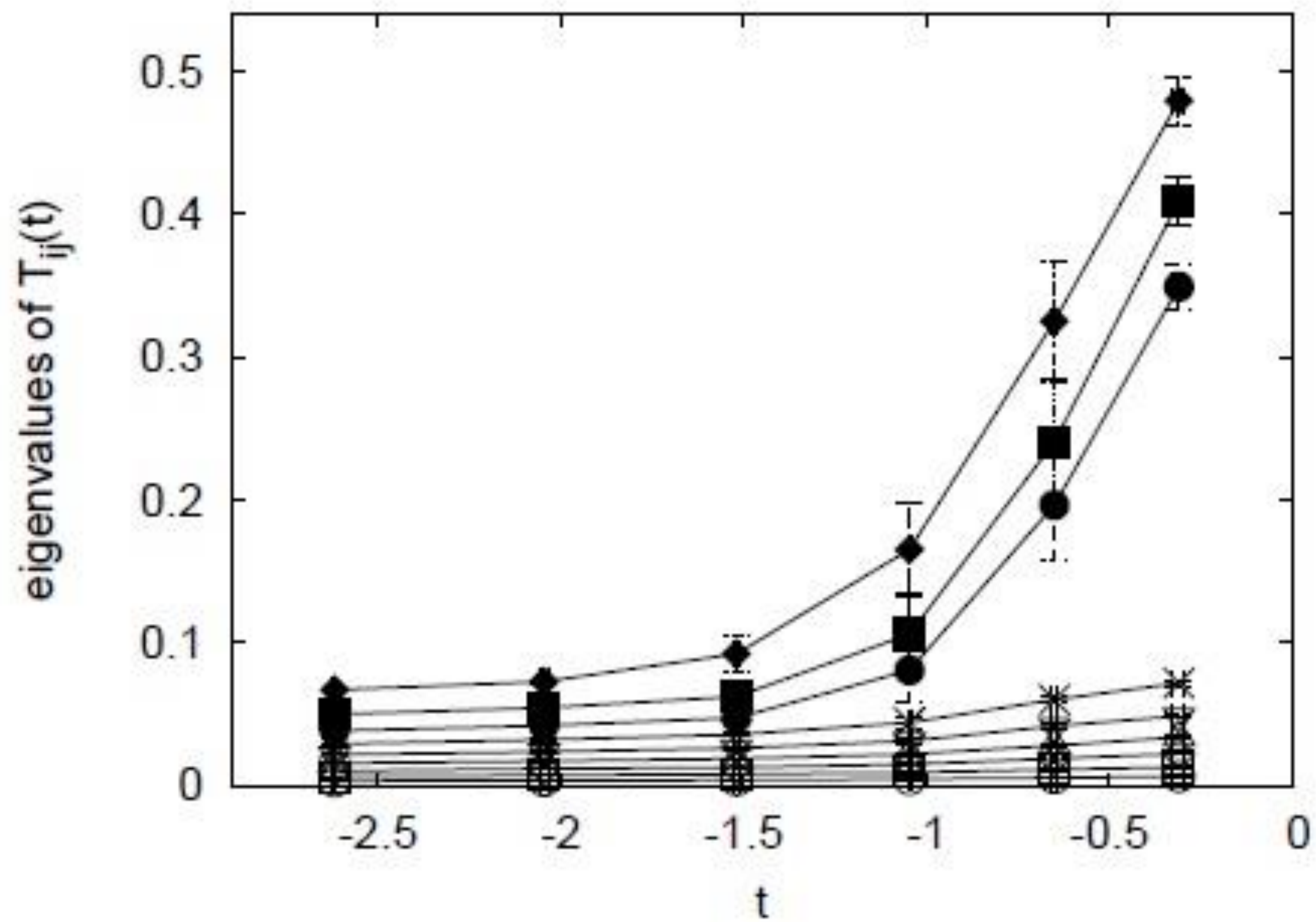
(ex.) **"Intersecting branes and a standard model realization in matrix models."**
A. Chatzistavrakidis, H. Steinacker, and G. Zoupanos.
JHEP09(2011)115

"An extended standard model and its Higgs geometry from the matrix model,"
H. Steinacker and J. Zahn,
PTEP 2014 (2014) 8, 083B03

(ex.) Expandind universe

S.-Kim, J. Nishimura, A. Tsuchiya
Phys.Rev.Lett. 108 (2012) 011601

They did a numerical simulation for the IIB matrix model with Lorentzian signature.
By identifying the eigenvalue of A_0 with time they found that the eigenvalues of only 3 A_i's become large, which



can be regarded as an expanding 3+1 D universe.

(ex.) Expanding universe 2

Recently another interesting picture of the expanding universe has been obtained by considering fuzzy manifolds.

"Quantized open FRW cosmology from Yang-Mills matrix models"
H. Steinacker,
Phys.Lett. B782(2018) 176-180

"The fuzzy 4-hyperboloid H_n^4 and higher-spin in Yang-Mills matrix models"
Marcus Sperling, H. Steinacker,
arXiv:1806.05907

4. Diffeomorphism invariance
and
Gravity

Diff. invariance and gravity

Because we have exact N=2 SUSY, it is natural to expect to have graviton in the spectrum of particles.

Actually there are some evidences.

(1) Gravitational interaction appears from one-loop integral.

(2) Emergent gravity by Steinacker. Gravity is induced on the non-commutative back ground.

However, it would be nicer, if we can understand how the diffeomorphism invariance is realized in the matrix model.

I would like to introduce an attempt, although it is not complete.

Covariant derivatives as matrices

Y. Kimura,
M. Hanada and HK

The basic question :

In the large-N reduced model, a background of simultaneously diagonalizable matrices $A_\mu^{(0)} = P_\mu$ corresponds to the flat space, if the eigenvalues are uniformly distributed.

In other words, the background $A_\mu^{(0)} = i\partial_\mu$ represents the flat space.

How about curved space?

Is it possible to consider some background like

$$A_\mu^{(0)} = i\nabla_\mu \text{ ?}$$

Actually, there is a way to express the covariant derivatives on any *D*-dim manifold by *D* matrices.

More precisely, we consider

M: any D-dimensional manifold,

φ_α: a regular representation field on M.

Here the index α stands for the components of the regular representation of the Lorentz group $SO(D-1,1)$.

The crucial point is that for any representation r, its tensor product with the regular representation is decomposed into the direct sum of the regular representations:

$$V_r \otimes V_{reg} \simeq V_{reg} \oplus \cdots \oplus V_{reg} .$$

In particular the Clebsh-Gordan coefficients for the decomposition of the tensor product of the vector and the regular representaion

$$V_{vector} \otimes V_{reg} \cong V_{reg} \oplus \cdots \oplus V_{reg}$$

are written as $C_{(a)\alpha}{}^{b,\beta}, \ (a = 1,..,D).$

Here *b* and β are the dual of the vector and the regular representation indices on the LHS. (a) indicates the a-th space of the regular representation on the RHS, and α is its index.

Then for each $\boldsymbol{a}$ $(\boldsymbol{a} = 1..\boldsymbol{D})$

$$\psi_\alpha = C_{(a)\alpha}{}^{b,\beta}\,\nabla_b \varphi_\beta$$

is a regular representation field on $\boldsymbol{M}$.

In other words, if we define $\boldsymbol{\nabla}_{(\boldsymbol{a})}$ by

$$\left(\nabla_{(a)}\varphi\right)_\alpha = C_{(a)\alpha}{}^{b,\beta}\,\nabla_b \varphi_\beta$$

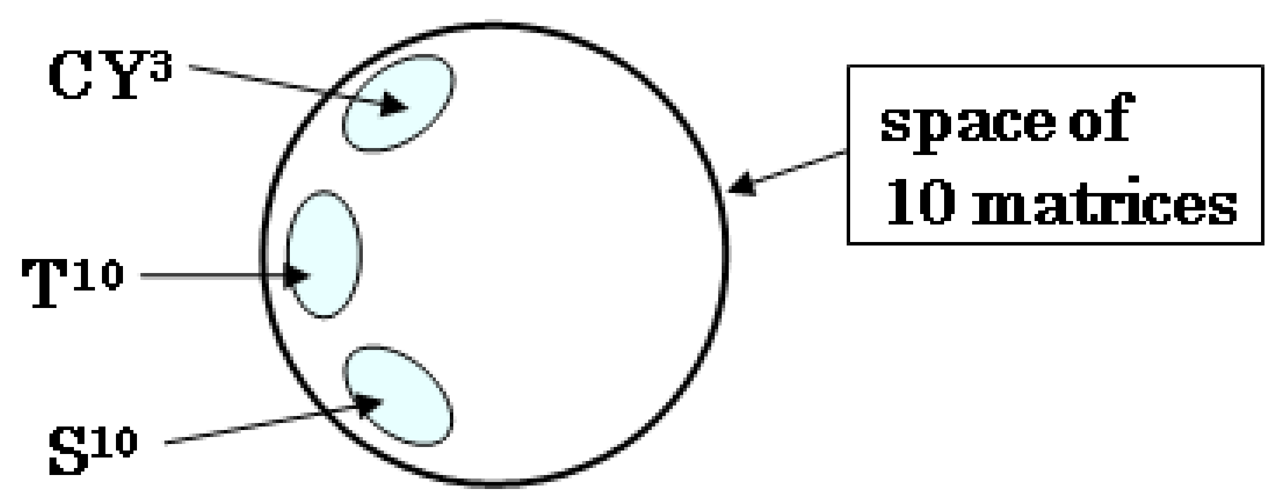
each $\boldsymbol{\nabla}_{(\boldsymbol{a})}$ is an endomorphism on the space of the regular representation field on $\boldsymbol{M}$.

Thus we have seen that any covariant derivative on any $\boldsymbol{D}$ dimensional manifold can be expressed by $\boldsymbol{D}$ matrices.

Therefore any *D*-dimensional manifold *M* with $D \leq 10$ can be realized in the space of the IIB matrix model as

$$A_a{}^0 = \begin{cases} \nabla_{(a)}, & a = 1, \cdots, D \\ 0, & a = D+1, \ldots, 10 \end{cases},$$

where $\nabla_{(a)}$ is the covariant derivative on *M* multiplied by the C-G coefficients.



Good points and bad points

Good point 1

Einstein equation is obtained at the classical level.

In fact, if we impose the Ansatz $A_a = i\nabla_{(a)}$
on the classical EOM $\left[A_a\left[A_a, A_b\right]\right] = 0,$
we have

$$\left[\nabla_{(a)}\left[\nabla_{(a)}, \nabla_{(b)}\right]\right] = 0$$

$$\Leftrightarrow \ 0 = \left[\nabla_a, \left[\nabla_a, \nabla_b\right]\right]$$

$$= \left[\nabla_a, R_{ab}{}^{cd} O_{cd}\right] = (\nabla_a R_{ab}{}^{cd}) O_{cd} - R_{ab}{}^{ca} \nabla_c$$

$$\Leftrightarrow \ \nabla_a R_{ab}{}^{cd} = 0\,,\ R_{ab} = 0 \quad \Leftrightarrow \ R_{ab} = 0\,.$$

Any Ricci flat space with $D \leq 10$ is a classical solution of the IIB matrix model.

Good point 2

Both the diffeomorphism and local Lorentz invariances are manifestly realized as a part of the $SU(N)$ symmetry.

In fact, the infinitesimal diffeomorphism and local Lorentz transformation act on φ_α as

$$\varphi \to \left(1 + \xi^\mu \partial_\mu\right)\varphi \qquad \text{and}$$

$$\varphi \to \left(1 + \varepsilon^{ab} O_{ab}\right)\varphi \text{ , respectively.}$$

Both of them are unitary because they preserve the norm of φ_α

$$\|\varphi_\alpha\|^2 = \int d^D x \sqrt{g}\, \varphi^{\alpha *} \varphi_\alpha$$

Bad points

Fluctuations around the classical solution

$$A_a^{(0)} = i\nabla_{(a)}$$

1. **contain infinitely many massless states.**
2. **Positivity is not guaranteed.**

This can be seen by considering the fluctuations around the flat space.
In this case the background is equivalent to

$$A_a^{(0)} = i\partial_a \otimes 1_{reg} \ ,$$

where 1_{reg} is the unit matrix on the space of the regular representation.

Because the unit matrix 1_{reg} is infinite dimensional, we have infinite degeneracy, and in particular we have infinitely many massless states. **→ bad point 1**

In general, the regular representation contains infinite tower of higher spins, and we have many negative norm states. It is not clear whether we have sufficiently many symmetries to eliminate those negative norm states.

→ bad point 2

One possible way out is to consider a non-commutative version.
What we have done is to regard the matrices as endomorphisms on the space of the regular representation fields.
It is easy to show that this space is equivalent to the space of the functions on the frame bundle of the spin bundle.

If we can construct a non-commutative version of such bundle, we can reduce the degrees of freedom significantly without breaking the diffeomorphism and local Lorentz invariance.

Topology change of space-time

1. Low energy effective action of Quantum gravity/string theory

Low energy effective action of IIB matrix model

A. Tsuchiya, Y. Asano and HK

We have seen that any *D*-dim manifold is contained in the space of *D* matrices. Therefore IIB matrix model should contain the effects of the topology change of space-time.

As was pointed out by Coleman some years ago, such effects give significant corrections to the low energy effective action.

It is interesting to consider the low energy effective action of the IIB matrix model.

Actually we can show that if we integrate out the heavy states in the IIB matrix model, the remaining low energy effective action is not a local action but has a special form, which we call *the multi-local action*:

$$S_{\text{eff}} = \sum_i c_i S_i + \sum_{ij} c_{ij} S_i S_j + \sum_{ijk} c_{ijk} S_i S_j S_k + \cdots,$$

$$S_i = \int d^D x \sqrt{g(x)} O_i(x).$$

Here O_i are local scalar operators such as

$$1, R, R_{\mu\nu}R^{\mu\nu}, F_{\mu\nu}F^{\mu\nu}, \bar{\psi}\gamma^{\mu}D_{\mu}\psi, \cdots.$$

S_i are parts of the conventional local actions.
The point is that S_{eff} is a function of S_i's.

This is essentially the consequence of the well-known fact that the effective action of a matrix model contains multi trace operators.

More precisely, we first decompose the matrices A_a into the background ${A^0}_a$ and the fluctuation ϕ :

$$A_a = {A^0}_a + \phi_a .$$

Here we assume that the background ${A^0}_a$ contains only the low energy modes, and ϕ contains the rest. We also assume that this decomposition can be done in a SU(N) invariant manner.

Then we integrate over ϕ to obtain the low energy effective action.

Substituting the decomposition into the action of the IIB matrix model, and dropping the linear terms in ϕ, we obtain

$$S=\frac{1}{4}Tr\Big(\left[A_a^0,A_b^0\right]^2$$
$$+2\left[A_a^0,\phi_b\right]^2+\left[A_a^0,A_b^0\right]\left[\phi_a,\phi_b\right]-2\left[A_a^0,\phi_b\right]\left[A_b^0,\phi_a\right]$$
$$+4\left[A_a^0,\phi_b\right]\left[\phi_a,\phi_b\right]+\left[\phi_a,\phi_b\right]^2+\text{fermion}\Big).$$

In principle, the 0-th order term $S_0=\frac{1}{4}Tr\left(\left[A_{(a)}^0,A_{(b)}^0\right]^2\right)$ can be evaluated with some UV regularization, which should give a local action.

The one-loop contribution is obtained by the Gaussian integral of the quadratic part.

Then the result is given by a double trace operator as usual:

$$W = \sum K^{abc\cdots,\,pqr\cdots}\, Tr\left(A_a^0 A_b^0 A_c^0 \cdots\right) Tr\left(A_p^0 A_q^0 A_r^0 \cdots\right)$$

The crucial assumption here is that both of the diffeomorphism and the local Lorentz invariance are realized as a part of the SU(*N*) symmetry.

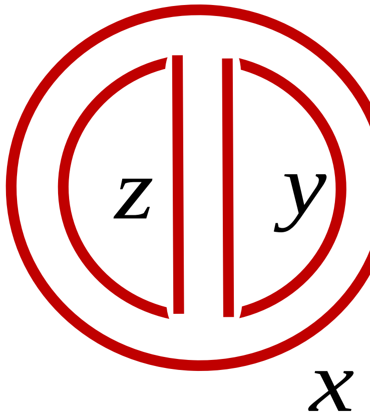
Then each trace should give a local action that is invariant under the diffeomorphisms and the local Lorentz transformations:

$$S_{\mathrm{eff}}{}^{\text{1-loop}} = \sum_{ij} \frac{1}{2} c_{ij} S_i S_j, \quad S_i = \int d^D x \sqrt{g(x)}\, O_i(x).$$

Similar analyses can be applied for higher loops.

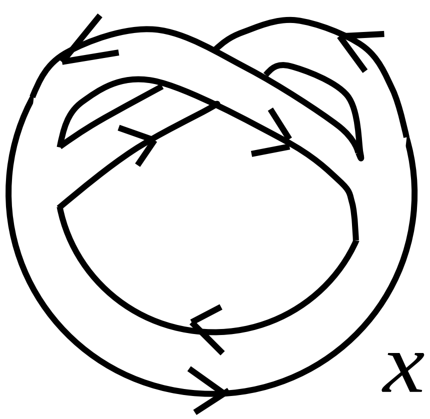
In the two loop order, from the planar diagrams we have a cubic form of local actions

$$S_{\text{eff}}^{\text{2-loop Planar}} = \sum_{i,j,k} \frac{1}{6} c_{ijk} S_i S_j S_k,$$



while non-planar diagrams give a local action

$$S_{\text{eff}}^{\text{2-loop NP}} = \sum_i c_i' S_i.$$



We have seen that

the low energy effective theory of the IIB matrix model is given by *the multi-local action*:

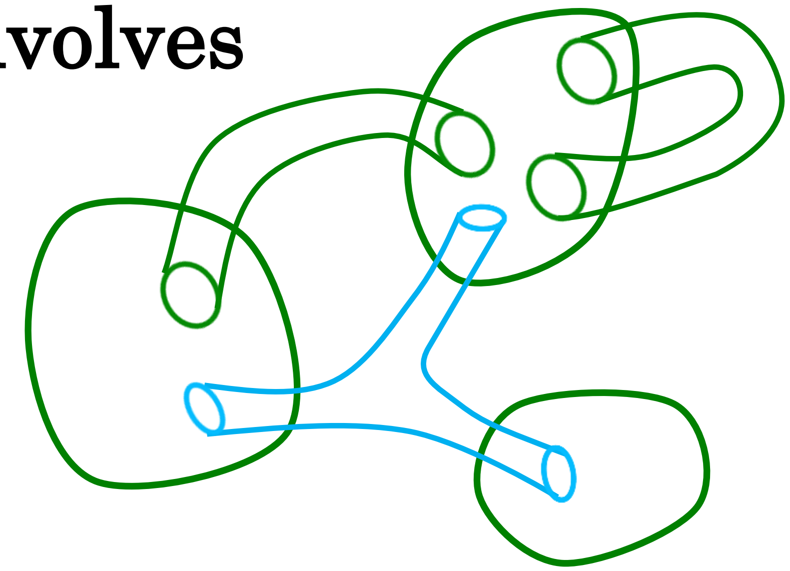
$$S_{\mathrm{eff}} = \sum_i c_i S_i + \sum_{ij} c_{ij} S_i S_j + \sum_{ijk} c_{ijk} S_i S_j S_k + \cdots,$$

$$S_i = \int d^D x \sqrt{g(x)} O_i(x).$$

This reminds us of the theory of baby universes by Coleman.

Coleman (1989)

Consider Euclidean path integral which involves the summation over topologies,



$$\sum_{\text{topology}} \int [dg] \exp(-S) \,.$$

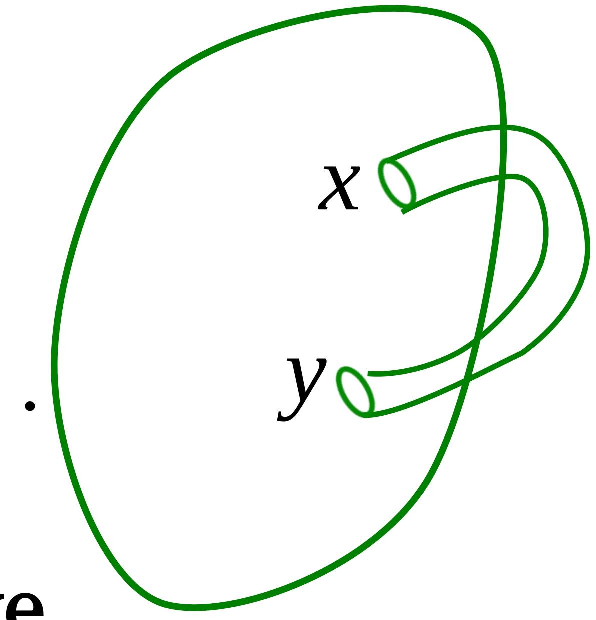
Then there should be a wormhole-like configuration in which a thin tube connects two points on the universe. Here, the two points may belong to either the same universe or different universes.

If we see such configuration from the side of the large universe(s), it looks like two small punctures.

But the effect of a small puncture is equivalent to an insertion of a local operator.

Therefore, a wormhole contributes to the path integral as

$$\int [dg] \sum_{i,j} c_{ij} \int d^4x\, d^4y \sqrt{g(x)}\sqrt{g(y)}\, O^i(x)\, O^j(y)\ \exp(-S)\,.$$



Summing over the number of wormholes, we have

$$\sum_{N=0}^{\infty} \frac{1}{n!} \left(\sum_{i,j} c_{ij} \int d^4x\, d^4y \sqrt{g(x)}\sqrt{g(y)}\, O^i(x)\, O^j(y) \right)^n$$

$$= \exp\left(\sum_{i,j} c_{ij} \int d^4x\, d^4y \sqrt{g(x)}\sqrt{g(y)}\, O^i(x)\, O^j(y) \right).$$

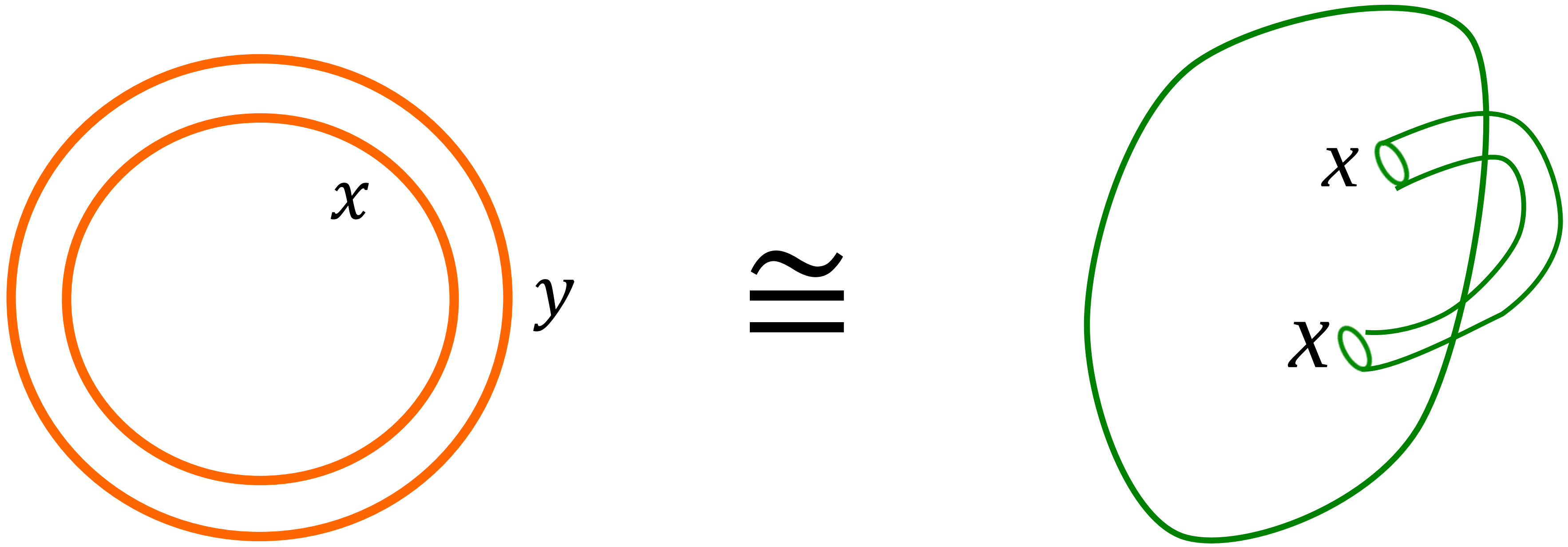
Thus wormholes contribute to the path integral as

$$\int [dg] \exp\left(-S + \sum_{i,j} c_{ij} \int d^4x\, d^4y \sqrt{g(x)}\sqrt{g(y)}\, O^i(x)\, O^j(y) \right).$$

bifurcated wormholes

$\Rightarrow$ cubic terms, quartic terms, …

Although there is no precise correspondence, the loops in the IIB matrix model resemble the wormholes.



Probably this phenomenon occurs universally. We may say that if the theory involves gravity and topology change, its low energy effective action becomes the multi-local action.

2. Fine tunings by nature itself

Multi-local action may provide a mechanism of automatic fine tunings and give a solution to the naturalness problem.

Physics of the multi-local action

We consider the action given by

$$S_{\mathrm{eff}} = \sum_i c_i S_i + \sum_{ij} c_{ij} S_i S_j + \sum_{ijk} c_{ijk} S_i S_j S_k + \cdots,$$

$$S_i = \int d^D x \sqrt{g(x)} O_i(x).$$

Because S_{eff} is a function of S_i's , we can express $\exp(iS_{\mathrm{eff}})$ by a Fourier transform as

$$\exp\left(iS_{eff}\left(S_1, S_2, \cdots\right)\right) = \int d\lambda\, w\left(\lambda_1, \lambda_2, \cdots\right) \exp\left(i \sum_i \lambda_i S_i\right),$$

where λ_i's are Fourier conjugate variables to S_i's, and w is a function of λ_i's .

Then the path integral for S_{eff} can be written as

$$Z = \int [d\phi] \exp(iS_{\text{eff}}) = \int d\lambda\, w(\lambda) \int [d\phi] \exp\left(i \sum_i \lambda_i S_i \right).$$

The last integral is the ordinary path integral for the action

$$\sum_i \lambda_i S_i = \int d^D x \sqrt{g} \sum \lambda_i O_i.$$

Because O_i are local scalar operators such as

$$1\,,\, R\,,\, R_{\mu\nu} R^{\mu\nu}\,,\, F_{\mu\nu} F^{\mu\nu}, \bar{\psi} \gamma^\mu D_\mu \psi\,,\, \cdots.$$

$\sum_i \lambda_i S_i$ is an ordinary local action and λ_i are nothing but the coupling constants.

Therefore the system we are considering is very close to the ordinary field theory, but we have to integrate over the coupling constants with weight $w(\lambda_1, \lambda_2. \cdots)$.

If $\int[d\phi]\exp\left(i\sum_i \lambda_i S_i\right)$ **has a sharp peak at** $\lambda = \lambda^{(0)}$**, we can say that the coupling constants are fixed to** $\lambda^{(0)}$**.**

However, it is not clear how to define the value of the path integral

$$\int[d\phi]\exp\left(i\sum_i \lambda_i S_i\right)$$

in the Lorentzian theory, because we do not know a priori the initial and final states of the universe.

Instead, we can take a working hypothesis:

Maximum Entropy Principle (MEP):

Coupling constants are tuned so that the entropy of the universe becomes maximum.

Suppose that we pic up a universe randomly from the multiverse. Then the most probable universe is expected to be the one that has the maximum entropy. **(T. Okada and HK)**

If the cosmological evolution is completely understood, we can calculate the total entropy of the universe, and in principle all of the independent low-energy couplings are determined by maximizing it.

For example if we accept the inflation scenario in which universe pops out from nothing and then inflates, most of the entropy of the universe is generated at the stage of reheating just after the inflation stops. Therefore the potential of the inflaton should be tuned so that inflation occurs as much as possible.

Furthermore, if we assume that Higgs field plays the role of inflaton, the above analysis tells that Higgs potential should become flat at some high energy scale.

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Actually from the recent experimental data, we see that

(1) the parameters of the SM indeed seem to be chosen such that Higgs potential becomes flat around the Planck scale.

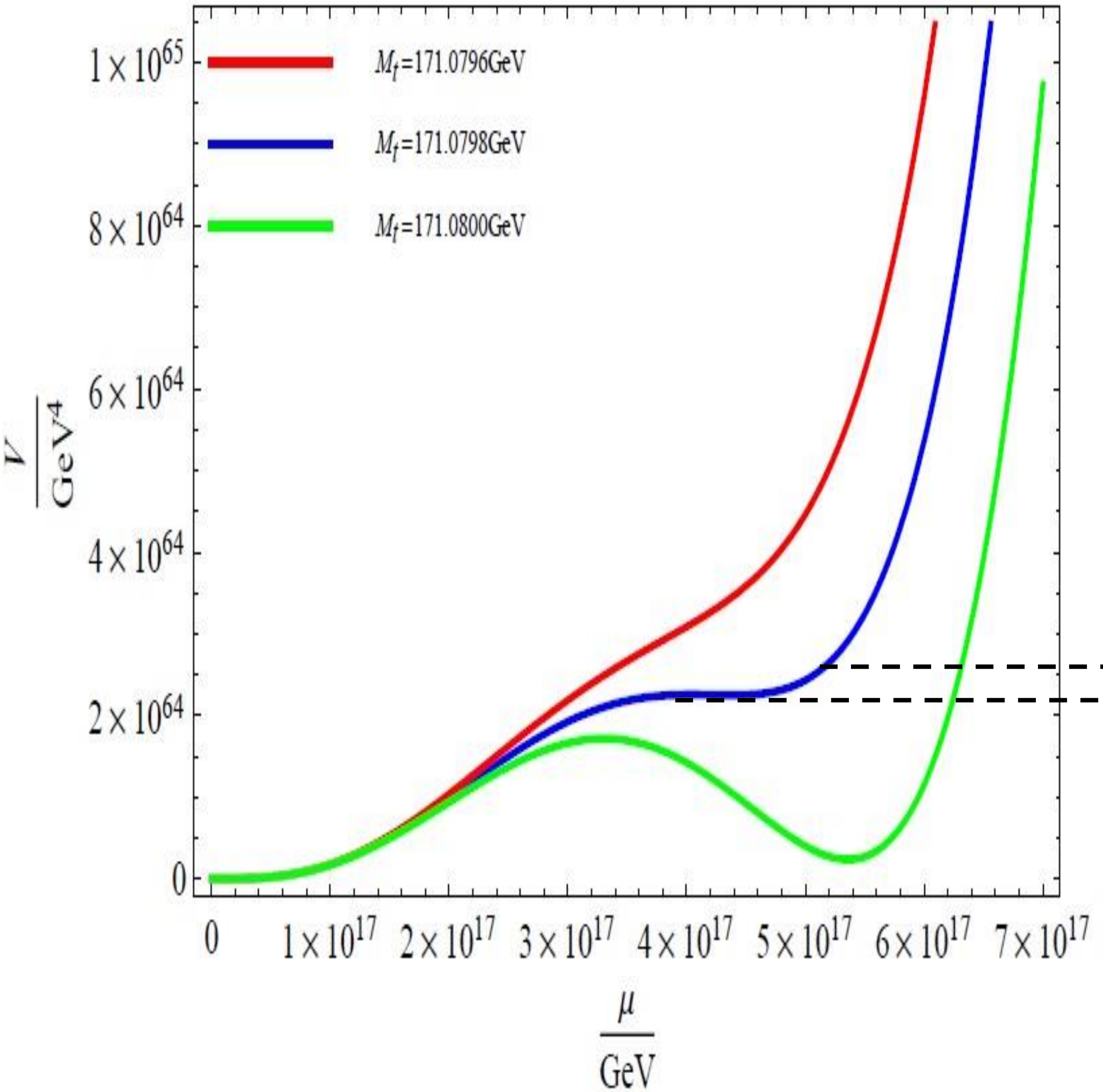
(2) We can obtain realistic cosmological model of Higgs inflation.

Critical Higgs inflation

If we introduce a non-minimal coupling $\xi R\varphi^2$

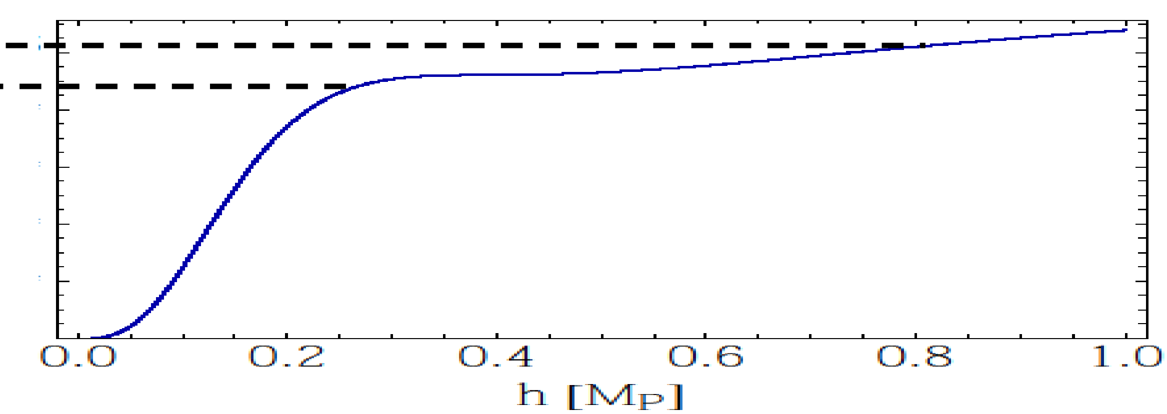
a realistic Higgs inflation is possible.

ξ can be as small as 10.



In the Einstein frame the effective potential becomes

$$V(\varphi_h), \quad \varphi_h = \frac{h}{\sqrt{1+\xi h^2/M_P{}^2}}.$$



Summary and conclusions

- **It is natural to expect that space-time emerges from matrices.**
- **But there are various possibilities.**
- **It is also important to understand the time evolution of universe.**
- **In particular matrix models may describe the very beginning of the universe.**
- **It is important to develop numerical techniques to solve space-time matrix models.**
- **Topology change of universe is automatically included in matrix models, and it may give a clue to resolve the naturalness problem.**