# One-sided versus two-sided dependence

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# The difference between one-sided and two-sided points of view.

**One-sided versus two-sided.**
Stochastic Systems (Processes).
Two flavours:
Time, discrete.
(Dynamical Systems, asymmetric description).
Past and future, one-sided (SRB). Non-equilibrium...
versus
Space, discrete, here one-dimensional.
(Mathematical Physics, symmetric description).
Left and right, two-sided (DLR). Equilibrium.
**Question:**
When are both descriptions equivalent?
When not equivalent?

**Introduction:**

Simple background.

Markov modeling (for the short-sighted..)

*Time*:

Probability and Statistics (Markov chains).

Future independent of past, given the *present*.

Ergodic Theory, Dynamical Systems.

Ahistoric, forget history.

(Henry Ford: All history is bunk...)

First Emperor Qin Shi Huang: "Erase history."

*Space*:

Statistical (Mathematical) Physics.

Markov: Inside independent of outside,
given the *border*.

(Take control of your borders,
Forget and be ignorant about everywhere else.)

2-state Markov chains
-timelike-
versus
1-dimensional, nearest neighbour, spin
(e.g. Ising) models
-spacelike-.

Probability measures on e.g.
two-symbol sequences,
configuration space $\Omega = \{-, +\}^{Z}$.
**Theorem**:
(well-known, see e.g.
Wikipedia lemma "Markov Property",
see further Georgii).
Stationary Markov chains, i.e.
invariant Markov measures on histories,
and n.n. Gibbs measures,
in dimension 1,
are the **same** objects.
(Brascamp,Spitzer,...)
**Warning**: This is about objects (measures)
on *infinite* time/space.

**Question**:
If we try to be more far-sighted
does this sameness stay true?
Case 1:
General Ergodic Processes.
**Answer**: NO!
Gurevich, Ornstein-Weiss constructed examples
which are one-sided random,
two-sided deterministic.
Thus one-sided entropy (Kolmogorov-Sinai) *positive*
different from two-sided entropy which is *zero*,
and one-sided tail (*trivial*) different
from two-sided tail (*contains everything*).
**Remark**: Ergodic (invariants) are one-sided objects.

Case 2:
Continuous (Quasilocal) Ergodic Processes.
What if we change independent of Markov to
weakly dependent(continuous, quasilocal, almost Markov),
does this sameness between one-sided and two-sided remain true?
Then we exclude the Gurevich and Ornstein-Weiss examples,
which are not continuous (in the product topology).
Various aspects studied by various people.
(Fernández, Gallo, Maillard, Verbitskiy, Redig,
Pollicott, Walters, den Hollander-Steif, Tempelman...)

**Answers:**

Sameness with extra regularity conditions: Yes.

One-sided equals two-sided.

(SRB, Thermodynamic Formalism..).

Without those: NO!

**Neither** class includes the other.

One direction known (since 2011),

(Fernández, Gallo, Maillard)

other direction new (here).

**Time version:**

Class of Stochastic Processes,
rediscovered repeatedly,
under a variety of names:
( $g$-measures=
Chains of Infinite Order=
Chains with Complete Connections=
Uniform Martingales/Random Markov
Processes).
(Keane 70's, Harris 50's,
Onicescu-Mihoc and Doeblin-Fortet 30's,
Kalikow 90's).
Studied in Ergodic Theory, Probability.

**Spatial version**:
Gibbs (=DLR) measures=
Gibbs fields=
" almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm$, for all $i \in Z$.
(Can be much more general.)
**Warning:**
DLR Gibbs $\neq$ SRB Gibbs.

**Gibbs measures:**

Let $G$ be an infinite graph, here $Z$.
Configuration space:
Space of sequences: $\Omega = \{-,+\}^G$.
Probability measures on $\Omega$,
labeleled by **interactions**.
An interaction is a collection of functions,
$\Phi_X(\omega)$, dependent on $\{-,+\}^X$,
where the $X$ are subsets of $G$.
Let $\Lambda$ be a finite subset of $G$.
We write $\Omega_\Lambda = \{-,+\}^\Lambda$.

Energy (**Hamiltonian**)
$H_{\Lambda}^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.
A measure $\mu$ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL configurations $\omega$,
boundary conditions $\tau$
and finite volumes $\Lambda$.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
Nearest-neighbour interaction means that
$\Phi(X) = 0$,
except when $X = \{i, i+1\}$ or $X = i$,
for some $i \in Z$.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_{\{1,\ldots,n\}} | \tau_{\{1\ldots,n\}^c}) = \mu(\omega_{\{1\ldots,n\}} | \tau_0 \tau_{n+1})$.
Conditioned on the border spins,
at 0 and $n+1$,
*inside* and *outside* are independent.

A two-state Markov chain is again
a measure on the same sequence space $\Omega$.
Now it has to satisfy the " ordinary"
(timelike) Markov property:
$\mu(\omega_{\{1...n\}}|\tau_{\{-\infty,....,0\}}) = \mu(\omega_{\{1...n\}}|\tau_0)$.
One can describe this via a product
of 2-by-2 stochastic matrices $P$
with non-zero entries:
$P(k,l) = P(\omega_i = k \to \omega_{i+1} = l)$.
Here $k, l = \pm$ and $i$ is any site (=time) in $Z$.
There is a one-to-one connection between
stationary (time-invariant)
2-state Markov Chains
and (space-translation-invariant) nearest-neighbor
Ising Gibbs measures.

Continuity (=almost Markov = quasilocality).
Product topology:
Two sequences are close if they are
equal on a large enough finite interval.
Topology metrisable,
metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where $n$ is the site with
minimal distance from origin such that
$\omega_n \neq \omega'_n$.
A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
(or not too long ago)
whatever it is.

**Processes** (time):

$\mu(\sigma_0 = \omega_0 | \omega_{Z-}) = g(\omega_0 \omega_{Z-})$,

with $g$-function continuous.

Probability of getting $\omega_0$, given the past.

Continuous dependence on the **past**.

Continuity studied since the 30's

(Doeblin-Fortet).

**Claim!?**:

Continuity implies uniqueness (Harris(50's)).

Mistake in proof pointed out by Keane (70's).

Counterexamples due to Bramson-Kalikow (90's).

Sharper criterion Berger-Hoffman-Sidoravicius (2003-2017).

**Gibbs measures:**
Continuity of conditional probabilities
corresponds to summability of interactions.
$\sum_{0 \in X} ||\Phi_X|| < \infty$.
Continuous dependence on **outside**
beyond the border.
*(Quasilocality).*
No action at a distance.
(No observable influence from behind the moon)
**Plus:** "non-nullness".
Any finite change in the -infinite- system
costs a finite amount of energy.
Any configuration in finite domain
occurs with finite probability,
whatever is happening outside.
Gibbs measures satisfy (equivalently) a
finite-energy condition.
Equivalence holds (Kozlov-Sullivan):
Finite-energy + continuity = Gibbs.

**Our Counterexample:**
(Gibbs, non-g-measure).
Gibbs measures for Dyson models.
Low temperatures.
Long-range Ising models.
Ferromagnetic pair interactions.
$\Phi_{i,j}(\omega) = -J|i-j|^{-\alpha}\omega_i\omega_j$.
Interesting regime $1 < \alpha \leq 2$.
Phase transition for large J,
at low temperatures:
There exist then **two** different
Gibbs measures, for the same interaction,
called $\mu^+$ and $\mu^-$, for such $\Phi$.
Microscopic interfaces **don't exist**.
Spatially continuous conditional probabilities.
**Warning:**
Impossible for Markov Chains or Fields,
always uniqueness.

**Claim:**

At low T and for $\alpha^* < \alpha < 2$

Dyson Gibbs measures are not g-measures.

Here technical condition $\alpha^* = 3 - \frac{\ln 3}{\ln 2}$.

Proof uses technically rather hard Input,

perturbative, cluster expansions, from others,

giving the $\alpha^*$ condition,

plus three simple Observations.

**Input**:

Interface result for Dyson models
(Cassandro, Merola, Picco, Rozikov).
Take interval $[-L, +L]$,
all spins to the left are minus,
all spins to the right are plus.
Then there is an interface point **IF**, such that:
1) To the left of the interface
we are in the minus phase ($\mu^-$),
to the right of the interface
we are in the plus phase ($\mu^+$).
2) With overwhelming probability the location
of the interface is at most $O(L^{\frac{\alpha}{2}})$ from the center.
$... - - - - - m....|$**IF**$| + m.....| + + + + + ...$

**Observation 1:**
If I change all spins left of a length-N interval of minuses, the effect from the left on the central O(L) interval is bounded by $O(LN^{1-\alpha})$, thus small for $N$ large.

**Consequence:**
A large interval of minuses (size $N$) will have a moderately large (size $L$) interval of minus phase on both sides. Interfaces are pushed away.

**Observation 2:**
If I decouple a comparatively small interval,
of size $L_1 = o(L)$,
in the beginning of my minus-phase interval,
this hardly changes the interface location.
(Cost of **IF** shift by $\varepsilon L$ is larger, namely $O(L^{2-\alpha})$.
Shown by Cassandro et al.)

**Observation 3:**
If I make in this $L_1$ interval
an alternating configuration
+-+-+-+-+-...
then the total energy (influence)
on its complement
is bounded by the double sum
$\sum_{i=1....L_1, j>L_1}(|j-i|^{-\alpha} - |j+1-i|^{-\alpha})=$
$\sum_{i=1....L_1, j>L_1}(O(|j-i|^{-(\alpha+1)})=$
$\sum_{i=1....L_1} O(|i|^{-\alpha})$
which is bounded, uniformly in $L_1$.
Therefore finite, small effect.

**Remark:**
Effect only at positive temperature.
**Entropic Repulsion**.
A large alternating interval,
preceded by a MUCH
larger interval of minuses,
cannot shield the influence
of this homogeneous minus interval.
But this means precisely that
the conditional probability of finding a plus (or a minus),
at a given site, conditioned on an alternating past,
is not continuous.
Thus two-sided continuity
occurring at the same time
as one-sided discontinuity.

Alternating configuration is discontinuity point,
due to cancellations of pluses and minuses.
Set of discontinuity points may have measure zero, but nonremovable.

$\ldots - - - - - + - + - + - X$ (- N, $alt_L$ intervals)

versus

$\ldots + + + + - + - + - + - X$ (+N, $alt_L$ intervals)

Expected value of X differs,
by more than cst,
uniformly in L and N(L).
Direct influence from Deep Past.

Analogies with higher-dimensional
Gibbs measures.
Analogy $g$-measures:
Global Markov property.
Conditioning on infinite-volume
(like half-line) events.
There are Markov fields
which are not Globally Markov.
Other analogy:
There are Markov fields which depend
discontinuously on lexicographic past.

**Open Question** (Bethuelsen-Conache),
trigger: Schonmann projection
(one-dimensional marginal,
of 2d Ising measures).
Entropic repulsion in one or two directions?
Is it a g-measure?
Partial results so far, suggesting different behaviour.
Non-Gibbs, possibly g-measure.

**Conclusion:**

Two-sided continuous dependence
-spacelike- does *not* imply
one-sided continuous dependence
-timelike.

**Summary:**

Controlling borders is NOT the same as
control of history,
except for the shortsighted.

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arXiv 1705.03156, Comm. Math. Phys., 363, 767–788.

Further questions:
1) Get rid of the technical restriction on $\alpha$,
and large n.n.term,
with Bissacot, Endo, Kimura, Ruszel.
(Kimura, Littin-Picco)
2) Understand $\alpha = 2$ case (open).
3) Other Dyson model questions,
a) add possibly decaying
*inhomogeneous* external fields, deterministic or random.
b) Metastability,
c) metastates for random boundary conditions.

with Endo, Le Ny, Kimura, Ruszel, Spitoni,

Littin (3a).

# Happy Birthday, Francois!