Shaken dynamics for the 2d Ising model

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Outline:

- Definition of the dynamics
- A geological motivation
- Main results
  - invariant measure
  - relation with Gibbs measure
  - convergence to equilibrium at low temperature
- Conclusions

The dynamics

Λ two-dimensional $L \times L$ square in $\mathbb{Z}^2$
$\mathcal{X}_\Lambda$ the set of spin configurations in Λ., i.e., $\mathcal{X}_\Lambda = \{-1, 1\}^\Lambda$
In Λ we fix a set B with fixed spins:
state space $\mathcal{X}_{\Lambda,B} = \{\sigma \in \mathcal{X}_\Lambda : \ \sigma_x = +1 \quad \forall x \in B\}$.

Pair Hamiltonian on (σ, τ) on $\mathcal{X}_{\Lambda,B} \times \mathcal{X}_{\Lambda,B}$:

$$H(\sigma, \tau) = -\sum_{x \in \Lambda} [J\tau_x(\sigma_{x^\downarrow} + \sigma_{x^\leftarrow}) + q\tau_x\sigma_x - \lambda(\sigma_x + \tau_x)]$$

where $x^\downarrow, x^\leftarrow$ are down, left neighbors of the site x on Λ with periodic b.c., $J > 0$, $q > 0$ is an inertial constant and $\lambda > 0$ an external field.

$$H(\sigma,\tau) = -\sum_{x\in\Lambda} [J\tau_x(\sigma_{x^\downarrow} + \sigma_{x^\leftarrow}) + q\tau_x\sigma_x - \lambda(\sigma_x + \tau_x)] =$$

$$-\sum_{x\in\Lambda} [J\sigma_x(\tau_{x^\uparrow} + \tau_{x^\rightarrow}) + q\sigma_x\tau_x - \lambda(\sigma_x + \tau_x)]$$

where $x^\uparrow, x^\rightarrow$ are up, right neighbors of x. We have

$$H(\sigma,\sigma) = H(\sigma) - q|\Lambda| + \lambda \sum_{x\in\Lambda} \sigma_x$$

where $H(\sigma)$ is the usual Ising hamiltonian with magnetic field $-\lambda$

$$H(\sigma) = -\sum_{\langle x,y\rangle\in\mathcal{B}_\Lambda} J\sigma_x\sigma_y + \lambda\sum_{x\in\Lambda}\sigma_x$$

Note also that $H(\sigma,\tau) \neq H(\tau,\sigma)$, interaction in opposite directions:

$$dl : x^\downarrow, x^\leftarrow \Longleftrightarrow ur : x^\uparrow, x^\rightarrow$$

[ISS '07], [GSSV '11], [DSS '12], [PSS '16], [DSS '15]

Define

$$\overrightarrow{Z}_\sigma = \sum_{\sigma' \in \mathcal{X}_{\Lambda,B}} e^{-H(\sigma,\sigma')} \qquad \overleftarrow{Z}_\sigma = \sum_{\sigma' \in \mathcal{X}_{\Lambda,B}} e^{-H(\sigma',\sigma)}$$

and the two asymmetric parallel updatings on $\mathcal{X}_{\Lambda,B}$

$$P^{dl}(\sigma,\sigma') := \frac{e^{-H(\sigma,\sigma')}}{\overrightarrow{Z}_\sigma} = \prod_{x\in\Lambda} \frac{e^{h_x^{dl}(\sigma)\sigma'_x}}{2\cosh(h_x^{dl}(\sigma))}$$

with

$$h_x^{dl}(\sigma) = \Big[J(\sigma_{x^\downarrow} + \sigma_{x^\leftarrow}) + q\sigma_x - \lambda\Big],$$

and

$$P^{ur}(\sigma,\sigma') := \frac{e^{-H(\sigma',\sigma)}}{\overleftarrow{Z}_\sigma} = \prod_{x\in\Lambda} \frac{e^{h_x^{ur}(\sigma)\sigma'_x}}{2\cosh(h_x^{ur}(\sigma))}.$$

with

$$h_x^{ur}(\sigma) = \Big[J(\sigma_{x^\uparrow} + \sigma_{x^\rightarrow}) + q\sigma_x - \lambda\Big]$$

The shaken dynamics is the composition of these two steps, with interactions in opposite directions:

$$P(\sigma,\tau)=\sum_{\sigma'\in\mathcal{X}_{\Lambda,B}} P^{dl}(\sigma,\sigma')P^{ur}(\sigma',\tau)=\sum_{\sigma'\in\mathcal{X}_{\Lambda,B}} \frac{e^{-H(\sigma,\sigma')}}{\overrightarrow{Z}_{\sigma}}\frac{e^{-H(\tau,\sigma')}}{\overleftarrow{Z}_{\sigma'}}$$

Every step of the shaken dynamics: two asymmetric parallel updatings

A geological motivation

Terrestrial tides: amplitude: ~ 0.3 m, period: $\sim$ 12h

The source of this very fast deformation of the Earth is clear in the non-inertial frame in which the x-axis is directed toward the Sun/Moon. The tides are due to the gradient of the gravitational and of the centrifugal forces.

$$F_c(r) = m\omega^2 r \qquad F_g(r) = -\frac{KmM}{r^2}$$

$F_c(R_{ES}) + F_g(R_{ES}) = 0$. Let $\frac{r_E}{R_{ES}} = \epsilon$. For small ϵ and $\rho \leq r_E$:

$$F_c(R_{ES} - \rho) \sim m\omega^2 R_{ES}\left[1 - \frac{\rho}{R_{ES}}\right] \qquad F_g(R_{ES} - \rho) = -\frac{KmM}{R_{ES}^2}\left[1 + 2\frac{\rho}{R_{ES}}\right]$$

Solar tide force:

$$F_c(R_{ES} - \rho) + F_g(R_{ES} - \rho) \sim -\frac{KmM}{R_{ES}^2} 3 \frac{\rho}{R_{ES}}$$

A geological model

In [Doglioni et al., 2006] it has been proposed that tides are one of the driving forces, beside convection, of the drift of the tectonic plates.

It is introduced a model to describe friction between lithosphere and mantle in terms of discrete bonds subject to random fractures.
Breakage occurs with exponential distribution with rate depending on the number of active present bonds (mean field model without geometry).

A toy model with tide effect

Assume that the bonds are located on a square lattice, and that the exponentially small probability of each fracture has a dependence on the state of the nearest neighbor bonds. Bonds surrounded by broken bonds breaks with slightly higher probability. Bonds with intact neighbors tend to remain (ferromagnetic interaction).

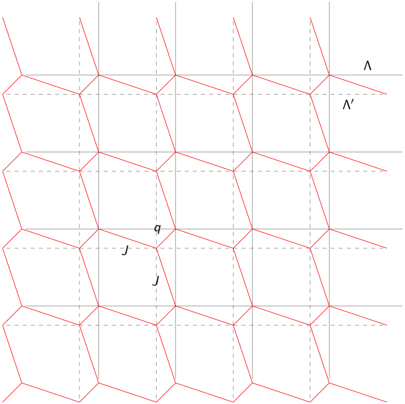
The main forces (convection, gravitation,...) responsible of the drift of tectonic plates are much more relevant but correspond to very large time scales in the evolution. We represent their effect in our toy model with an external field (possibly not constant) and with an inertial constant in the dynamics.

Since the plates are driven back and forth to the faults by the tides, we assume that twice a day each bond may be fractured first on the basis of the state of the bonds above and to its right, and then below and to its left: a shaken dynamics.

A fracture is an earthquake. Interpret $\sigma_x = +1$ an earthquake at location x. Faults can be described as boundary condition B. We assume $J \ll \lambda < q$.

The pair hamiltonian on $\mathcal{X}_{\Lambda,B} \times \mathcal{X}_{\Lambda,B}$

$$H(\sigma, \sigma') = -\sum_{x \in \Lambda} [J\sigma'_x(\sigma_{x\downarrow} + \sigma_{x\leftarrow}) + q\sigma'_x\sigma_x - \lambda(\sigma_x + \sigma'_x)]$$



From the square to the hexagonal lattice

$$\mathcal{X}_{\Lambda,B} \times \mathcal{X}_{\Lambda,B} = \mathcal{X}_{\mathbb{H}}$$

First results

The stationary measure of the shaken dynamics is

$$\pi_{\Lambda,B}(\sigma) = \frac{\overrightarrow{Z}_\sigma}{Z} \quad \text{with} \quad \overrightarrow{Z}_\sigma = \sum_{\sigma' \in \mathcal{X}_{\Lambda,B}} e^{-H(\sigma,\sigma')}, \; Z = \sum_{\sigma \in \mathcal{X}_{\Lambda,B}} \overrightarrow{Z}_\sigma$$

and reversibility holds. This stationary measure is the marginal of the measure on $\mathcal{X}_{\Lambda,B} \times \mathcal{X}_{\Lambda,B} = \mathcal{X}_{\mathbb{H}}$ defined by:

$$\pi_2(\sigma,\tau) := \frac{1}{Z} e^{-H(\sigma,\tau)}.$$

Since $\mathbb{H}$ is a bipartite graph, the vertex set V of $\mathbb{H}$ can be decomposed is two sets $V = V^1 + V^2$, with $|V^i| = |\Lambda|$, $i = 1, 2$ and each $\boldsymbol{\sigma} \in \mathcal{X}_{\mathbb{H}}$ can be written as $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ with $\sigma^i \in \mathcal{X}_{V^i,B}$, $i = 1, 2$. The shaken dynamics on $\mathcal{X}_{\Lambda,B}$ corresponds to an alternate dynamics on $\mathcal{X}_{\mathbb{H}}$ in the following sense

$$P^{sh}(\sigma^1, \tau^1) = \sum_{\tau^2 \in \{-1,+1\}^{V^2}} P^{alt}(\boldsymbol{\sigma}, \boldsymbol{\tau})$$

where

$$P^{alt}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \frac{e^{-H(\sigma^1,\tau^2)}}{\overrightarrow{Z}_{\sigma^1}} \frac{e^{-H(\tau^1,\tau^2)}}{\overleftarrow{Z}_{\tau^2}} = \prod_{x\in V^2} \frac{e^{h_x(\sigma^1)\tau_x^2}}{2\cosh(h_x(\sigma^1))} \prod_{x\in V^1} \frac{e^{h_x(\tau^2)\tau_x^1}}{2\cosh(h_x(\tau^2))}$$

with

$$h_x(\sigma^i) = J(\sigma_{z_1} + \sigma_{z_2}) + q\sigma_{z_3} - \lambda,$$

where $z_1, z_2, z_3 \in V^i$ nearest neighbors of x, and the measure $\pi_2(\sigma^1, \sigma^2)$ is the non reversible stationary measure of P^{alt}.

Noting that $P^{alt}(\boldsymbol{\sigma},\boldsymbol{\tau})$ does not depend on σ^2, we can define the evolution of X_t^{sh} as a marginal of the evolution of the alternate process X^{alt}. Given a path $\boldsymbol{\omega}$ for the process X^{alt}

$$\boldsymbol{\omega} : \ \boldsymbol{\omega}(0), \boldsymbol{\omega}(1), \dots, \boldsymbol{\omega}(t)$$

and the associated path for the process X^{sh} ω

$$\omega : \ \boldsymbol{\omega}^1(0), \boldsymbol{\omega}^1(1), \dots, \boldsymbol{\omega}^1(t)$$

we have

$$\mathbb{P}^{sh}(\omega) = \sum_{\boldsymbol{\omega}^2(1),\dots,\boldsymbol{\omega}^2(t)} \mathbb{P}^{alt}(\boldsymbol{\omega}).$$

Different regimes in q

On the hexagonal anisotropic lattice $\mathbb{H}$, with two bonds J and one bond q exiting from each site, different regimes:

- $q = J$ hexagonal regular lattice;
- $q \to \infty$ in any $x \in \Lambda$ with large probability $\sigma_x^1 = \sigma_x^2$, so $\sigma^1 = \sigma^2 \in \mathcal{X}_\Lambda$: configurations on the square lattice;
- $q \to 0$ in any $x \in \Lambda$ the spin interact only with two nearest neighbor sites, i.e. independent copies of one dimensional systems.

Distance between π_Λ and π_Λ^G

From now on let $B = \emptyset$, i.e., we consider standard periodic boundary conditions.

$$\pi_\Lambda^G(\sigma) = \frac{e^{-H(\sigma)}}{Z_G} \qquad \text{with} \qquad Z_G = \sum_{\sigma \in \mathcal{X}_\Lambda} e^{-H(\sigma)}$$

$$\|\pi_\Lambda - \pi_\Lambda^G\|_{TV} = \frac{1}{2} \sum_{\sigma \in \mathcal{X}_\Lambda} |\pi_\Lambda(\sigma) - \pi_\Lambda^G(\sigma)|.$$

Theorem1

Set $\delta = e^{-2q}$, and let δ be such that

$$\lim_{|\Lambda|\to\infty} \delta^2 |\Lambda| = 0,$$

there exist J_0 such that for any $J > J_0$

$$\lim_{|\Lambda|\to\infty} \|\pi_\Lambda - \pi_\Lambda^G\|_{TV} = 0$$

Extension of [PSS1] where $\lambda = 0$.

Remark (finite volume regime)

Let Λ be fixed and finite. There exist J_0 sufficiently large and $\eta \in (0,1)$ such that if $J_0 < J < q(1-\eta)$ then there exists a constant $C = C(J_0, \eta, |\Lambda|)$ such that

$$\|\pi_\Lambda - \pi_\Lambda^G\|_{TV} \leq C\delta^2$$

Convergence to equilibrium at low temperature

We will use the following parametrization:

$$J = \frac{\beta}{2},\ q = \beta,\ \lambda = \frac{\epsilon\beta}{2}$$

and suppose ϵ small but fixed, $\beta \to \infty$ and Λ large ($|\Lambda| > \frac{1}{\epsilon^2}$) fixed, i.e., independent of β.
In this low temperature regime π_Λ concentrates on -1, parallel to the external magnetic field $-\lambda$:

-1 is the stable state;
$+1$ is a metastable state.

In this low temperature regime

$$\pi_\Lambda(\sigma) \propto \sum_\tau e^{-H(\sigma,\tau)} \approx e^{-\mathcal{H}(\sigma)}$$

with $\mathcal{H}(\sigma) = \min_\tau H(\sigma,\tau)$.

The configuration $+1$ corresponds to a local minimum of the energy $\mathcal{H}(\sigma)$ and there is a local drift to this minimum given by typical transitions of probability of order one.

To leave $+1$ the process has to go "against" this drift, with transitions of exponentially small probability.

Indeed small clusters of minus spin in a see of positive spins, have the tendency to shrink, and there is a critical size of cluster of minus spin to overcome in order to prefer to grow.

Theorem 2

Let τ_{-1}^{sh} be the first hitting time to the configuration -1 for the shaken dynamics $\mathbb{P}_{+1}(\tau_{-1}^{sh} > t)$ its distribution starting from the configuration $+1$.
For β sufficiently large and for any $\alpha > 0$ arbitrarily small we have

$$\mathbb{P}_{+1}(\tau_{-1}^{sh} > T^{sh} e^{\alpha\beta}) < \exp\left\{ - e^{a\beta} \right\} \tag{1}$$

for some $a > 0$, with $T^{sh} = e^{E_c^{sh}\beta}$ and

$$E_c^{sh} = 4l_c - 2\epsilon(l_c - 1)l_c - \epsilon, \quad \text{where} \quad l_c = \frac{1}{\epsilon}$$

Note that with the same parameters J and λ for the usual Glauber single spin flip dynamics we have that for β large and $\alpha > 0$ arbitrarily small (see e.g. [OV], [BdH])

$$\mathbb{P}_{+1}\Big(T^{Gl} e^{-\alpha\beta} < \tau_{-1}^{Gl} < T^{Gl} e^{\alpha\beta}\Big) \sim 1 \qquad \text{for large} \quad \beta$$

with

$$T^{Gl} = e^{E_c^{Gl}\beta}, \qquad \text{where} \qquad E_c^{Gl} = 4l_c^{Gl} - \epsilon(l_c^{Gl} - 1)l_c^{Gl} - \epsilon \quad \text{with} \quad l_c^{Gl} = \frac{2}{\epsilon}$$

so that $E_c^{sh} \sim \frac{2}{\epsilon} = \frac{1}{2} \cdot \frac{4}{\epsilon} \sim \frac{1}{2} E_c^{Gl}$ and therefore

$$T^{sh} \asymp \sqrt{T^{Gl}}.$$

The advantage is not due to parallelization. Indeed even if the shaken dynamics is parallel, so that

$$P^{sh}(+1,-1) > 0$$

at low temperature (β large) the shaken dynamics behaves with large probability like a single spin flip dynamics when starting from $+1$. Parallelization has positive effects when the dynamics is going along and not against the drift.

The gain is due to geometrical reasons, related to the fact that shaken dynamics is equivalent to the alternate dynamics on the hexagonal lattice, and the potential barrier on this lattice is lower than the corresponding barrier on the square lattice.

Some idea of the proofs

First resuts DBC for shaken:

$$\sum_{\sigma' \in \mathcal{X}_{\Lambda,B}} \frac{e^{-(H(\sigma,\sigma')+H(\tau,\sigma'))}}{\overleftarrow{Z}_{\sigma'}} = \overrightarrow{Z}_{\sigma} P^{sh}(\sigma,\tau) = \overrightarrow{Z}_{\tau} P^{sh}(\tau,\sigma) =$$

$$\sum_{\sigma' \in \mathcal{X}_{\Lambda,B}} \frac{e^{-(H(\tau,\sigma')+H(\sigma,\sigma'))}}{\overleftarrow{Z}_{\sigma'}}$$

Invariant measure of the alternate dynamics:

$$\sum_{\sigma^1,\sigma^2} \pi_2(\sigma^1,\sigma^2) P^{alt}(\boldsymbol{\sigma},\boldsymbol{\tau}) =$$

$$\sum_{\sigma^1,\sigma^2} \frac{e^{-H(\sigma^1,\sigma^2)}}{Z} \frac{e^{-H(\sigma^1,\tau^2)}}{\overrightarrow{Z}_{\sigma^1}} \frac{e^{-H(\tau^1,\tau^2)}}{\overleftarrow{Z}_{\tau^2}} = \frac{e^{-H(\tau^1,\tau^2)}}{Z} = \pi_2(\tau^1,\tau^2)$$

even if in general no reversibility

$$\pi_2(\sigma^1,\sigma^2) P^{alt}(\boldsymbol{\sigma},\boldsymbol{\tau}) \neq \pi_2(\tau^1,\tau^2) P^{alt}(\boldsymbol{\tau},\boldsymbol{\sigma}).$$

Theorem 1
Standard cluster expansion argument as in [PSS'16]

Theorem 2
Metastability in Freidlin Wentzel regime in an anisotropic hexagonal lattice (vertical bonds $q = 2J$).

$$P^{alt}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \frac{e^{-H(\sigma^1,\tau^2)}}{\overrightarrow{Z}_{\sigma^1}} \frac{e^{-H(\tau^1,\tau^2)}}{\overleftarrow{Z}_{\tau^2}} =$$

$$\prod_{x\in V^2} \frac{e^{h_x(\sigma^1)\tau^2_x}}{2\cosh(h_x(\sigma^1))} \prod_{x\in V^1} \frac{e^{h_x(\tau^2)\tau^1_x}}{2\cosh(h_x(\tau^2))} \asymp e^{-\Delta(\boldsymbol{\sigma},\boldsymbol{\tau})\beta}.$$

$$\Delta(\boldsymbol{\sigma},\boldsymbol{\tau}) = \sum_{x^2\in V^2} \Delta_{x^2}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \sum_{x^1\in V^1} \Delta_{x^1}(\boldsymbol{\sigma},\boldsymbol{\tau})$$

where

$$\Delta_{x^2}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \left[\tau^2_{x^2} 2h_{x^2}(\sigma^1)\right]_-, \qquad \Delta_{x^1}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \left[\tau^1_{x^1} 2h_{x^1}(\tau^2)\right]_-$$

$$h_{x^2}(\sigma^1) = \frac{1}{2}\left(\sigma^1_{z_1} + \sigma^1_{z_2} + 2\sigma^1_{z_3} - \epsilon\right)$$

with $z_1, z_2, z_3 \in V^1$ the nearest neighbors of x^2, z_3 being the neighbor related to x^2 by a q-bond and similarly for $h_{x^1}(\tau^2)$.

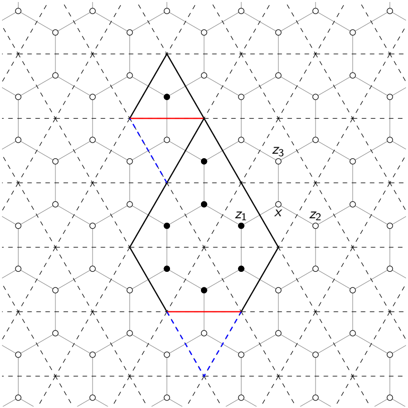


Figure : A spin configuration on portion of the hexagonal lattice with its Peierls' contour. Black dots represent "minus" spins and white dots "plus" spins. Red edges in the contour have cost q whereas black edges have cost J.

cost to create a new line: $2 - \epsilon$
cost to erase a line of length l: $2\epsilon l - \epsilon$

critical side:

$$2 - \epsilon = 2\epsilon l_c - \epsilon$$

Typical time to reach $-\mathbf{1}$ is related to minimal cost to reach size l_c
Isoperimetric problem: maximal area for rhombus.
Area of rhombus of side l is $2l^2$.

$$E_c^{sh} = 4l_c - 2\epsilon(l_c - 1)l_c - \epsilon$$

Final remark

Different is the case of shaken dynamics with $J = q$ (hexagonal regular lattice).
Relevant configuration are hexagons. We expect

$$E_c^{sh(J=q)} \sim 6l_c - 6\epsilon l_c^2, \quad \text{where} \quad l_c = \frac{1}{2\epsilon}$$

$$E_c^{sh(J=q)} \sim \frac{3}{2\epsilon}$$

Conclusion

The advantages of the shaken dynamics can be summarized as follows:

- the dynamics is actually on a different lattice so there is a gain of a square root when moving "against the drift" in the small temperature regime;
- the dynamics is parallel so there is a gain in the efficiency proportional to $|\Lambda|$ when moving "along the drift", moreover the dynamics can be efficiently simulated on computers with parallel architecture;
- by Theorem 1 and Theorem 2 we can conclude that in small temperature regime the shaken dynamics is an efficient tool for Gibbs sampling within a given error depending on β;
- there is an additional parameter q tuning the geometry of the system.