Phase transitions for a Solid-on-Solid interface interacting with a potential

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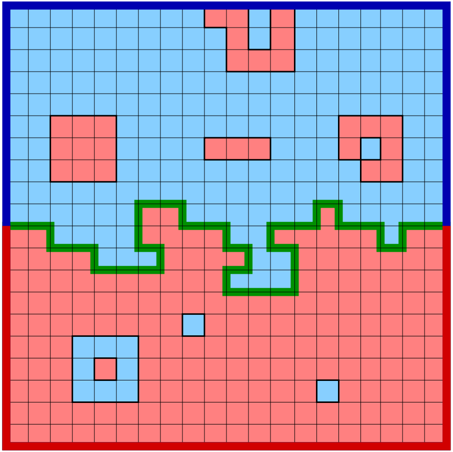
Phase coexistence/Interfaces for lattice models

For the Ising model at low temperature with mixed boundary condition, we can define a notion of interface between the $+$ and $-$ phase.

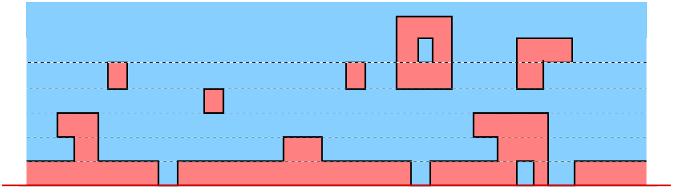


The Solid-on-solid simplification

To study the qualitative behavior of these interfaces, it seems reasonable to consider the following simplificated picture

(A) We ignore the effect of small clusters, in the $+$ and $-$ phases.

(B) We assume that the interphase is the graph of a function.



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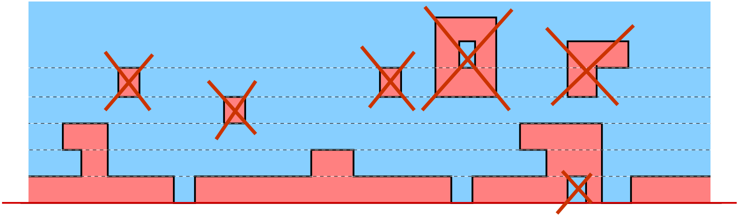

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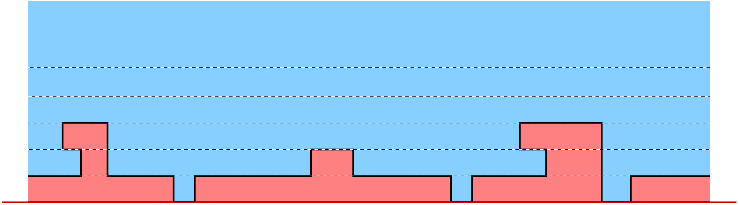

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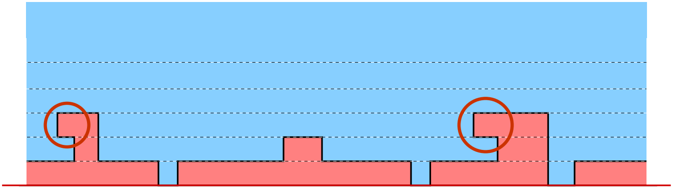

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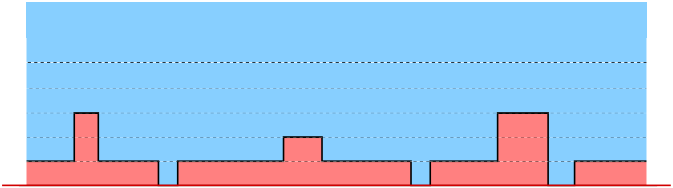

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Solid-On-Solid Definition

Set $\Lambda_N = [\![-N, N]\!]^2$ and $\Omega_N := \{\phi \ : \ [\![-N, N]\!]^2 \to \mathbb{Z}\}$ and define

$$\mathcal{H}_N^n(\phi) := \sum_{\substack{\{x,y\}\subset\Lambda_N \\ x\sim y}} |\phi(x) - \phi(y)| + \sum_{\substack{\{x\in\Lambda_N, y\in\partial\Lambda_N\} \\ x\sim y}} |\phi(x) - n|.$$

The Solid-On-Solid (SOS) measure $\mathbf{P}_{N,\beta}$ is defined on Ω_N by

$$\mathbf{P}_{N,\beta}^n(\phi) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N^n(\phi)}$$

where

$$Z_{N,\beta} := \sum_{\phi\in\Lambda} e^{-\beta \mathcal{H}_N^n(\phi)}$$

Rigidity at low temperature

If $\beta \geq 1$, then one has for every N every $x \in \Lambda_N$

$$\mathbf{P}_{N,\beta}[|\phi(x)| \geq n] \leq Ce^{-4\beta n}$$

where $C > 0$ does not depend on β.

Theorem (Brandenberger, Wayne '82)

If β is sufficiently large ($\beta \geq 5$ works), then there exists $\mathbf{P}_\beta$ a probability on $\Omega_{\mathbb{Z}_2}$ such that for every bounded local function f (i.e. function that depends only on a finite number of coordinate)

$$\lim_{N \to \mathbb{Z}^2} \mathbf{E}_{N,\beta}[f(\phi)] = \mathbf{E}_\beta[f(\phi)]$$

Under $\mathbf{P}_\beta$ the level set $\phi^{-1}\{0\}$ percolates,

The pinning for SOS $(\beta \geq 1)$

We add pinning potential $h \in \mathbb{R}$ at level 0,

$$\mathbf{P}^h_{N,\beta}(\phi) := \frac{1}{Z^h_{N,\beta}} e^{-\beta \mathcal{H}_N(\phi) + h|\phi^{-1}\{0\}|} \text{ with } Z^h_{\Lambda,\beta} := \sum_{\phi \in \Omega_N} e^{-\beta \mathcal{H}_N(\phi) + h|\phi^{-1}\{0\}|}.$$

Proposition (Excess free-energy)

$$\mathrm{F}_\beta(h) := \lim_{N\to\infty} \frac{1}{(2N+1)^2} \log \frac{Z^h_{N,\beta}}{Z_{N,\beta}} = \lim_{N\to\infty} \frac{1}{(2N+1)^2} \log \mathbf{E}_{N,\beta}[e^{h|\phi^{-1}\{0\}|}]$$

exists, it is convex and non-decreasing in h. We have

$$\mathrm{F}_\beta(h) = 0, \text{ when } h \leq 0, \quad \mathrm{F}_\beta(h) > 0, \text{ when } h > 0. \tag{1}$$

We also have $\mathrm{F}_\beta(h) \overset{h\to 0}{\sim} c_\beta h$. We have at every differentiability point

$$\partial_h \mathrm{F}_\beta(h) = \lim_{N\to\infty} (2N+1)^{-2} \mathbf{E}^h_{N,\beta}[|\phi^{-1}(0)|] \tag{2}$$

We wish to investigate how this phase transition is modified when introducing one of the following modifications

(A) Half-space confinement: we restrict ourselves to positive trajectories.

$$Z_{N,\beta}^{h,+,n} := \sum_{\phi \in \Omega_N^+} e^{-\beta \mathcal{H}_\Lambda^n(\phi) + h|\phi^{-1}\{0\}|},$$

with

$$\Omega_N^+ := \{\phi \;:\; [\![-N,N]\!]^2 \to \mathbb{Z}_+\}.$$

(B) Introduction of disorder: we replace the energy reward h by $h + \omega_x$ when $\phi(x) = 0$ where $(\omega_x)_{x \in \mathbb{Z}^2}$ is an IID field satisfying $\mathbb{E}\left[e^{\omega}\right] = 1$ and $\mathbb{E}[e^{\alpha\omega}] < \infty$ for all $\alpha \in \mathbb{R}$,

$$Z_{N,\beta}^{h,\omega} := \sum_{\phi \in \Omega_N} e^{-\beta \mathcal{H}_N(\phi) + \sum_{x \in [\![-N,N]\!]^2} (h+\omega_x)\delta_x} \quad \text{with } \delta_x := \mathbf{1}_{\{\phi(x)=0\}}.$$

Effect on the free-energy

To investigate the large scale asymptotic behavior, define as before the excess free-enery as

$$\mathrm{F}^{\omega}_{\beta}(h):=\lim_{N\to\infty}\frac{1}{(2N)^2}\mathbb{E}\log\frac{Z^{h,\omega}_{N,\beta}}{Z_{N,\beta}}=\lim_{N\to\infty}\frac{1}{(2N)^2}\mathbb{E}\log\mathbf{E}_{N,\beta}[e^{\sum_{x\in[\![1,N]\!]^2}(h+\omega_x)\delta_x}],$$

$$\mathrm{F}^{+}_{\beta}(h):=\lim_{N\to\infty}\frac{1}{(2N)^2}\log\frac{Z^{h,+,n}_{N,\beta}}{Z_{N,\beta}}=\lim_{N\to\infty}\frac{1}{(2N)^2}\log\mathbf{E}_{N,\beta}[e^{h|\phi^{-1}\{0\}|}\mathbf{1}_{\Omega^+_N}]. \tag{3}$$

Both quantities are still convex and non-decreasing and we have

$$0\le\mathrm{F}^{\omega}_{\beta}(h),\ \mathrm{F}^{+}_{\beta}(h)\le\mathrm{F}_{\beta}(h).$$

Effect on phase transition

We want to know if $\mathrm{F}^{\omega}_{\beta}(h)$ and $\mathrm{F}^{+}_{\beta}(h)$ display the same kind of transition as $\mathrm{F}(h)$.

Critical points and critical behavior

Let $h_\beta^+ := \inf\{h \ : \mathrm{F}_\beta^+(h) > 0\}$ and $h_\beta^\omega := \inf\{h \ : \mathrm{F}_\beta^+(h) > 0\}$ be the critical point associated with half space problem

Proposition (L' 17)

For every $\beta \geq 1$ *we have*

$$h_\beta^+ = \log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right), \ \textit{and} \ \forall u \in [0,1], \quad c_1(\beta)u^3 \leq \mathrm{F}_\beta^+(h_\beta^+ + u) \leq C_1(\beta)u^3.$$

Proposition (L' 19)

For every $\beta \geq 1$ *we have*

$$h_\beta^\omega = 0, \ \textit{and} \ \forall h \in [0,1], \quad c_2(\beta,\omega)h^2 \leq \mathrm{F}_\beta^\omega(h) \leq C_2(\beta,\omega)h^2.$$

More accurate behavior of the free-energy

Theorem

When $\beta \geq 5$ we have

(A) *[L' 17] There exists two constant c_1 and c_2 (depending on β such that*

$$\mathrm{F}^{+}_{\beta}(h^{+}_{\beta} + u) \overset{u \to 0+}{\sim} \max_{n \geq 1} \left[c_1 e^{-4\beta n} u - c_2 e^{-6\beta n} \right]$$

(B) *[L' 19] There exists two constant c_3 and c_4 depending on β and (for c_3) on the distribution of ω such that*

$$\mathrm{F}^{\omega}_{\beta}(h) \overset{h \to 0+}{\sim} \max_{n \geq 1} \left[c_3 e^{-4\beta n} h - c_4 e^{-8\beta n} \right]$$

Heuristic for these expressions (1)

Interfaces wants to remain mostly flat: the strategies that gives the largest contribution to Z are obtained by

(i) Fixing a height n, place most of the interface at this height.

(ii) Produce rare spikes to collect the benefits from being at height 0.

The probability making spikes of width 1 is of order $c_1 e^{-4\beta n}$.
The probability of making a spike of larger width is of order $c_2 e^{-6\beta n}$.

Heuristic for these expressions (2)

(A) In the half-space case, the model is found to be equivalent to a model with no constraint where spikes of width receive rewards $u(= h - h_\beta^+)$ and other large of spikes yield an energetic penalty.

$$c_1 e^{-4\beta n} u - c_2 e^{-6\beta n}$$

(B) In the disordered case, the second negative term e^ω corresponds to lack of averaging

$$c_3 e^{-4\beta n} h - c_4 e^{-8\beta n}$$

Theorem (L '18)

There exists a sequence $(u_n^*)_{n\geq 1}$ *such that*

(1) $\mathrm{F}_\beta^+(h_\beta^+ + u)$ *is* C^∞ *on* $(0,\infty)\setminus\{u_n^*\}_{n\geq 1}$, *and not differentiable at* u_n^*.

(2) *There exists* c *such that* $u_n^* \overset{n\to\infty}{\sim} ce^{-2\beta n}$.

(3) *When* $u \in (u_{n+1}^*, u_n^*)$,

$$\lim_{\Lambda\to\mathbb{Z}^2} \mathbf{P}_{\Lambda,\beta}^{h,+,m} = \mathbf{P}_\beta^h, \quad \forall m \leq n.$$

$\phi^{-1}\{n\}$ *percolates almost surely under* $\mathbf{P}_\beta^h$

(4) *When* $u = u_n^*$,

$$\lim_{\Lambda\to\mathbb{Z}^2} \mathbf{P}_{\Lambda,\beta}^{h,+,m} = \begin{cases} \mathbf{P}_\beta^{h+} & \text{if } m \leq n-1, \\ \mathbf{P}_\beta^{h-} & \text{if } m = n. \end{cases}$$

$\phi^{-1}\{n-1\}$ *percolates almost surely under* $\mathbf{P}_\beta^{h+}$.
$\phi^{-1}\{n\}$ *percolates almost surely under* $\mathbf{P}_\beta^{h-}$.