Homogenization in amorphous media and applications

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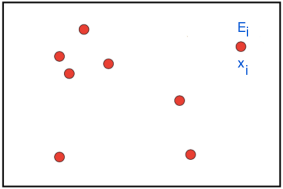
Motivations

- **Population dynamics**
- **Mott variable range hopping**
  - Fundamental hopping mechanism of electron transport in strongly disordered systems, as doped semiconductors
  - In the regime of low impurity density, one encodes the electron interactions into the jump rates and considers independent random walkers.
  - Final object: random walk on a marked simple point process

Marked simple point process

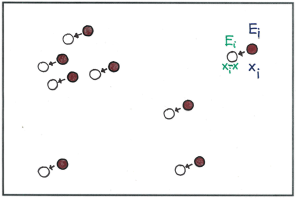


- $\{\color{red}\bullet\} = \{x_i\}$: simple point process,
  random locally finite subset of $\mathbb{R}^d$
- E_i: mark of x_i, real random variable
- $\omega = \{(x_i, E_i)\}$ marked simple point process
- Ω space of possible configurations ω

Action of the group $\mathbb{R}^d$ by translations

- Given $x \in \mathbb{R}^d$ and $\omega = \{(x_i, E_i)\}$, we set

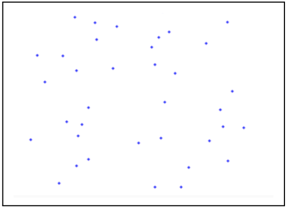
$$\tau_x \omega := \{(x_i - x, E_i)\}$$



- $\mathbb{P}$: law of $\omega = \{(x_i, E_i)\}$
- **$\mathbb{P}$ stationary and ergodic w.r.t. spatial translations**

Example 1: marked Poisson point process

- **Sample $\hat{\omega} := \{x_i\}$ as PPP on $\mathbb{R}^d$ with density λ**



  - $\big|\{x_i\} \cap A\big| \sim$ Poisson rv with mean $\lambda \ell(A)$,
    $\ell(\cdot)$: Lebesgue measure
  - $A \cap B = \emptyset \implies |\{x_i\} \cap A|$ and $|\{x_i\} \cap B|$ are independent

- **Mark the points $x_i' s$ with i.i.d. random variables E_i**

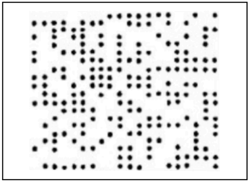
Example 1: marked Poisson point process

- Sample $\hat{\omega} := \{x_i\}$ as PPP on $\mathbb{R}^d$ with density λ
- Mark $x_i's$ with i.i.d. random variables E_i

If $E_i \sim \nu$, then $\mathbb{P}$ is called the ν–randomization of the simple point process on $\mathbb{R}^d$.

Example 2: marked diluted crystal

- Let $\{z_i\} \subset \mathbb{Z}^d$ be the vertexes of site percolation



- U: uniformly distributed random vector in $[0, 1)^d$
- Set $x_i := z_i + U$, $\forall i$
- Mark the points $x_i' s$ with i.i.d. random variables E_i

Palm distribution $\mathbb{P}_0$

- $\omega = \{(x_i, E_i)\}$ marked simple point process
- Ω: space of possible configurations ω
- Ω_0: space of configurations ω with $0 \in \{x_i\}$
- $\mathbb{P}_0$**: Palm distribution associated to** $\mathbb{P}$

  Probability with support in Ω_0

  Roughly, $\mathbb{P}_0 = \mathbb{P}(\cdot | 0 \in \{x_i\})$

Palm distribution $\mathbb{P}_0$

Expectations: $\mathbb{P} \to \mathbb{E}$, $\mathbb{P}_0 \to \mathbb{E}_0$

Due to ergodicity:

Fact

For $\mathbb{P}$–a.a. $\omega = \{(x_i, E_i)\}$ it holds

$$\lim_{k \to \infty} Av_{\substack{x:|x| \leq k \\ x \in \{x_i\}}} f(\tau_x \omega) = \mathbb{E}_0[f] \,.$$

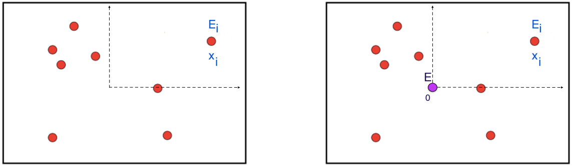
Av=Average

Example

$\mathbb{P}$: ν-randomization of a PPP

Then, $\mathbb{P}_0$ is the law of ω obtained as follows:

- Sample $\{(x_i, E_i)\}$ with law $\mathbb{P}$
- Sample independently a r.v. E with distribution ν
- Set $\omega := \{(x_i, E_i)\} \cup \{(0, E)\}$



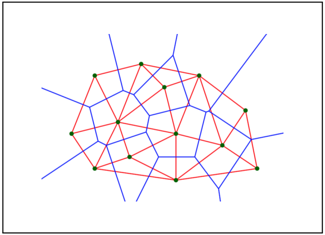
Random walk $(X_t^\omega)_{t\geq 0}$

- $\omega = \{(x_i, E_i)\}$ random environment
- $(X_t^\omega)_{t\geq 0}$ continuous time random walk
- State space $\hat{\omega} = \{x_i\}$
- $\mathbb{P}(X_{t+dt}^\omega = x_j \,|\, X_t^\omega = x_i) = c_{x_i,x_j}(\omega)dt, \qquad i \neq j$
- Symmetric jump rates: $c_{x_i,x_j}(\omega) = c_{x_j,x_i}(\omega)$
- Covariant jump rates: $c_{x_i,x_j}(\omega) = c_{x_i-z,x_j-z}(\tau_z\omega)\ \forall z \in \mathbb{R}^d$
- Irriducible random walk

Examples

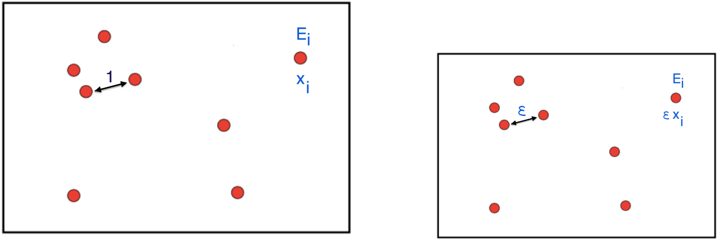
- Mott v.r.h.
  $c_{x_i,x_j}(\omega) = \exp\{-|x_i - x_j| - (|E_i| + |E_j| + |E_i - E_j|)\}$

- Nearest–neighbor random walk on the Delaneauy triangulation



ε–rescaling

- $\varepsilon > 0$ and $\omega = \{(x_i, E_i)\}$



- μ_ω^ε: measure on $\mathbb{R}^d$, $\mu_\omega^\varepsilon := \varepsilon^d \sum_i \delta_{\varepsilon x_i}$

ε–rescaling

- Intensity: $m := \mathbb{E}\big[\,|\{x_i\} \cap [0,1)^d|\,\big]$
- Due to ergodicity: $\mu_\omega^\varepsilon \to m\,dx$
- Ω_0: space of configurations ω with $0 \in \{x_i\}$

Proposition

Given $\varphi \in C_c(\mathbb{R}^d)$ and $g : \Omega_0 \to \mathbb{R}$ in $L^1(\mathbb{P}_0)$, for $\mathbb{P}$–a.a. ω it holds

$$\lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x)\varphi(x)g(\tau_{x/\varepsilon}\omega) = \int \varphi(x) m dx \cdot \mathbb{E}_0[g]\,. \qquad (1)$$

- In (1) the spatial variables x appears on "2 scales" : macroscopic $(x = \varepsilon x_i)$ / microscopic $(x/\varepsilon = x_i)$

Diffusively rescaled generator

- $\omega = \{(x_i, E_i)\}$, $\hat{\omega} = \{x_i\}$, $\varepsilon\hat{\omega} := \{\varepsilon x_i\}$
- **Rescaled Markov generator of the random walk**

$$\mathbb{L}^{\varepsilon}_{\omega} f(\varepsilon x_i) := \varepsilon^{-2} \sum_j c_{x_i,x_j}(\omega)\big(f(\varepsilon x_j) - f(\varepsilon x_i)\big)\,, \quad \varepsilon x_i \in \varepsilon\hat{\omega}\,,$$

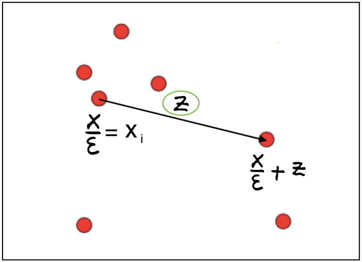
- $\mathbb{L}^{\varepsilon}_{\omega}$ self–adjoint operator in $L^2(\mu^{\varepsilon}_{\omega})$.

Amorphous gradient

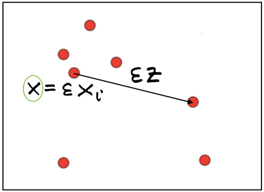
$u : \varepsilon\hat{\omega} \to \mathbb{R}$,

$$\nabla_\varepsilon u(x,z) := \frac{u(x+\varepsilon z) - u(x)}{\varepsilon}\,, \qquad x \in \varepsilon\hat{\omega}\,, \qquad \frac{x}{\varepsilon} + z \in \hat{\omega}\,.$$

Warning: x **macroscopic**, z **microscopic**



ε–rescaling $\rightsquigarrow$



Measure ν_ω^ε

- ν_ω^ε: atomic measure
- above (x,z) $\rightsquigarrow$ weight $\varepsilon^d c_{\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z}(\omega)$
- $\mu_\omega^\varepsilon = \varepsilon^d \sum_{x\in\varepsilon\hat{\omega}} \delta_x$
- Key identity

$$\langle -\mathbb{L}_\omega^\varepsilon f, g\rangle_{\mu_\omega^\varepsilon} = \frac{1}{2}\langle \nabla_\varepsilon f, \nabla_\varepsilon g\rangle_{\nu_\omega^\varepsilon}$$

Weak solution of Poisson equation

- $H^1_{\omega,\varepsilon}$ space: $\{u \in L^2(\mu^\varepsilon_\omega) \,:\, \nabla_\varepsilon u \in L^2(\nu^\varepsilon_\omega)\}$
- norm in $H^1_{\omega,\varepsilon}$: $\|u\|_{L^2(\nu^\varepsilon_\omega)} + \|\nabla_\varepsilon u\|_{L^2(\nu^\varepsilon_\omega)}$

Definition

Let $f \in L^2(\mu^\varepsilon_\omega)$, $\lambda > 0$.
$u \in H^1_{\omega,\varepsilon}$ is weak solution of

$$-\mathbb{L}^\varepsilon_\omega u + \lambda u = f \,,$$

if

$$\frac{1}{2}\langle \nabla_\varepsilon v, \nabla_\varepsilon u\rangle_{\nu^\varepsilon_\omega} + \lambda \langle v, u\rangle_{\mu^\varepsilon_\omega} = \langle v, f\rangle_{\mu^\varepsilon_\omega} \qquad \forall v \in H^1_{\omega,\varepsilon} \,.$$

Lax–Milgram theorem: u exists, unique

Effective diffusion matrix D

- Given $f : \Omega \to \mathbb{R}$, set $\nabla f(\omega, x) := f(\tau_x \omega) - f(\omega)$
- D: $d \times d$ symmetric matrix D such that

$$a \cdot Da = \inf_{f \in L^\infty(\mathbb{P}_0)} \frac{1}{2} \int d\mathbb{P}_0(\omega) \int_{x \in \hat{\omega}} c_{0,x}(\omega) \left(a \cdot x - \nabla f(\omega, x)\right)^2$$

- Macroscopic equation: $-\mathrm{div} D \nabla u + \lambda u = f$

Weak/strong convergence

- Fix $\omega \in \Omega$, $\{v_\varepsilon\}$ with $v_\varepsilon \in L^2(\mu_\omega^\varepsilon)$, $v \in L^2(mdx)$
- $v_\varepsilon \rightharpoonup v$:

$$\begin{cases} \sup \|v_\varepsilon\|_{L^2(\mu_\omega^\varepsilon)} < +\infty \,, \\ \lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x) v_\varepsilon(x) \varphi(x) = \int dx \, m v(x) \varphi(x) \,, \end{cases}$$

for all $\varphi \in C_c(\mathbb{R}^d)$.

Weak/strong convergence

- $v_\varepsilon \to v$:

$$\begin{cases} \sup \|v_\varepsilon\|_{L^2(\mu^\varepsilon_\omega)} < +\infty\,, \\ \lim_{\varepsilon\downarrow 0} \int d\mu^\varepsilon_\omega(x) v_\varepsilon(x) g_\varepsilon(x) = \int dx\, m v(x) g(x)\,, \end{cases}$$

for all $\forall g_\varepsilon \rightharpoonup g$

Weak/strong convergence

Example:

- take $v \in C_c(\mathbb{R}^d)$
- $v \in L^2(\mu_\omega^\varepsilon)$ and $v \in L^2(m dx)$
- set $v_\varepsilon := v$
- then $v_\varepsilon \to v$

Homogenization

- $\lambda_k(\omega) := \sum_i c_{0,x_i}(\omega)|x_i|^k$

Theorem

Assume $\mathbb{E}_0[\lambda_0^2] < \infty$, $\mathbb{E}_0[\lambda_2] < \infty$, D *strictly positive.*

Then $\exists \Omega_{\text{typ}} \subset \Omega$ *with* $\mathbb{P}(\Omega_{\text{typ}}) = 1$ *such that* $\forall \omega \in \Omega_{\text{typ}}$ *the following holds:*

Let $\lambda > 0$, $f_\varepsilon \in L^2(\mu_\omega^\varepsilon)$ *and* $f \in L^2(mdx)$.

Consider the weak solutions u_ε, u *of*

$$-\mathbb{L}_\omega^\varepsilon u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon \,,$$
$$-\text{div} D\nabla u + \lambda u = f \,.$$

...

Homogenization

$$-\mathbb{L}_\omega^\varepsilon u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon \, ,$$
$$-\,\mathrm{div} D \nabla u + \lambda u = f \, .$$

Theorem (Continuation)

Then:

(i) **Convergence of solutions**

$$f_\varepsilon \rightharpoonup f \implies u_\varepsilon \rightharpoonup u \, ,$$
$$f_\varepsilon \to f \implies u_\varepsilon \to u \, .$$

Homogenization

$$-\mathbb{L}_\omega^\varepsilon u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon \,,$$
$$-\operatorname{div} D \nabla u + \lambda u = f \,.$$

Theorem (Continuation)

Then:

(ii) **Convergence of flows**:

$$f_\varepsilon \rightharpoonup f \implies \nabla_\varepsilon u_\varepsilon \rightharpoonup \nabla u$$
$$f_\varepsilon \to f \implies \nabla_\varepsilon u_\varepsilon \to \nabla u$$

Homogenization

$$-\mathbb{L}_{\omega}^{\varepsilon} u_{\varepsilon} + \lambda u_{\varepsilon} = f_{\varepsilon}$$

$$-\operatorname{div} D \nabla u + \lambda u = f$$

Theorem (Continuation)

Then:

(iii) **Convergence of energies***:*

$$f_{\varepsilon} \to f \implies \langle \nabla u_{\varepsilon}, \nabla u_{\varepsilon} \rangle_{\nu_{\omega}^{\varepsilon}} \to \int dx\, m \nabla u(x) \cdot D \nabla u(x)$$

Convergence of semigroups

- $P^\varepsilon_{\omega,t}$: Markov semigroup of diffusively rescaled random walk
- P_t: Markov semigroup of Brownian motion with diffusion matrix D

Theorem

For any $\omega \in \Omega_{\rm typ}$, $t \geq 0$ *and* $f \in C_c(\mathbb{R}^d)$, *it holds*

$$\lim_{\varepsilon \downarrow 0} \int \left| P^\varepsilon_{\omega,t} f(x) - P_t f(x) \right|^2 d\mu^\varepsilon_\omega(x) = 0$$

$$\lim_{\varepsilon \downarrow 0} \int \left| P^\varepsilon_{\omega,t} f(x) - P_t f(x) \right| d\mu^\varepsilon_\omega(x) = 0\,.$$

2-scale convergence

- The proof is based on 2–scale convergence.
- In definition of weak/strong convergence, replace

$$\lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x) v_\varepsilon(x) \varphi(x)$$

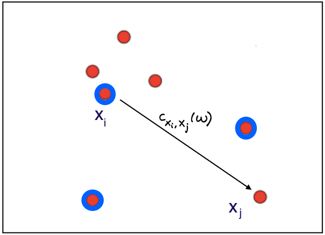
with

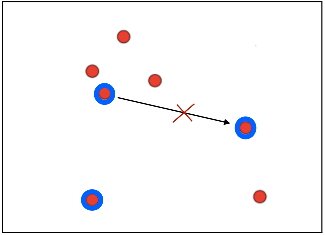
$$\lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x) v_\varepsilon(x) \varphi(x) g(\tau_{x/\varepsilon}\omega) \,.$$

- V.V. Zhikov, A.L. Pyatnitskii; *Homogenization of random singular structures and random measures.* Izv. Math. 70, (2006).
- F. Flegel, M. Heida, M. Slowik. Random conductance model. Weaker form of Thm.1–(i) under stronger assumptions.

Exclusion process

- **Population dynamics**
- **Interacting random walks: site–exclusion constraint**





Exclusion process

- $\eta \in \{0,1\}^{\hat{\omega}}$: particle confguration
- $c_{x,y}(\omega) \leq g(|x-y|)$, $g \in L^1(dx)$
- $\rho_0 : \mathbb{R}^d \to [0,1]$ macroscopic density profile
- $\rho(x,t)$ solution of Cauchy system

$$\begin{cases} \partial_t \rho = \mathrm{div}(D \cdot \nabla \rho)\,, \\ \rho(x,0) = \rho_0(x) \end{cases}$$

- $\mathfrak{m}_\varepsilon$: initial distribution of the exclusion process

Theorem

Let $\mathbb{P}$ *be the law of a marked Poisson point process.*

- *Suppose that* $\{\mathfrak{m}_\varepsilon\}$ *corresponds to* ρ_0,
  i.e. $\forall \delta > 0$ *and* $\forall \varphi \in C_c(\mathbb{R}^d)$

$$\mathfrak{m}_\varepsilon\Big(\Big|\varepsilon^d \sum_{x\in\hat{\omega}} \varphi(\varepsilon x)\eta_x - \int_{\mathbb{R}^d} \varphi(x)\rho_0(x)dx\Big| > \delta\Big) \to 0\,.$$

- *Then for all* $t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ *and* $\delta > 0$ *we have*

$$\mathbb{P}_{\omega,\mathfrak{m}_\varepsilon}\Big(\Big|\varepsilon^d \sum_{x\in\hat{\omega}} \varphi(\varepsilon x)\eta_x(\varepsilon^{-2}t) - \int_{\mathbb{R}^d} \varphi(x)\rho(x,t)dx\Big| > \delta\Big) \to 0\,.$$

Hydrodynamic limit of exclusion processes with symmetric jump rates

- K. Nagy; *Symmetric random walk in random environment.* Period. Math. Hung. 45, (2002).
- **If one looks only at a finite family of times, one mainly needs a weak form of convergence of semigroup (see Thm.2):**
  - A. Faggionato; *Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit.* EJP 13 (2008).
  - A. Faggionato; *Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster.* EJP 15 (2010).
- F. Redig, E. Saada, F. Sau; *Symmetric simple exclusion process in dynamic environment: hydrodynamics.* arXiv:1811.01366
  Tool: the invariance principle