Chaos propagation for ball into bins dynamics

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Outline

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Initial Motivation

- Work in progress with G. Posta.
- The terminology *Propagation of Chaos* comes from Kac.
- Initial motivation: try to investigate the connection between a detailed and a reduced description of particles' evolution.
- Quite often, we are not interested in a detailed description of a system of a large number of particles but rather in his collective behaviour

Initial Motivation

- It is necessary to introduce procedures leading to simplified models, retaining the interesting features of the original system, cutting away redundant informations
- Propagation of chaos is a central concept of kinetic theory introduced to relate the equations of Boltzmann and Vlasov to the dynamics of many-particle systems.
- The equations of kinetic theory are obtained by studying the limiting behaviour of N-particle systems as N tends to infinity. A key concept in such studies is the propagation of chaos.

Chaos Propagation

The idea proposed by Kac is

- One picks a *chaotic* initial distribution of particles:

$$u_N(0, v_1, \ldots, v_N) = u_0(v_1)...u_o(v_N)$$

  $u_N(0, v_1, \ldots, v_N)$ is the velocity density of N particles at time $t = 0$ with velocities v_i, $i = 1, \ldots, N$
- For fixed N the evolution due to the master equations will in general destroy the independence property of $v_1, \ldots, v_N$ at time t.

Chaos Propagation

- If one focuses on the reduced distribution at time t of the first k components, it should approximately happen

$$u_N(t, v_1, \ldots, v_k) \to u_t(v_1), \ldots, u_t(v_k) \quad N \to \infty$$

- $u_t(v)$ solution of a *non linear equation* initial condition $u_0(v)$
- In this sense independence (or chaos) propagates

Chaos Propagation

- The consequence is that the study of one individual gives information on the behavior of the group.
- The price to pay for this simplification is that the limiting process evolves accordingly to a nonlinear equation.

Some Past Literature on Chaos Propagation

- Propagation of chaos is a largely studied topic in literature see for example A-S. Sznitman: Saint-Flour XIX-1989.
- General results on propagation of chaos include diffusions with jumps under suitable Lipschitz conditions on coefficients see for example C. Graham 1992.

Simultaneous jumps

- In recent years, different applicative contexts have motivated the introduction of models whose components are allowed to jump simultaneously, and are therefore not covered by the results of Graham
- Propagation of chaos is not obvious in these models since simultaneous jumps could in principle interfere with asymptotic independence.

Outline

GRBB process

The *General Repeated Balls into Bins (GRBB)* process

$$(\eta^L(t))_{t \geqslant 0}$$

is a Markov chain with values in $\mathbb{Z}_+^L$:

- Consider a finite numbers of balls initially placed in an arbitrary way in L bins.
- At each time step $t = 1, 2, \ldots$ a ball is taken from each non-empty bin.
- Then, at time $t + 1$, each ball is reassigned into bins according to law which depends on the particle distribution at time t.
- The total number of balls is conserved.

GRBB process

- Is a conservative interacting particles system in discrete time with parallel updates.
- We can think the GRBB process as a zero-range process on the complete graph with constant jump rates and parallel updates
- Because of the parallel updating it is not reversible.
- For this reason its invariant measure is difficult to compute in most cases.

GRBB process

The GRBB process appears naturally in different applicative contexts

- We can think to balls in every bin as customers in a queue. Customers are served at discrete times and each served customer is reassigned to a random queue.
- The parallel updating is justified by thinking to customers as tasks (or tokens) in a network of parallel CPU which are reassigned at every round.

GRBB process

More precisely, define

- $\mathcal{P}(\mathbb{Z}_+)$ the space of probability measures on $\mathbb{Z}_+$ with the *total variation distance* $\|\cdot\|$.
- The *Empirical measure function* $\pi_L\colon \mathbb{Z}_+^L \to \mathcal{P}(\mathbb{Z}_+)$ as

$$\pi_L(\xi) := \frac{1}{L}\sum_{x=1}^{L} \delta_{\xi_x},$$

- The *map* $w^L\colon \mathbb{Z}_+^L \to \{0,1\}^L$

$$w^L(\xi) := (\mathbf{1}(\xi_1 > 0), \ldots, \mathbf{1}(\xi_L > 0)).$$

GRBB process

- Assume that for some $t \geqslant 0$ we have $\eta^L(t) = \xi$ and $q := \pi_L(\xi)$ then

$$\eta^L(t+1) = \xi - w^L(\xi) + B^{L,q},$$

- The random vector $B^{L,q}$ is independently generated at each time step t and satisfies

$$\sum_{x=1}^{L} B_x^{L,q} = (1 - q(\{0\}))L,$$

- so that *conservation of the number of particles* of the GRBB process holds.

The Non Linear Process

The *μ-nonlinear process* $(\eta(t))_{t \geqslant 0}$ is defined as

- Let q be the distribution of $\eta(t)$ for some $t \geqslant 0$
- Let B^q be a random variable with distribution μ^q independent from everything
- Assume $\eta(t) = \eta$

$$\eta(t+1) = \eta - \mathbf{1}(\eta > 0) + B^q.$$

- $\mathbb{E}(B^q) = m_{\mu^q} = 1 - q(\{0\})$.

Chaos Propagation

- Models whose components are allowed to jump simultaneously are not covered by the results of Graham
- Propagation of chaos is not obvious in these models, since simultaneous jumps could in principle interfere with asymptotic independence.

Definition of Chaos Propagation (GRBB process)

Chaos Propagation for the distribution of η^L (Sznitman):

- Consider a *chaotic* initial state for the GRBB process, *i.e.* an initial state sequence $\eta^L(0)$ with symmetric distribution such that the components become independent as $L \to +\infty$.
- The GRBB process preserves this property at each time t and if we look at the time evolution of $\eta^L(t)$ for $t \in [0, T]$, in the limit $L \to +\infty$, it behaves as the product of independent copies of the nonlinear process defined above.
- For all bounded functions Φ_k

$$\lim_{L \to +\infty} \mathbb{E}\left[\prod_{k=1}^{n} \Phi_{k\,t}(\eta^L)\right] = \prod_{k=1}^{n} \mathbb{E}\left[\Phi_{k\,t}(\eta)\right],$$

Goals on the GRBB process

We are interested in

- Proving *chaos propagation* for the GRBB process.
- A *quantitative estimate of the rate of convergence* of the empirical measure of the GRBB process

Outline

Sufficient Condition on the GRBB process

We introduce a sufficient condition to this aim

Sufficient Condition

- The distribution of $B^{L,q}$ is symmetric.
- For any $q \in \mathcal{P}(\mathbb{Z}_+)$, let $\nu_L^q \sim (B_1^{L,q}, B_2^{L,q})$. Assume there exists $\mu^q \in \mathcal{P}(\mathbb{Z}_+)$ and a constant C [1]such that

$$\sup_{q \in \mathcal{P}(\mathbb{Z}_+)} \|\nu_L^q - \mu^q \otimes \mu^q\| \leqslant \frac{C}{L}.$$

[1] *Constant* a positive number which does not depend on L

Theorem

Theorem - Chaos Propagation Rate - (N.C., G. Posta)

- $\eta^L(0)$ symmetric distribution and $\sup_L \mathbb{E}(\eta_1^L(0)) < +\infty$
- $B^q \sim \mu^q$ Lipschitz.
- $Q_L(t) := \pi_L(\eta^L(t))$ and $Q(t)$ distribution of $\eta(t)$.
- Assume $\exists C$ such that

$$\mathbb{P}\left(\|Q_L(0) - Q(0)\| > \delta\right) \leqslant \frac{C}{\sqrt{L}}.$$

- If the *Sufficient Condition* holds $\Longrightarrow \exists\, C'$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|Q_L(t) - Q(t)\| > \delta\right) \leqslant \frac{C'}{\sqrt{L}}.$$

Previous results for simultaneous jump mechanism

- Convergence of the empirical measure is equivalent to Chaos Propagation of Sznitman
- L. Andreis, P. Dai Pra, M. Fischer (2017) prove propagation of chaos for a wide class of models with simultaneous jumps
- Due to a different simultaneous jump mechanism, the ball into bins dynamics is not contained in this class and a different approach is needed.

Outline

Sketch of the Proof

- It is enough to show that for any $t \in [0, T]$ there exists a constant C such that

$$\mathbb{P}\left(\|Q_L(t) - Q(t)\| > \delta\right) \leqslant \frac{C}{\sqrt{L}}.$$

- We prove it by induction on t.
- By hypothesis it is true for $t = 0$.
- We assume it holds for some $t \geqslant 0$ and prove it for $t + 1$.
- Given $Q(t)$ then $Q(t+1) := F(Q(t))$ is Lipschitz by hypothesis

Sketch of the Proof

- Adding and subtracting terms we have that

$$\|Q_L(t+1) - Q(t+1)\| \leqslant \|Q_L(t+1) - \mathbb{E}[Q_L(t+1)|Q_L(t)]\| + \\ + \|\mathbb{E}[Q_L(t+1)|Q_L(t)] - F(Q_L(t))\| + \|F(Q_L(t)) - Q(t+1)\|,$$

- The third term can be bounded by Lipschitz condition.
- The second term is bounded observing that *Empirical Process* $(Q_L(t))_{t \geqslant 0}$ is a Markov chain with values in $\mathcal{P}(\mathbb{Z}_+)$ and using the sufficient condition.
- The bound the first term is a delicate calculation where is used the symmetry of the distribution of the initial state $\eta_L(0)$, conservation of the number of particles, and the sufficient condition.

Outline

Fermi-Dirac statistics

- $q \in \mathcal{P}(\mathbb{Z}_+)$
- μ^q Bernoulli distribution $p = 1 - q(\{0\})$
- $B^{L,q} \sim$ *Fermi-Dirac statistics* L sites, $N = (1 - q(\{0\}))L$ particles.
- Recall $X = (X_1, \dots, X_L) \sim$ *Fermi-Dirac statistics* with L sites $N \leqslant L$ particles

$$\mathbb{P}(X_1 = x_1, \dots, X_L = x_L) =$$

$$= \begin{cases} \binom{N}{L}^{-1} & \text{when } x_k \in \{0,1\}, k \in \{1,\dots,L\}, \text{ and } \sum_{k=1}^{L} x_k = N \\ 0 & \text{otherwise.} \end{cases}$$

Chaos Propagation Fermi-Dirac statistics case

- μ^q is 1-Lipschitz.
- The sufficient condition is assured by

  Theorem

  $X = (X_1, \dots, X_L) \sim$ Fermi-Dirac L sites, N particles.
  $\gamma_L^N \sim (X_1, X_2)$ and $\lambda^{N/L}$ Bernoulli $p = N/L$

$$\Longrightarrow \|\gamma_L^N - \lambda^{N/L} \otimes \lambda^{N/L}\| = \frac{2N}{L(L-1)}\left(1 - \frac{N}{L}\right).$$

- Which implies the sufficient condition

$$\sup_{q \in \mathcal{P}(\mathbb{Z}_+)} \|\nu_L^q - \mu^q \otimes \mu^q\| = \sup_{N/L \leqslant 1} \|\gamma_L^N - \lambda^{N/L} \otimes \lambda^{N/L}\| \leqslant \frac{1}{L}$$

Maxwell-Boltzmann statistics

- $q \in \mathcal{P}(\mathbb{Z}_+)$
- μ^q Poisson distribution $\lambda = 1 - q(\{0\})$
- $B^{L,q} \sim$ *Maxwell-Boltzmann statistics* L sites, $N = (1 - q(\{0\}))L$ particles.
- Recall $X = (X_1, \ldots, X_L) \sim$ *Maxwell-Boltzmann statistics* with L sites N particles

$$\mathbb{P}(X_1 = x_1, \ldots, X_L = x_L) = \binom{N}{x_1, \ldots, x_L} \frac{1}{L^N},$$

Chaos Propagation Maxwell-Boltzmann statistics case

- μ^q is Lipschitz.
- $B^{L,q} \sim$ *Maxwell-Boltzmann statistics* L sites, $N = (1 - q(\{0\}))L$ particles.
- The sufficient condition is assured by

  Theorem

  $X = (X_1, \ldots, X_L) \sim$ Maxwell-Boltzman L sites, N particles. $\gamma_L^N \sim (X_1, X_2)$ and $\lambda^{N/L}$ Poisson parameter $p = N/L$

$$\Longrightarrow \|\gamma_L^N - \lambda^{N/L} \otimes \lambda^{N/L}\| \leqslant \frac{4N}{L^2}.$$

- Which implies the sufficient condition

$$\sup_{q \in \mathcal{P}(\mathbb{Z}_+)} \|\nu_L^q - \mu^q \otimes \mu^q\| \leqslant \frac{1}{L}$$

Bose-Einsten statistics

- $q \in \mathcal{P}(\mathbb{Z}_+)$
- μ^q Geometric distribution on $\mathbb{Z}_+$ parameter $1/(2 - q(\{0\}))$
- $B^{L,q} \sim$ *Bose-Einstein statistics* L sites, $N = (1 - q(\{0\}))L$ particles.
- Recall $X = (X_1, \ldots, X_L) \sim$ *Bose-Einstein statistics* L sites N particles

$$\mathbb{P}(X_1 = x_1, \ldots, X_L = x_L) = \frac{\mathbf{1}\left(\sum_{k=1}^{L} x_k = N\right)}{\binom{L+N-1}{N}},$$

Chaos Propagation Bose Einstein statistics case

- μ^q is Lipschitz.
- $B^{L,q} \sim$ *Bose Einstein statistics* L sites, $N = (1 - q(\{0\}))L$ particles.
- The sufficient condition is assured by

  Theorem

  $X = (X_1, \dots, X_L) \sim$ Bose-Einstein L sites, N particles.
  $\gamma_L^N \sim (X_1, X_2)$ and $\lambda^{N/L}$ Geometric $p = 1/(1 + N/L$

  $$\implies \|\gamma_L^N - \lambda^{N/L} \otimes \lambda^{N/L}\| \leqslant \frac{14N}{L^2}.$$

- Which implies the sufficient condition

$$\sup_{q \in \mathcal{P}(\mathbb{Z}_+)} \|\nu_L^q - \mu^q \otimes \mu^q\| \leqslant \frac{1}{L}$$

Chaos Propagation and Equivalence of Ensemble

- The sufficient condition in these three cases can be seen as an equivalence of ensembles estimate for a non interacting unbounded spin system

$$\sup_{q\in\mathcal{P}(\mathbb{Z}_+)} \|\nu_L^q - \mu^q \otimes \mu^q\| \leqslant \frac{C}{L}.$$

  Is the difference between the canonical and the grand canonical mean of a local function.

- It seems interesting to investigate in which sense one can speak of chaos propagation in presence of interaction.