Random walk in a non-integrable random scenery time

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joint work with Marco Lenci & Françoise Pène

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Equilibrium and Non-equilibrium Statistical Mechanics
A conference in honor of François Dunlop

Outline

Anomalous diffusions

Anomalous diffusions are stochastic processes $X(t) \in \mathbb{R}^d$ that scale in time with exponent $\delta \neq 1/2$:

$$\mathbb{E}(|X(t)|^2) \sim t^{2\delta} \quad \text{for } t \to \infty, \quad \delta \neq 1/2$$

The behavior of superdiffusive processes ($\delta > 1/2$) characterizes many different natural systems and is mainly connected to motion in disorder media:

- light particle in an optical lattice;
- tracer in a turbolent flow;
- molecular diffusion in porous media.

Main features

- long ballistic “flights“
- short disorder motion

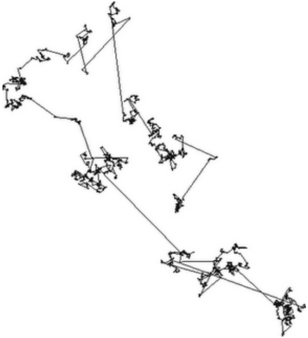

Figura : Typical Lévy flight

Models for anomalous diffusions

Schlesinger, Klafter['85] Schlesinger, Zaslavsky ['94], Barkai, Dubkov ['17]

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Lévy walks give rise to superdiffusive motion with

$$\mathbb{E}(|X(t)|^2) \sim \begin{cases} t^2 & \text{if } \alpha \in (0,1) \\ t^{3-\alpha} & \text{if } \alpha \in (1,2) \end{cases} \qquad \text{for } t \to \infty \qquad \text{(LÉVY SCHEME)}$$

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Good behavior but **naive models**: the lengths of the jumps are independent $\Longrightarrow$ the medium is renewed after each collision.

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$$n^{-1/\alpha}(\zeta_1+\ldots+\zeta_n)\xrightarrow{d} Z_1$$

with Z_1 α-stable r.v., $\alpha\in(0,2)$.

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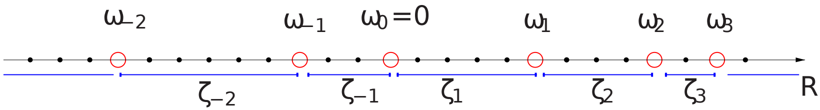

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(Lévy) Random environment

- Let $(\xi_j)_{j\in\mathbb{N}}$ i.i.d. integer r.v.'s with $\mathbb{E}(\xi_1)=0$ and $\mathbb{E}(\xi_1^2)<\infty$, and let

$$S=(S_n)_{n\in\mathbb{N}}\;:\quad S_0=0\,,\quad S_n=\sum_{j=1}^{n}\xi_j$$

Underlying RW on $\mathbb{Z}$

Discrete and continuous time processes

Discrete time process $Y = (Y_n)_{n\in\mathbb{N}}$ is the RW on ω coupled to S,

$$Y_n \equiv Y_n^{\omega} = \omega_{S_n}\,, \qquad \forall n \in \mathbb{N}$$

In other words Y_n is the position of point of ω labeled by S_n.

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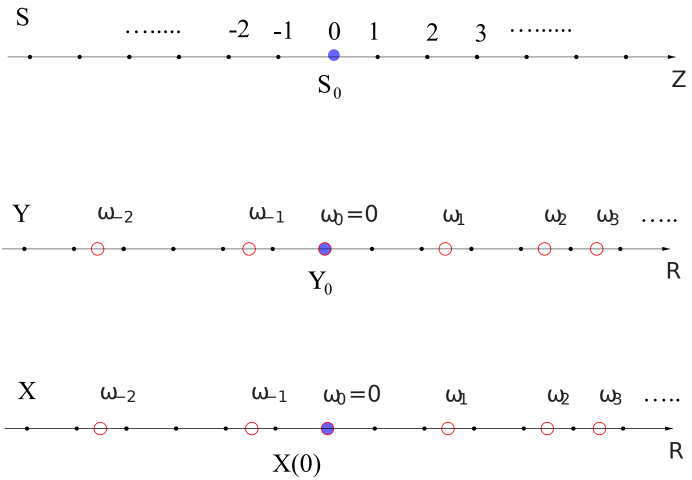
- for $t \in [T_n, T_{n+1})$, set

$$X_t := Y_n + \operatorname{sgn}(\xi_{n+1})(t - T_n),$$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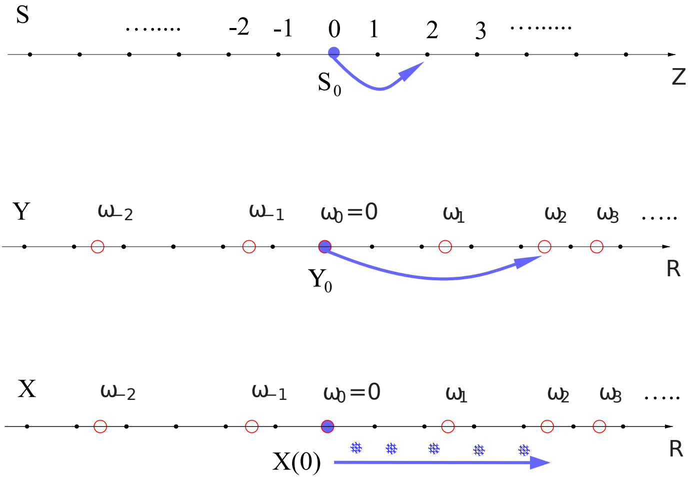
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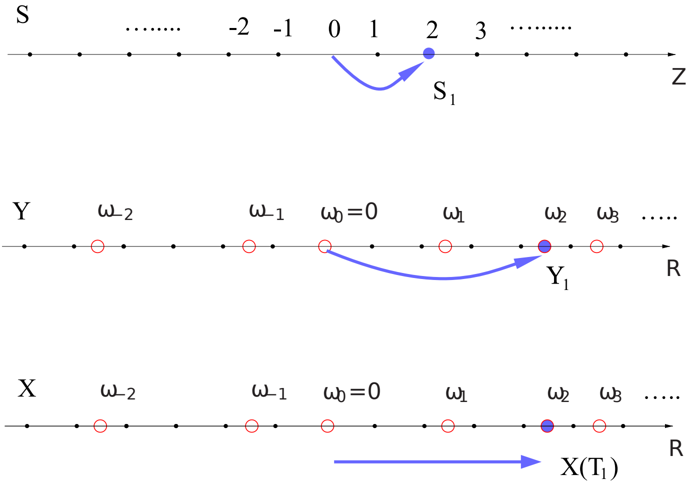
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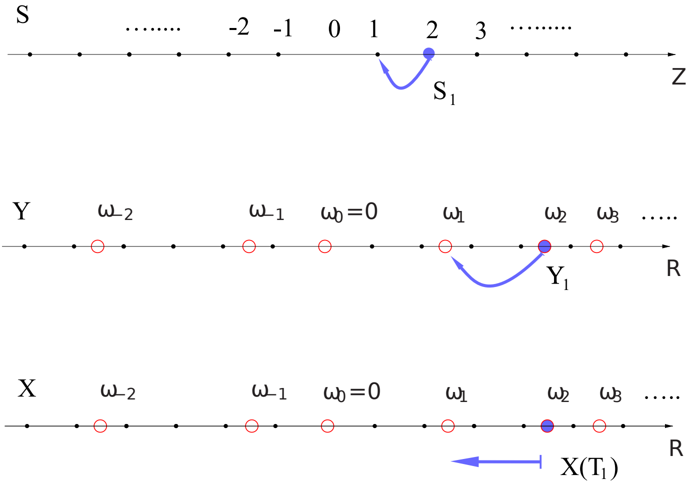
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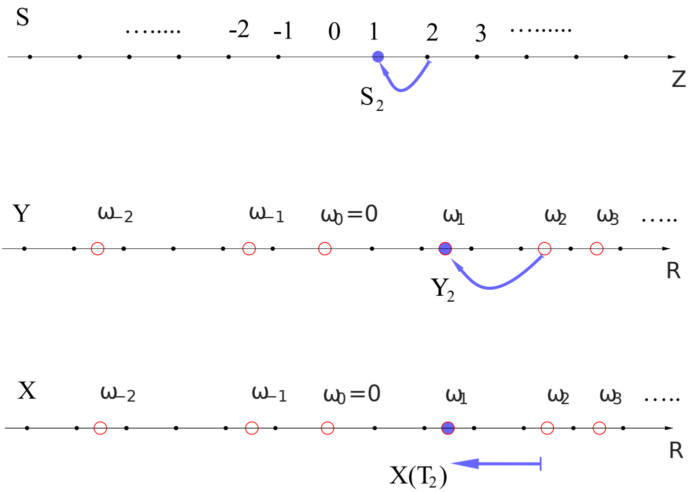
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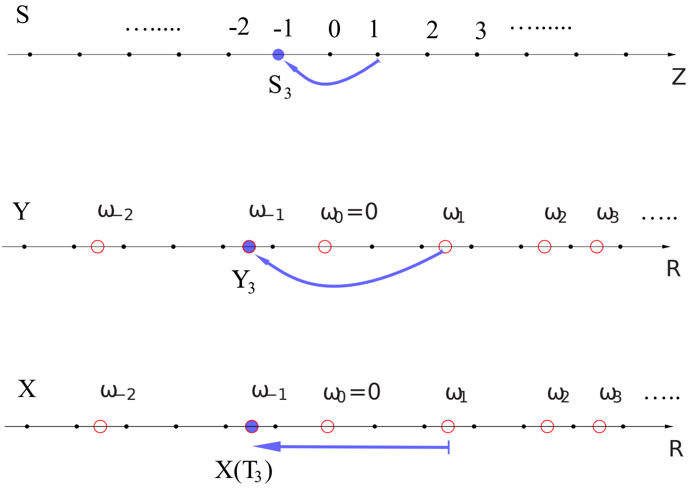
$X(T_2)$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We will consider:

- the quenched law of X_t denoted by P_ω, for any fixed environment ω.
- the annealed law of X_t denoted by $\mathbb{P}$ such that, for $G \in \mathcal{C}$, $F \in \Omega_{en}$

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Goal: Scaling limit of $(Y_n)_{n\in\mathbb{N}}$ and $(X_t)_{t\in\mathbb{R}^+}$.

Previous (annealed) results

- Barkai, Fleurer, Klafter ['00]: using Laplace transform and Tauberian theorem,

$$\mathbb{E}(X_t^2) \geq c(\alpha) t^{2-\alpha} \Longrightarrow \begin{cases} \text{if } \alpha \in (0,1) & \text{superdiffusive behavior} \\ \text{if } \alpha \in (1,2) & \text{at least diffusive behavior} \end{cases}$$

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- Beenakker, Groth, Akhmerov ['09,'12]: by heuristic arguments

$$\mathbb{P}(X_{\tau_{0,L}} = L)) \sim \begin{cases} L^{-\alpha} \log L & \text{if } \alpha \in (0,1) \quad \text{superdiffusive behavior} \\ L^{-1} & \text{if } \alpha \in (1,2) \quad \text{diffusive behavior} \end{cases}$$

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- Burioni, Caniparoli, Vezzani ['11]: heuristic arguments and simulations

$$\mathbb{E}(X_t^2) \sim \begin{cases} t^{\frac{2+2\alpha-\alpha^2}{1+\alpha}} & \text{if } \alpha \in (0,1) \quad \text{superdiffusive behavior} \\ t^{\frac{5}{2}-\alpha} & \text{if } \alpha \in [1, \frac{3}{2}] \quad \text{superdiffusive behavior} \\ t & \text{if } \alpha \in (\frac{3}{2}, 2) \quad \text{diffusive behavior} \end{cases}$$

Results are not in agreement for $\alpha \in (1,2)$.

Case $\alpha \in (1, 2)$: Finite mean, infinite variance

- Berger, Rosenthal ['13] show that if $\mu = \mathbb{E}(\zeta)$, then for P-a.e. ω

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Let $\mu = \mathbb{E}(\zeta)$. For P-a.e. ω

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  $\longrightarrow$ *convergence of finite-dimensional distributions follows*
  $\longrightarrow$ *The annealed CLT then follows trivially*
- *quenched moments convergence of $Y_n/\sqrt{n}$ and $X_t/\sqrt{t}$ to moments of their respective limit in law.*

Case $\alpha \in (0,1)$: infinite mean and variance

Collision times can be written as

$$T_n := \sum_{k=1}^{n} |\omega_{S_k} - \omega_{S_{k-1}}| = \sum_{k \in \mathbb{Z}\setminus\{0\}} \mathcal{N}_n(k)\zeta_k$$

where $\mathcal{N}_n(k) = \sharp\{j \in \{0,\dots,n\} : [k,k+1] \subseteq [S_{j-1}, S_j]\}$

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[Kesten, Spitzer '79] $\quad Z_n = \sum_{j=0}^{n} \zeta_{S_j} = \sum_{k\in\mathbb{Z}} N_n(k)\zeta_k\,, \quad n \in \mathbb{N}$

with $(\zeta_k)_{k\in\mathbb{Z}}$ i.i.d. , $(\xi_j)_{j\in\mathbb{N}}$ i.i.d. , $S_n = \sum_{j=1}^{n} \xi_j$ and
$N_n(k) = \sharp\{j \in \{0,\dots,n\} \,:\, S_j = k\}$.

Convergence of times $(T_n)_{n\in\mathbb{N}}$

Let $B = (B_t)_{t\in\mathbb{R}^+}$ be a **Brownian motion** with $B_t \sim N(0, \mathrm{Var}(\xi)t)$
Let $L_t = (L_t(x))_{x\in\mathbb{R}}$ its **local time** for $t \in \mathbb{R}^+$
Let $Z_\pm = (Z_\pm(x))_{x\in\mathbb{R}^+}$ two i.i.d α-stable processes s.t. $Z_\pm(1) \sim Z_1$
and $Z = \mathbb{1}_{\{}x > 0\}Z_+(x) - \mathbb{1}_{\{}x < 0\}Z_-(x)$. Finally set

$$\Delta(t) = \int_{-\infty}^{+\infty} L_t(x) dZ(x)$$ Kesten-Spitzer process

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Assumption on the underlying RW: $\mathbb{E}(|\xi_1|^{2/\alpha+\varepsilon}) < \infty$.

Proposition 1 (B., Lenci, Pène - SPA '19).

Let $\alpha \in (0,1)$. Under $\mathbb{P}$, and taking $n \to \infty$, it holds

$$\left(\frac{T_{[ns]}}{n^{\frac{1+\alpha}{2\alpha}}} \right)_{s\in\mathbb{R}^+} \xrightarrow{w} \Delta \quad \text{in } D(\mathbb{R}^+, J_1) .$$

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Goal: Convergence of processes

$$\bar{Y}^{(n)}(t) = \frac{Y_{[nt]}}{n^{1/2\alpha}}, \quad \bar{X}^{(n)}(t) = \frac{X_{nt}}{n^{1/(\alpha+1)}}$$

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But $\quad \bar{Y}^{(n)}(t) = \bar{\omega}^{(\sqrt{n})} \circ \bar{S}^{(n)}(t) \xrightarrow{w} Z \circ B$

- $\bar{\omega}^{(n)} = \left(\frac{\omega_{[nx]}}{n^{\frac{1}{\alpha}}}\right)_{x\in\mathbb{R}} \xrightarrow{w} Z$ α – stable process
- $\bar{S}^{(n)} = \left(\frac{S_{[nt]}}{n^{\frac{1}{2}}}\right)_{t\in\mathbb{R}^+} \xrightarrow{w} B$ invariance principle
- $\bar{T}^{(n)} = \left(\frac{T_{[ns]}}{n^{\frac{1+\alpha}{2\alpha}}}\right)_{s\in\mathbb{R}^+} \xrightarrow{w} \Delta$ proposition 1

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Convergence of processes Y and X

Theorem 2 (B., Lenci, Pène - SPA '19).

Let $\alpha \in (0,1)$. Under $\mathbb{P}$, and taking $n \to \infty$, the finite-dimensional distributions of

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  Indeed, due to non-integrable environment, the RW Y may cross a gap ζ_k of the order of the total sum of gaps visited up to there, and then jumps back and forth many times. In the limit, the variation of position across these gaps will be of finite order and create accumulation of discontinuities.

Case $\alpha \in (1, 2)$: Proof ideas

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   - By the ergodicity of the process seen from the particle:

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{} \mu, \qquad \frac{\ell(t)}{t} \xrightarrow[t\to\infty]{} 1/\mu, \qquad \mathbb{P} - a.s$$

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Convergence of finite dimensional distributions are based on characteristic functions.

Key point: (T_n) behaves in the limit as a RWRS, converging to the Kesten-Spitzer process Δ.

Characteristic function of T_n

- $\bar{T}^{(n)}(s) = \frac{1}{n^{(1+\alpha)/2\alpha}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{N}_{[ns]}(k)\zeta_k$
- $\phi_\zeta(\theta) = \exp[-c_1|\theta|^\alpha(1 - \imath c_2 \mathrm{sgn}\theta)]$ *(Hyp. $\zeta \sim \alpha$-stable,)*

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$$\mathbb{E}[\exp(\imath\theta \bar{T}^{(n)}(s)) \mid S] = \prod_{k \in \mathbb{Z}\setminus\{0\}} \phi_\zeta\left(\theta \frac{\mathcal{N}_{[ns]}(k)}{n^{(1+\alpha)/2\alpha}}\right)$$

$$= \exp\left(-c_1|\theta|^\alpha(1 - \imath c_2 \mathrm{sgn}\theta) \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{\left(\mathcal{N}_{[ns]}(k)\right)^\alpha}{n^{(1+\alpha)/2}}\right)$$

On the other hand

$$\mathbb{E}[\exp(\imath\theta\Delta(s))] = \mathbb{E}\left[\exp\left(-c_1(1-\imath c_2 \text{sgn}\theta)|\theta|^\alpha \int_{\mathbb{R}} (L_s(x))^\alpha dx\right)\right]$$

and one has to show

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- $\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathbb{E}\left[|\mathcal{N}_n(k) - \mathbb{E}[|\xi|]N_n(k)|^\alpha\right] = o(n^{(1+\alpha)/2})$
- results and strategy implemented in [Kesten Spitzer, ’79]

Conclusions

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- For $\alpha \in (1,2)$ (integrable environment) we prove quenched CLT and quenched moment convergence for discrete and continuous time process .
  $\Longrightarrow$ quenched diffusive behavior.
- For $\alpha \in (0,1)$ (non-integrable environment) we establish a functional limit theorem for discrete and continuous time.
  $\Longrightarrow$ annealed superdiffusive behavior.

Open problems

- Annealed moments
  - comparison with previous estimates and simulations;
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- Moment assumption over the underlying RW.
- What happens in dimension $d \geq 2$?
  - definition of a 2 D-Lévy environment, with distances between points provided by i.i.d. α-stable r.v.;
  - comparison with 2D and 3D- models on Lévy-like environments. Buonsante, Burioni, Vezzani ['11]

Thank you for your attention!