Functional Equations in LIMoges (FELIM) 2019

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Functional Equations in LIMoges (FELIM) 2019 is the twelfth in a series of annual international gatherings for researchers in functional equations. This conference, held annually at the University of Limoges since 2008, aims to present recent advances in symbolic or symbolic-numeric algorithms which treat systems of linear or nonlinear, ordinary or partial, differential equations, (q-)difference equations,... Additionally, FELIM emphasizes on the development state of related software implementations and the publicity of such codes. Topics include, but are not limited to, ordinary differential equations, difference equations, partial differential equations; integration of dynamical systems; methods for local solving (formal, symbolic-numeric, modular); methods for global solving or simplification (e.g., decomposition, factorisation); applications and software applications.

The conference website with links to the previous editions can be found at:

https://indico.math.cnrs.fr/event/4208/

1 Invited talks abstracts

Veronika Pillwein (Johannes Kepler University, Linz, Austria)
Functional equations in inequality proving.

`Monday, March 25, 10:00-11:00`

Expressions that satisfy difference, differential, or mixed relations appear very commonly in applications in Natural Sciences. It is well known that for many of these objects, defining identities can be derived and proven algorithmically. When it comes to proving inequalities involving expressions of this type, there are only few symbolic approaches to date. In this talk we present several examples of computer proofs that heavily rely on the application of algorithms for discovering and proving difference and differential equations.

Carole El Bacha (Lebanese University, Beirut, Lebanon)
On the computation of simple forms and regular solutions of linear difference systems.

`Tuesday, March 26, 9:30-10:30`

Let Δ denote the difference operator whose action on a function f is defined by

$$\Delta(f(z)) = (z-1)(f(z) - f(z-1)).$$

We consider first-order linear difference systems having a singularity at infinity of the form

$$(S): \quad D(z)\,\Delta(\mathbf{y}(z)) + A(z)\mathbf{y}(z) = 0,$$

where $\mathbf{y}(z)$ is the vector of unknowns and $D(z)$ and $A(z)$ are square matrices with factorial series entries such that $\det(D(z)) \not\equiv 0$. System (S) is said to be *simple* if

$$\det(D(\infty)\lambda + A(\infty)) \not\equiv 0.$$

In this talk, we develop an algorithm that transforms any first-order linear difference system with factorial series coefficients into a simple system. The computation of simple forms is crucial for the following two tasks. First, it allows to recognize the nature of the singularity at infinity: indeed, a simple system of the form (S) has a regular singular if and only if the matrix $D(\infty)$ is invertible. In this case, we are reduced to a system of the first kind. Second, it is a first step toward the computation of regular solutions. We then introduce new methods for computing regular solutions. We devote a particular study to systems of the first kind and provide a direct algorithm for computing a formal fundamental matrix of solutions of such systems. We also describe a new approach for computing regular solutions of simple linear difference systems. This method searches for regular solutions written as

$$\mathbf{y}(z) = \sum_{m\geq 0} z^{-[\rho+m]}\mathbf{y}_m(z), \qquad z^{-[\rho]} = \frac{\Gamma(z)}{\Gamma(z+\rho)}, \qquad \rho \in \mathbb{C},$$

where Γ stands for the usual Gamma function and $\mathbf{y}_m(z)$ is a finite linear combination of functions ϕ_i (to be defined) with constant vector coefficients. It reduces the problem of computing such solutions to solving linear difference systems with constant coefficients. Note that this method can be applied to any simple system whatever is the nature of the singularity $z = \infty$. This is a joint work with M. Barkatou and T. Cluzeau.

Fernando Sanz Sánchez (University of Valladolid, Spain)
Real Turrittin's theorem and applications to trajectories of vector fields.

`Tuesday, March 26, 14:30-15:30`

A classical result known as Turrittin's Theorem establishes that a system of linear ODEs with formal meromorphic coefficients

$$Y' = A(x)Y, \quad A(x) \in \mathcal{M}_{n\times n}(\mathbb{C}[[x]][x^{-1}])$$

can be transformed into a system in normal form $Z' = B(x)Z$, where all coefficients of $B(x)$ are diagonal matrices up to the Poincaré rank of the system, by means of polynomial gauge transformations and ramifications. In this talk, we present a version of this result in the case of the real base field, where we require that all the transformations are given by matrices with real coefficients. This is a joint work with M. Barkatou and F. A. Carnicero.

In the second part of the talk, we present a result concerning singularities of vector fields where the use of real Turrittin's Theorem is essential. Namely, we show that to any given formal invariant curve Γ of a real analytic vector field ξ at $(\mathbb{R}^n, 0)$ there corresponds an actual trajectory of ξ accumulating to 0 and asymptotic to Γ. This is a joint work with F. Cano and O. LeGal.

Marius van der Put (University of Groningen, The Netherlands)
Solutions of first order autonomous differential equations.

`Wednesday, March 27, 9:30-10:30`

The problem of algebraic dependence of solutions to first order autonomous equations over an algebraically closed field of characteristic zero is given a 'complete' answer. The geometry of curves and generalized Jacobians provides the key to algebraic dependence. Classification and formal solutions of autonomous equations are treated.

This contrasts recent work on this theme with similar results on algebraic dependence, where the model theory of differentially closed fields of characteristic zero is the main technical background.

2 Contributed talks abstracts

Antonio Jiménez-Pastor (Johannes Kepler University, Linz, Austria)
D^n-finite functions: a growing chain.

`Monday, March 25, 11:30-12:00`

A formal power series $f(x) \in K[[x]]$ is differentially definable over a ring R if it satisfies a linear differential equation with coefficients in such ring, and we say $f(x) \in D(R)$. This means, there are $d \in \mathbb{N}$ and $r_0, ..., r_d \in R$ such that:

$$r_d(x)f^{(d)}(x) + ... + r_0(x)f(x) = 0.$$

The classical example of these functions are the so called *D-finite* functions ($R = K[x]$). Using the closure properties of the differentially definable functions, we can iterate this construction. We call *D^n-finite* functions the nth iteration of this construction, namely, $D^n(K[x])$.

$$K[x] \subset D(K[x]) \subset D^2(K[x]) \subset \cdots \subset D^n(K[x]) \subset \ldots.$$

In this talk we are going to present the main properties of this chain of functions and, in particular, show that it is a proper chain (i.e., $D^n(K[x]) \subsetneq D^{(n+1)}(K[x])$) using some results from differential Galois theory.

Tarik Chakkour (INRA, Clermont-Ferrand, France)
A continuous-in-time financial model for public institutions.

`Monday, March 25, 14:30-15:00`

The continuous-in-time financial model, first introduced by Sundaresan in [1], constitutes a powerful tool for studying the development of continuous-in-time methods in finance. These include computational and estimation methods to test and implement continuous-time models.

In recent papers [2, 5] we have also constructed a continuous-in-time model which is designed to be used for the finances of public institutions and not for the financial market. This model permits to set out annual and multiyear budgets for any organization and describes working of loan and repayment in order to forecast its future financial plans. We build this model because of the default of a discrete model which is using tables. This model relies on measure theory and uses the mathematical tools such convolution and integration, etc. In addition, we check the mathematical consistency of this model in [3, 4, 6, 7]. We use the strategy elaboration phase to better adjust the project implementation. In this presentation, we will show the mathematical framework involving some partial differential equation (PDE).

Keywords: Mathematical computation; algorithms; discrete mathematics; software tool, PDE, inverse problem, measure theory.

References

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Luc Pirio (LMV, UMR 8100 - CNRS & Université Versailles - St. Quentin)
Polylogarithms, webs and cluster algebras.

`Monday, March 25, 15:00-15:30`

I will explain how, using methods from "web geometry", it is possible to obtain interesting results about functional equations of polylogarithms, in connection with the theory of cluster algebras.

Ana Rojo-Echeburúa (University of Kent, Canterbury, UK)
Variational systems with an euclidean symmetry using the rotation minimising frame.

`Monday, March 25, 15:30-16:00`

We show how to adapt the methods of Gonçalves and Mansfield [1, 2] to study variational systems with an Euclidean symmetry, using the Normal, or Rotation Minimising frame defined in Wang and Joe [3] and Wang, Jüttler, Zheng and Liu [4]. The Rotation Minimising frame has many advantages when considering the evolution of curves in computer aided design environments. Lie group based moving frames is the subject of many recent studies, however, the powerful symbolic computational methods derived for them which have proved so useful in the applications, do not apply to the Normal frame, as this frame is not defined by algebraic equations on the jet variables at a given point, but rather is defined by a differential equation. In this talk, we present the Rotation minimising frame and show how to use the known symbolic techniques despite the fact that it does not readily fit the known framework needed for these techniques. We derive the invariant differentiation formulae and the syzygy operator needed to obtain Noether's laws for variational problems with a Euclidean symmetry using the Rotation minimising frame and present some application in biological problems such as the modelization of proteins, nucleid acids and polymers.

References

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Tsvetana Stoyanova (Sofia University, Bulgaria)

Stokes matrices via monodromy matrices of a class of reducible equations.

`Monday, March 25, 16:30-17:00`

In this talk I will present a partial extension of the results obtained in [1]. In particular, we consider a third-order reducible equation

$$L\,y = L_3(L_2(L_1(y))) = 0 \tag{1}$$

$$L_j = \partial - \left(\frac{\alpha_j}{x} + \frac{\beta_j}{x^2}\right), \qquad \partial = \frac{d}{dx}\,,$$

where $\alpha_j, \beta_j \in \mathbb{C}$ are arbitrary such that $\beta_i \neq \beta_j$ for $i \neq j$ and $(\beta_1, \beta_2, \beta_3) \neq (0,0,0)$. The equation (1) has over $\mathbb{CP}^1$ two singular points: $x = 0$ is an irregular point of Poincaré rank 1 and $x = \infty$ is a regular point. Introducing a complex parameter ε, $0 < |\sqrt{\varepsilon}| < 1$, that split the irregular singularity into two finite Fuchsian singular points $x_L = -\sqrt{\varepsilon}$ and $x_R = \sqrt{\varepsilon}$, we perturb equation (1) to a third-order reducible Fuchsian equation

$$L(\varepsilon)\,y \;=\; 0, \quad L(\varepsilon) = L_3(\varepsilon) \circ L_2(\varepsilon) \circ L_1(\varepsilon), \tag{2}$$

$$\begin{aligned} L_j(\varepsilon) &= \partial - \left(\tfrac{\alpha_j}{2} + \tfrac{\beta_j}{2\sqrt{\varepsilon}}\right)\tfrac{1}{x-\sqrt{\varepsilon}} - \left(\tfrac{\alpha_j}{x} - \tfrac{\beta_j}{x^2}\right)\tfrac{1}{x+\sqrt{\varepsilon}}, \\ L_j(0) &= L_j\,. \end{aligned}$$

In concordance with equation (2), we consider equation (1) under assumption that it has at most two singular directions. Denote by θ_R this singular direction that belongs to $(-\pi/2, \pi/2]$, and by θ_L the singular direction $\theta_R + \pi$. Supposedly, $\theta_L \in (\pi/2, 3\pi/2]$. Denote by St_{θ_R} and St_{θ_L} the corresponding Stokes matrices. In this talk I will show that during a resonance when $\sqrt{\varepsilon} \to 0$ radially and at least one of the numbers $\alpha_2 - \alpha_1$, $\alpha_3 - \alpha_2$ belongs to $\mathbb{Z}_{\leq -2}$ the matrices $e^{2\pi T_j}$ $j = R, L$, where $M_j(\varepsilon) = e^{2\pi T_j} e^{\pi i(\Lambda + \frac{1}{x_j} Q)}$ are the monodromy matrices of the equation (2), tend to the Stokes matrices $St_{\theta_j}, j = R, L$ of the equation (1).

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References

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Cyril Banderier (CNRS, Université de Paris 13, LIPN, France)
PDEs, D-finite functions, urns model and limiting surface of Young tableaux.

`Monday, March 25, 17:00-17:30`

In order to analyze the limiting surface of random triangular Young tableaux, we introduce a generalization of Pólya urns, with some periodicity constraints.

It is in fact possible to encode the dynamics of the process via a differential polynomial, and thus the enumeration leads to a PDE. Our generalization kills the direct integrability of the associated PDE, which was the key of the seminal articles of Flajolet et al on Pólya urns.

However, the model is still solvable, and we get that periodic urns give PDEs, which are in fact non-trivially D-finite functions, related to sums of hypergeometric functions.

It is also interesting that the asymptotics are not given by a work of Flajolet and Lafforgue on differential equations (some confluence/resonance is messing the situation), so we get a new universal law, in terms of product of generalized gamma functions, which will systemically appear for this type of functional equations.

Joint work of Cyril Banderier/Philippe Marchal/Michael Wallner.

Ali El Hajj (Université de Limoges, XLIM, France)
Simple forms and rational solutions of pseudo-linear systems.

`Tuesday, March 26, 11:00-11:30`

Extended from the ideas developed in [1] and [2] for differential and difference systems, we develop a unified algorithm for computing simple forms of pseudo-linear systems of the form $A\delta Y + B\phi Y = 0$, where A and B are in some ring of power series, with δ a *pseudo derivation* and ϕ an automorphism holding suitable properties. We show that this permits to provide an efficient alternative to previous algorithms for computing rational solutions of pseudo-linear systems. Moreover, we develop a new algorithm for computing rational solutions of several pseudo-linear systems in several variables. This is a joint work with M. Barkatou and T. Cluzeau.

References

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Diego Dominici (Johannes Kepler University, Linz, Austria)
Orthogonal polynomial solutions of differential-difference equations.

`Tuesday, March 27, 11:30-12:00`

We consider solutions of a first-order differential-difference equation with polynomial coefficients. It is well known that all the classical orthogonal polynomials (Hermite, Laguerre, Jacobi, Bessel) satisfy such equations. In this talk we address the question: are these families the only ones? We use a combination of analytic and symbolic computation methods to find an answer. This is a joint work with V. Pillwein.

Thierry Combot (Université de Bourgogne, France)
Hyperexponential forms and the Poincare problem.

`Tuesday, March 27, 16:00-16:30`

Consider a solvable polynomial vector field X on the plane. The Poincaré problem is to find a bound on the degree of rational first integrals of X. Using the hypothesis of solvability, we build a bound on the degree of polynomial first integrals depending on X and its differential invariant F. We prove that such integral admits a compact representation, i.e. depending only on the degree of F, X. We will present the remaining challenges for the rational case.

Sergei A. Abramov (Dorodnicyn Computing Centre, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia)
When the search for solutions can be terminated.

`Tuesday, March 27, 16:30-17:00`

As a rule, search algorithms for those solutions of differential equations and systems that belong to a fixed class of functions are designed so that nonexistence of solutions of the desired type is detected only in the last stages of the algorithm. In some cases, performing additional tests on the intermediate results makes it possible to stop the algorithm as soon as these tests imply that no solutions of the desired type exist. We will consider these questions in connection with the search for rational solutions of linear homogeneous differential systems with polynomial coefficients. (Some approaches are already known for the case of scalar equations.)

Johannes Middeke (Johannes Kepler University, Linz, Austria)
A family of denominator bounds for first order linear recurrence systems.

`Wednesday, March 27, 11:00-11:30`

For linear recurrence systems, the problem of finding rational solutions is reduced to the problem of computing polynomial solutions by computing a content bound or a denominator bound. While sharper bounds lead to polynomial solutions of lower degrees, this advantage need not compensate for the time spent on computing that bound. To strike the best compromise (sharpness of the bound versus time spent on it) we will give a family of bounds. The setting for our content bounds includes the shift case, the q-shift case, the multi-basic case and others. We give two versions, a global version, and a version that bounds each entry separately. This is a joint work with Mark van Hoeij.

Camilo Sanabria (Universidad de los Andes, Bogota, Colombia)
Solving algebraic LODEs in terms of a finite family of functions.

`Wednesday, March 27, 11:30-12:00`

F. Klein showed that if the projective differential Galois group of a second order ordinary linear differential equation is finite then a full system of solutions can be expressed in terms of a finite family of hypergeometric functions precomposed by a rational function and multiplied by a hyperexponential function. Generalizations of this theorem for higher order equations have been obtained by M. Berkenbosch and by the speaker. The drawback of these generalizations is that they sacrifice the finiteness of the family of functions necessary to express the solutions. We overcome this drawback by allowing linear combinations of a finite family of functions precomposed by an algebraic function of tamed degree and multiplied by a hyperexponential function.