$\mathcal{D}$-modules and Bernstein-Sato polynomials.

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1 Introduction to functional equations.

Let us consider a ring R equal to $K[X_1, \cdots, X_n]$ where K is a field of characteristic zero, or to $K[X_1, \cdots, X_n]_a \subset K(X_1, \cdots, X_n)$, the set of algebraic function regular at $a \in K^n$, or to the set $\mathcal{O} = \mathbb{C}\{x_1, \cdots, x_n\}$ of convergent power series at the origin of $\mathbb{C}^n$, or more generally $\mathcal{O} = \mathcal{O}_{X,p}$, or to the ring $K[[X_1, \cdots, X_n]]$. We consider the ring $\mathcal{D}_R$ of of finite order linear differential operators with coefficients in R.

One can also replace R by the ring $\mathcal{O}$ of holomorphic functions defined on a open subset $U \subset \mathbb{C}^n$, and consider again a ring of operators $\mathcal{D}(U)$. However the next theorem is valid only on $\mathcal{D}(U')$, for relatively compact stein open set $U' \subset U$.

Theorem 1.1 *Given any $f \in R$, there exists a non zero polynomial $e(s)$ and a differential operator $P(s) \in \mathcal{D}_R[s]$ polynomial in the indeterminate s such that*

$$\boxed{P(s)f^{s+1} = e(s)f^s} \tag{1}$$

The set of polynomials $e(s)$ for which an equation of the type (1) exists is clearly an *ideal* B_f of $\mathbb{C}(s)$, which is principal since $\mathbb{C}[s]$ is a principal domain.

The *Bernstein-Sato polynomial* of f is by definition the monic generator of this ideal denoted $b_f(s)$ or simply $b(s)$:

$$B_f = \mathbb{C}[s] \cdot b_f(s)$$

Even if this definition is essentially punctual we shall systematically work with the sheaves $\mathcal{O}_X$ and $\mathcal{D}_X$ on a smooth algebraic variety or an analytic manifold.

Two generalisations :

I) Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{C}^n}$- module and $m \in \mathcal{M}$ the germ of a section in $\mathcal{M}$. Then working in $\mathcal{M}\left[\frac{1}{f}, s\right] = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}\left[\frac{1}{f}, s\right]$ we have :

Theorem 1.2 *Given any $f \in \mathcal{O}_{\mathbb{C}^n,0}$ there exists a non zero polynomial $e(s)$ and a differential operator $P(s) \in \mathcal{D}[s]$ polynomial in the indeterminate s such that*

$$\boxed{P(s)mf^{s+1} = e(s)mf^s} \tag{2}$$

We denote $b_m(s)$ the monic generator of the ideal of those e.

II) Let $f_1, \cdots, f_p$ be p elements in R, excluding the case of $K[[X_1, \ldots, x_n]]$. Then there exists a non zero polynomial $b(s_1, \cdots, s_p) \in \mathbb{C}[s_1, \cdots, s_n]$, and a functional equation :

$$\boxed{b(s_1,\cdots,s_p)f_1^{s_1}\cdots f_p^{s_p} = P(s_1,\cdots,s_p)f_1^{s_1+1}\cdots f_p^{s_p+1}} \tag{3}$$

with

$$b\in\mathbb{C}[s_1,\cdots,s_p]\,,\quad P\in\mathcal{D}[s_1,\cdots,s_p]$$

The set of polynomials $b(s_1,\cdots,s_p)$ as in (3) is an ideal $\mathcal{B}_{(f_1,\cdots,f_p)}$ of $K[s_1,\cdots,s_p]$.

In [10] Budur and al define the notion of Bernstein polynomial along an arbitrary variety $Z\subset X$ defined by $f_1,\ldots,f_p$ as a subvariety of a smooth manifold. We set $s = s_1+\cdots+s_p$ and let b be the smallest monic polynomial such that

$$b_f(s)f_1^{s_1}\cdots f_p^{s_p} = \sum_{a_1+\cdots+a_p=+1} P_a(s_1,\cdots,s_p)f_1^{s_1+a_1}\cdots f_p^{s_p+a_p}$$

with a finite sum in the RH side. The existence of such an equation follows from an application to $L(s)=s_1+\cdots+s_p$ of the methods developped by Sabbah in [29] to build step by step a multivariable b-function.

Theorem 1.3 *The polynomial $b_Z(s)=b_f(s-codim_C Z)$, in intrinsic i.e. depend neither of the embedding neither of the choice of an equation defining Z with is possibly non euced structure.*

History of the existence theorem

This polynomial was simultaneously introduced by Mikio Sato in a different context in view of giving functional equations for relative invariants of prehomogeneous spaces and of studying zeta functions associated with them, see [32], [33]. The name b-function comes from this theory and the so-called a,b,c functions of M. Sato, see [34] for definitions. The existence of a nontrivial equation as stated in (1), was first proved by I.N. Bernstein in the polynomial case, see [3]. The polynomials $b_f(s)$ are called Bernstein-Sato polynomials in order to take this double origin into account. The analytic local case is due to Kashiwara in [17]. An algebraic proof by Mebkhout and Narvaez can be found in [27]. The formal case is given by Björk in his book [6].

Nontrivial polynomials as in (3) were first introduced by C. Sabbah, see [29]. In the algebraic case the proof is a direct generalisation of the proof of Bernstein. For the analytic case see [29], completed by Bahloul in [2] where is shown the necessity of using a division theorem proved in [1]. There is by [29], [2] a functional equation (3) in which the polynomial $b(s_1,\cdots,s_p)$ is a product of a finite number of affine forms. It is as far as I know an unsolved problem to state the existence of a system of generators of $\mathcal{B}_{f_1,\cdots f_p}$ made of polynomials of this type.

1.1 A review of a elementary facts about b-functions.

- The functional equation (1) is an identity in $R[s,\frac{1}{f}]f^s$ which is the rank one free module over the ring $R[s,\frac{1}{f}]$, with s as an indeterminate. As for the structure of a $\mathcal{D}_R[s]$-module, the action on the generator f^s is :

$$\frac{\partial}{\partial x_i}g(x,s)f^s = \left[\frac{\partial g}{\partial x_i} + g\frac{s\frac{\partial f}{\partial x_i}}{f}\right]f^s$$

- **Remark 1.4** *We consider the submodule $\mathcal{D}[s]f^s$ generated by f^s. The equation (1) means that the $\mathcal{D}$-linear action of s on the quotient :*

$$\boxed{\tilde{s}\ :\frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]f^{s+1}}\longrightarrow\frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]f^{s+1}}}$$

has a minimal polynomial.

- If f is a unit in R then $b_f = 1$.

  By setting $s = -1$ in the functional equation : $P(-1)(1) = P(-1)f^0 = b(-1)\frac{1}{f}$, and this implies $b(-1) = 0$. We write usually in this case $b(s) = (s+1)\tilde{b}(s)$, and the relation $P(-1)(1) = 0$ yields :

$$P(s) = (s+1)Q(s) + \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}$$

  Carrying this over to the functional equation leads to the following result :

  Lemma 1.5 *The polynomial $\tilde{b}(s)$ is the minimal polynomial such that there is a functional equation*[1] *:*

$$\tilde{b}(s) f^s = \left[\sum_{i=1}^{n} Q(s) \cdot f + A_i(s) \frac{\partial f}{\partial x_i} \right] f^s \in \mathcal{D}[s](f + J(f)) f^s$$

  This last equation means that $\tilde{b}(s)$ is the minimal polynomial of the $\mathcal{D}$-linear action of s on the quotient

$$\boxed{(s+1)\frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]f^{s+1}} \simeq \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s](f+J(f))f^s}}$$

  where $J(f)$ is the jacobian ideal $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

1.2 A first list of examples

- Exercise When $f^{-1}(0) \neq \emptyset$ is smooth we have $b_f(s) = s + 1$. In the local case, take $f = x_1$. On $\mathbb{C}[x_1, \dots, X_n]$ we use the nullstellensatz for the ideal $f + j(f)$.

  The converse is true : the equality $b_f(s) = s + 1$ may only happen in the smooth case, when $R = \mathbb{C}[x_1, \dots, X_n]$, or $\mathbb{C}\{x_1, \dots, x_n\}$. This result can be found in the article [7] by Briançon and Maisonobe, and is surprisingly much more complicate.

- When $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ the obvious equation does yield the minimal polynomial :

$$\frac{1}{\prod_{i=1}^{n} \alpha_i^{\alpha_i}} \prod_{i=1}^{n} \left(\frac{\partial}{\partial_{x_1}}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial_{x_n}}\right)^{\alpha_n} \cdot f^{s+1} = \prod_{i=1}^{n} \left(\prod_{k=0}^{\alpha_i - 1} \left(s + 1 - \frac{k}{\alpha_i} \right) \right) \cdot f^s$$

  This example is a basic ingredient in the proof by Kashiwara of the rationality of the zeroes of b_f.

- Let $f = x_1^2 + \cdots + x_n^2$ and let Δ be the Laplacian operator then :

$$\Delta f^{s+1} = (s+1)(4s+2n) \cdot f^s$$

  is indeed minimal.

[1] Beware of the fact that $Q(s)f$ or $Q(s) \circ f$ is here a product of operators, distinct from $Q(s)(f)$ the result of applying the operator to f. When the symbol f^s appears there cannot be any ambiguity.

- Let X_{ij} be n^2 indeterminates and let $\partial_{ij} = \frac{\partial}{X_{ij}}$ the corresponding partial derivatives in the polynomial ring in these variables. Then we have the Cayley identity

$$(\det \partial_{ij})\,(\det X_{ij})^{s+1} = (s+1)\dots(s+n)\,(\det X_{ij})^{s}$$

For a proof see for example [11]. The two last examples are of semi-invariant sof an algebraic group, respectively $O_{\mathbb{C}}(n) \rtimes \mathbb{C}^*$ and $GL(n) \times GL(n)$, with the complement of $f = 0$ an open orbit.

Next example due to Kashiwara (unpublished see also Yano [38]), is much less trivial even it can still be treated with elementary methods :

- **1.2.1 Quasi homogeneous germs of an isolated singularity.**

We consider a system of weights $\underline{w} = (w_1, \cdots, w_n) \in (\mathbb{Q}_+^\star)^n$, and for $I \in \mathbb{N}^n$ we denote $\langle \underline{w}, I\rangle = w_1 I_1 + \cdots w_n I_n$

Definition 1.6 *The polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is quasi-homogeneous of weight (we say also w-degree) ρ if its expansion has the form :*

$$f = \sum_{\langle \underline{w}, I\rangle = \rho} f_I x^I$$

We denote $\mathbb{C}[X]_\rho$ the finite dimensional space of quasi-homogeneous polynomials of degree ρ

Definition 1.7 *The function germ f defines an isolated singularity if $\{0\}$ is an isolated point in the set defined by the equations*

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

The ideal $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ generated by the partial derivatives is called the Jacobian ideal. The hypothesis of isolated singularity implies that the quotient $\frac{\mathcal{O}}{J(f)}$ is finite-dimensional. Let M be a monomial basis of this quotient. Then we have :

Proposition 1.8 *Let f be a quasi homogeneous polynomial with an isolated singularity at $(0,0)$ and w-weight equal to 1. Let $|w| = \sum w_i$, and let Π be the set of weights without repetition of elements of M. Then :*

$$b_f(s) = (s+1) \prod_{\rho \in \Pi} (s + |w| + \rho)$$

This result can be deduced from the result of Malgrange in [24]. However the approach in [4] and [5] has the advantage of being completely elementary and to give an algorithm including a determination of an operators $P(s)$ in a much larger classes of nondegenerate singularities.

We end this section by a few explicit examples. In the first one about $y^2 + x^3$, we give a complete calculation including an explicit operator, and it is easy to work out in the same way the general quasi homogeneous case. There is in this case a symmetry of the roots of b around -1. In this case the set of root of $b(-s)$ is equal to the spectrum of the singularity which always has this property. The fact that the roots of b are in $\mathbb{Q}_{>0}$ was seen first in these types of examples and is a general fact to be developped in the next section.

- Let $f = x^2 + y^3$. Its Bernstein polynomial is $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$. We have $|w| = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, and an Euler operator $\chi = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{3}y\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x}x + \frac{1}{3}\frac{\partial}{\partial y}y - \frac{5}{6}$. Since $\chi f^s = sf^s$ we get:

$$(s+\frac{5}{6})f^s = (\frac{1}{2}\frac{\partial}{\partial x}x + \frac{1}{3}\frac{\partial}{\partial y}y)f^s = \frac{1}{4}\frac{\partial}{\partial x}\frac{\partial f}{\partial x}f^s + \frac{1}{3}\frac{\partial}{\partial y}yf^s \tag{4}$$

  The monomial y is not yet in the ideal of partial derivatives, but we can treat the term yf^s in the same way as f^s, using $\chi(yf^s) = (s+\frac{1}{3})yf^s$

$$(s+\frac{7}{6})yf^s = (s+\frac{1}{3}+\frac{5}{6})yf^s = (\frac{1}{2}\frac{\partial}{\partial x}x + \frac{1}{3}\frac{\partial}{\partial y}y)yf^s \tag{5}$$

$$= \frac{1}{6}\frac{\partial}{\partial x}y\frac{\partial f}{\partial x}f^s + \frac{1}{9}\frac{\partial}{\partial y}\frac{\partial f}{\partial y}f^s \tag{6}$$

  By multiplying the first relation by $s+\frac{7}{6}$ and replacing $(s+\frac{7}{6})yf^s$ by RHS of the second relation we find two operators $A, B \in \mathcal{D}[s]$ such that

$$(s+\frac{5}{6})(s+\frac{7}{6})f^s = (A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y})f^s$$

  and multiplying by $s+1$, and using $(s+1)\frac{\partial f}{\partial x_i})f^s = \frac{\partial}{\partial x_i}f^{s+1}$, yields the functional equation. The explicit result is :

$$(s+1)(s+\frac{5}{6})(s+\frac{7}{6})f^s = \left[\frac{1}{12}\frac{\partial}{\partial x}\frac{\partial}{\partial y}y\frac{\partial}{\partial x} + \frac{1}{27}\left(\frac{\partial}{\partial y}\right)^3 f^s + \frac{1}{4}(s+\frac{7}{6})\left(\frac{\partial}{\partial x}\right)^2\right] f^{s+1}$$

  The operator on the right can be modified by any element in the annihilator of f^s which in this case is generated by $s - \chi$ and by $\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$. In particular we can make it independent of s.

- Let $f = x^4 + y^5$. A direct application of theorem 1.8 gives :

$$\tilde{b}(s) = \prod_{1\le i\le 3; 1\le j\le 4} (s+\frac{i}{4}+\frac{j}{5})$$
$$= (s+\tfrac{9}{20})(s+\tfrac{13}{20})(s+\tfrac{7}{10})(s+\tfrac{17}{20})(s+\tfrac{9}{10})(s+\tfrac{19}{20})(s+\tfrac{21}{20})(s+\tfrac{11}{10})(s+\tfrac{23}{20})(s+\tfrac{13}{10})(s+\tfrac{27}{20})(s+\tfrac{31}{20})$$

  We notice the symmetry around -1 as expected.

- Let $f = x^4 + y^5 + txy^4$. For $t \neq 0$ the Bernstein-Sato polynomial is different only by the last factor $(s+\frac{31}{20})$ which is changed into $(s+\frac{11}{20})$. The algorithm in [4] applies in this case.

1.3 Relationship between different b-functions.

Let $f \in K[x_1, \cdots, x_n]$ be a polynomial with coefficients in K. For a given $a \in K^n$ we may consider various Bernstein-Sato polynomials :

1) The usual Bernstein-Sato polynomial $b_f = b_{alg}$, such that :

$$b_f(s)f^s \in A_n(K)[s] \cdot f^{s+1}$$

2) The local algebraic Bernstein-Sato polynomial b_{loc}, such that :

$$b_{loc,a}(s)f^s \in A_n(K)_{\mathfrak{m}_a}[s] \cdot f^{s+1}$$

with a functional equation having its coefficients in the algebraic local ring at a.

3) If $K = \mathbb{C}$ we may consider the local analytic Bernstein-Sato polynomial b_{an} at a, characterised by

$$b_{an,a}(s)f^s \in \mathcal{D}_{\mathbb{C}^n,a}[s] \cdot f^{s+1}$$

Obviously we have the divisibility relations $b_{an,a}|b_{loc,a}$, and $b_{loc,a}|b_{alg}$, but there is in in fact a much more precise relation

Proposition 1.9

$$b_{an,a} = b_{loc,a}\,, \quad b_f = \operatorname*{lcm}_{a \in K^n}(b_{loc,a}) \text{ when } K \text{ is algebraically closed}$$

The proof of these results may be found in a more general setting in [8] unpublished, including the case of multivariables Bernstein-Sato polynomials $b(s_1, \cdots, s_p)$. See also [12].

The adaptation to an analytic function in the neighbourhood of a relatively compact Stein open set follows rather easily from the local analytic case :

Theorem 1.10 *There exists a non trivial functional equation on any Stein U open subset in X such that $\overline{U}$ is compact, and f defined in a neighbourhood of $\overline{U}$. Any such Bernstein polynomial is the g.c.m. of the local ones.*

The argument consists in glueing together operators on the element of a covering conveniently modified using that $H^1(\overline{U}, (Ann_{\mathcal{D}[s]}f^s)_{<N}) = 0$ for a large enough N.

The fact that the roots of b_f are negative rational numbers in the local analytic case is transmitted via these results to all the Bernstein polynomials considered above except maybe $K[[X_1, \ldots, X_n]]$. The case of an arbitrary fields of caracteristic zero is considered in [8], unpublished.

We end this first section by quoting a result of analytic continuation. The original proof using resolution of singularities is completely different from the one sketched here, where the use of Berstein polynomial makes it much simpler.

1.4 Roots of Bernstein- polynomials and analytic continuation of f_+^s

In my talk I will skip this section already developped in Sabbah course.

The problem of the analytic continuation was posed as early as 1954 by Gelfand in Amsterdam congress. More detail about the proof as well as a survey of related topics like meromorphiic continuation vis Mellin transform, or division of distribution, can be found in [12]. We assume that the fonction f is real on an open subset U of $\mathbb{R}^n$:

$$f \ : \mathbb{R}^n \to \mathbb{R}$$

and admits a global Bernstein-Sato polynomial denoted by b.

For $\Re s > 0$ let us define the locally integrable function $Y(f)f^s$ or f_+^s

$$Y(f)f^s = \begin{cases} \exp(s \log f) \text{ if } f(x) > 0 \\ 0 \text{ if } f(x) \leq 0 \end{cases}$$

Proposition 1.11 *As a distribution $Y(f)f^s$ admits an analytic continuation with poles on the set :*

$$A - \mathbb{N} := \{s \in \mathbb{C} \,|\, \exists i \in \mathbb{N}, b(s+i) = 0\}$$

where A is the set of zeros of b.

Summarized proof
The evaluation on a test function $\varphi \in \mathcal{C}_c^\infty$ is :

$$\langle Y(f)f^s, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x)Y(f)f^s dx$$

convergent and holomorphic for $\Re s > 0$. Multiplying both sides by $b(s)$, we find :

$$\boxed{\langle Y(f)f^s, \varphi\rangle = \frac{1}{b(s)}\langle Y(f)f^{s+1}, P(s)^\star(\varphi(x))\rangle}$$

The LHS has therefore a meromorphic continuation as a *meromorphic function* defined on the half-plane $\{s \,|\, \Re s > -1\}$ with poles included in the zeros of b. Iterating this process with $b(s+1, b(s+2)\ldots$ we obtain the result. $\square$.

2 On existence theorems and rationality of the roots.

2.1 The algebraic case.

Let us prove the existence theorem due to Bersntein in the algebraic case. By contrast the analytic case is much heavier and will occupy most of this section.

Let $f \in R = K[X_1, \ldots, X_n]$. We work in $K[x_1, \cdots, x_n, \frac{1}{f}]$, with its natural structure of a module over the algebra $A_n(K)$, and also with the modules $K[s][x_1, \cdots, x_n, \frac{1}{f}]f^s$ and $K(s)[x_1, \cdots, x_n, \frac{1}{f}]f^s$ seen as modules over the algebras $A_n(K)[s]$ and $A_n(K)(s)$. We remark that $A_n(K)[s]$ is not a Weyl algebra but that $A_n(K)(s)$ is the Weyl algebra for the field $K(s)$ **so that we can apply dimension theory to this field as well.**

Lemma 2.1 *The module* $M = K\left[x_1, \cdots, x_n, \frac{1}{f}\right]$ *is a holonomic module over* $A_n(K)$. *Similarly,* $M_f = K\left[x_1, \cdots, x_n, \frac{1}{f}\right](s)f^s$ *is a holonomic as a module over* $A_n(K(s))$.

Let N be the total degree of the polynomial f. The result follows then easily from the existence of the respective good filtrations , and bounding their dimensions in view bounding the degree of Hilbert polynomials :

$$F_k(M) = \left\{\frac{g(x)}{f(x)^k}f^s \mid \deg_x g \leq k(N+1)\right\}$$

$$F_k(M_f) = \left\{\frac{g(x,s)}{f(x)^k}f^s \mid \deg_x g \leq k(N+1)\right\}$$

As a consequence, the module M_f contains the descending sequence of submodules

$$M_f \supset A_n(K(s)) \cdot f \cdot f^s \supset \cdots \supset A_n(K(s)) \cdot f^m \cdot f^s \supset \cdots$$

Since the module M_f is holonomic hence of finite length this sequence is stationary. There is an integer m and a differential operator $P_1(s)$ with coefficients in $K(s)$ such that

$$f^m \cdot f^s = P_1(s) \cdot f^{m+1} \cdot f^s \tag{7}$$

We can make a shift in this relation and we get $f^s = P_1(s-m) \cdot ff^s = \frac{P(s)}{b(s)}f^{s+1}$ with $P(s) \in K[s]$, if $b(s)$ is a common denominator of the coefficients. This is the desired functional equation.

Remarks 2.2 *1. Let $f_1, \cdots, f_p \in K[x_1, \cdots, x_n]$ be p non zero polynomials. The proof for the existence of a functional multivariables equation is essentially the same.*

2. *Let M be a holonomic $A_n(K)$-module and let $u \in M$. We can prove again the existence of an equations of the following type :*

$$P(\underline{s})uf^{\underline{s}+1} = b(\underline{s})uf^{\underline{s}}$$

with a proof of the same type.

3. *In the analytic case, the algebraic proof of Bernstein fails. The reason as explained in [27] is that there is not a simple way to involve the field $K(s)$, because the formation of rings of analytic operators with series coefficients does not commute with base change $K(s)\otimes_K$. In loc.cit. This failure is in a sense repaired, and a purely algebraic proof of the existence of a Bernstein-Sato polynomial is given, but with much more sophisticated tools than in the proof above. I am not aware of such a proof in the analytic multifunctions case.*

2.2 The main results.

In this section and in the two next one we will state and prove two classical results about the roots of the Bernstein polynomials, in the analytic case. For an isolated singularity the link with the monodromy due to Malgrange will be developped. The first result includes a proof of the existence of a non trivial functional equation, which can be adapted to a functional equation $P(s)mf^s \in \mathcal{D}[s]mf^{s+1}$, for an element of a holonomic $\mathcal{D}$-module.

We consider X an analytic manifold, $p \in X$ and since the purely local statement contain the essential difficulty we shall work with $(X, p) = (\mathbb{C}^n, 0)$, in all such statements. As we already mentionned in the introduction we have :

Theorem 2.3 *Let $f \in \mathcal{O}_{X,p}$ be an holomorphic germ. Then f admits a Bernstein-sato polynomial b_f. The roots of b_f are negative rational numbers.*

We have for an isolated singularity a more precise result, proved by Malgrange in [24]:

Theorem 2.4 *Let $f \in \mathcal{O}_{X,p}$ be an holomorphic germ with an isolated singularity. Then the reduced Bernstein polynomial $\tilde{b}_f = \frac{b(s)}{s+1}$ is the minimal polynomial of $-\partial_t t$ acting on the saturated Brieskorn Lattice $\tilde{H}''_f = \sum_{i \geq 0} (t\partial_t)^i H''_f$.*

The Brieskorn Lattice is $H''_f = \frac{\Omega^n_X}{df \wedge d\Omega^{n-2}_X}$. This is by the action of $t = \times f$ a free $\mathbb{C}\{t\}$-modules of rank μ with a connection with regular singularity determined by $\partial_t(df \wedge \eta = d\eta)$. The monodromy of this connection coincide with the monodromy T on the highest cohomology of the Milnor fibre $H^{n-1}(F_f, p)$. From this follows a property of the set $\tilde{R}_f$ of roots of $\tilde{b}_f(-s)$

Corollary 2.5 *The set of eigenvalues of T is $exp(-2i\pi R_f)$.*

2.3 Proof of the existence of b_f.

We still work in this section with $f \in \mathcal{O}_{X,p}$ a non constant holomorphic germ.

Following Malgrange we denote $\mathcal{M}_f = \mathcal{D}_X[s]f^s = \mathcal{D}_X[s]/\mathcal{J}_f$, or $\mathcal{M}$ for short when there is no possible confusion. We shall also write M for its fibre at the origin. Here $\mathcal{J}_f$

is the annihilator $\mathrm{ann}_{\mathcal{D}_X[s]} f^s$. We define on $\mathcal{O}[\frac{1}{f}, s]f^s$ an endomorphism of $\mathcal{D}_X$ modules, which was first introduced in [24] and also in [17] :

$$t : a(s)f^s \mapsto a(s+1)f^{s+1}$$

The map t is obviously invertible on $\mathcal{O}[\frac{1}{f}, s]f^s$, and leave $\mathcal{M}_f$ stable. The quotient $\mathcal{M}/t\mathcal{M}$ is a $\mathcal{D}_X[s]$-modules. In other words, the mutiplication by s is an endomorphism of $\mathcal{D}_X$-module and since $\mathcal{M}/t\mathcal{M} = \frac{\mathcal{D}_X[s]f^s}{\mathcal{D}_X[s]f^{s+1}}$, a Bernstein-Sato polynomial for f is nothing but a minimal annihilating polynomial for this endomorphism [2] The module $\mathcal{M}$ is in fact a $\mathcal{D}_X[s,t]$ with the relation:

$$[t, s] = t.$$

We can extend this structure to a $\mathcal{D}_{X\times\mathbb{C}}$, on $\mathcal{O}[\frac{1}{f}, s]f^s$ with $\frac{\partial}{\partial_t} = -(s+1)t^{-1}$ and we define $\mathcal{M}_f \subset \mathcal{N}_f = \mathcal{D}_{X\times\mathbb{C}} \cdot f^s$. This module identifies with the direct image $i_{f,+}\mathcal{L}$ introduced by Sabbah for the case $\mathcal{L} = \mathcal{O}$.

Lemma 2.6 *The annihilator in $\mathcal{D}_{\underline{x},t}$ of f^s is the left ideal generated by $t - f$, and the vector fields $\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t}$.*

Proof. The annihilator of f^s clearly contains the ideal $\langle t - f, \frac{\partial}{\partial x_i} + \{\frac{\partial f}{\partial x_i}\frac{\partial}{\partial t}\}\rangle$. The latter is maximal as a left ideal, which forces the equality in the lemma, since $\mathcal{N}_f \neq 0$. □

As a consequence the module $\mathcal{N}_f$ is simple (=it has no submodule). In fact it is isomorphic to the quotient $\mathcal{O}\left[\frac{1}{t-f}\right]/\mathcal{O}]$. We denote by $\delta(t-f)$ in this quotient. Then using the relations in the lemma, $\mathcal{N}_f$ can be written as a direct sum :

$$\mathcal{N}_f = \mathcal{D}_{\underline{x}t}\delta = \bigoplus_{k\geq 0} \mathcal{O}_X \partial_t^k \delta$$

and the functional equation is equivalent to the following :

$$b_f(-\partial_t t)\delta(t-f) = P(x, \partial_x, t\partial_t)t\delta(t-f).$$

Let us now start the proof of the existence theorem.

Case 1 We assume first an Euler homogeneity condition, namely : there is a vector field ξ such that $\xi(f) = f$, and we prove

Proposition 2.7 *Under the above condition $\mathcal{M}_f$ is a coherent subholonomic (=the codimension of char$(\mathcal{M}_f)$ is $n-1$)*

Proof. First we have $sf^s = \xi(f^s)$, which implies that $\mathcal{M}_f = \mathcal{D}_X f^s$.Then $\mathcal{M}_f$ admits a good filtration by the $\mathcal{O}_X$- coherent submodules [3]:

$$\mathcal{M}_k = \mathcal{D}_X(k)f^s \subset \sum_{i+j\leq k} \mathcal{O}_X \left(s^i \frac{1}{f^j}\right)$$

The $\mathcal{D}_X$ coherence follows from the existence of this good filtration. See for example in [13].

[2]In [17], these modules $\mathcal{M}$ and $\mathcal{M}/t\mathcal{M}$ are denoted respectively by $\mathcal{N}_f$ and $\mathcal{M}_f$. Since Malgrange focuses on the isolated singularity case he only consider in fact the fibre M of $\mathcal{M}_f$.

[3]Goodness is obvious and $\mathcal{O}$ coherence follows from being finitely generated over $\mathcal{O}$ and from the inclusion.

The restriction of $\mathcal{M}_f$ to the complement of $f^{-1}(0)$ or even to $U := X \setminus sing(f^{-1}(0))$ is clearly subholonomic. Indeed at any $q \in U$, $df(q) \neq 0$. Wet set $f_i := \frac{\partial f}{\partial x_i}$ and assume for simplicity that $f_1(q) \neq 0$. Then locally $\mathrm{ann}_{\mathcal{D}_X} f^s$ is generated by $\frac{\partial}{\partial x_i} - \frac{f_i}{f_1}\frac{\partial}{\partial x_1}$ and we have for $char\mathcal{M}$ the local equations :

$$\xi_i - \frac{f_i}{f_1}\xi_1 = 0; \text{ for } i = 1, \ldots, n$$

There is a greatest subholonomic module $\mathcal{M}' \subset \mathcal{M}_f$. We shall admit this fact for which we have to use biduality in $\mathcal{D}_X$-modules and the corresponding spectral sequence. See for example [13].

As a consequence, $\mathcal{M}_f/\mathcal{M}'$ is a coherent with support in $f^{-1}(0)$. By the analytic nullstellensatz, there exist $k > 0$, such that $f^k.(f^s(mod\mathcal{M}')) = 0$. Therefore $\mathcal{D}_X.f^{s+k} \subset \mathcal{M}'$ is subholonomic. Since t^k sends isomorphically $\mathcal{M}_f = \mathcal{D}_X.f^s$ to $\mathcal{D}_X.f^{s+k}$ we are done.

Corollary 2.8 *The $\mathcal{D}$-module $\mathcal{M}_f/t\mathcal{M}_f$, is holonomic.*

Indeed we have $char\mathcal{M}_f/t\mathcal{M}_f \subset char\mathcal{M}_f$ and due to the behaviour of the multiplicity along any $n+1$-dimensional component of $char\mathcal{M}_f$ in the exact sequence

$$0 \longrightarrow \mathcal{M}_f \xrightarrow{t} \mathcal{M}_f \longrightarrow \mathcal{M}_f/t\mathcal{M}_f \longrightarrow 0$$

none of this component remains in $char\mathcal{M}_f/t\mathcal{M}_f$. $\square$

End of the proof of the existence theorem: Under the hypothesis in case one : we just have to recall that for any holonomic module $\mathcal{L}$ and any endomorphism $h : \mathcal{L} \to \mathcal{L}$ has a minimal polynomial. In [13] we prove this fact by elementary methods by induction on the number of components of $char\mathcal{L}$ and direct analysis of the case of a unique component $T_Y * X$, with Y smooth. We can also like in [17] use a more sophisticated argument : the sheaf fiber $End(\mathcal{L}, \mathcal{L})_p$ at any point p is finite dimensional over $\mathbb{C}$ by the theorem of constructibility [16] of Kashiwara.

General case We consider the germ at $(p, 0)$, $g(\underline{x}, y) = e^y f(x)$. We are in case 1 by $\frac{\partial}{\partial t} g = g$ and we can write a functional equation :

$$b_g(s)e^{sy}f^s = P(\underline{x}, y, \partial_{x_1}, \ldots, \partial_{x_n}, s)e^{(s+1)y}f^{s+1}.$$

We could make P independent of ∂_y by $\partial_y^\ell(g^s) = (s+1)^\ell g^s$. We can therefore set $y = 0$ in the equation and this give the result.

2.4 Rationality of the roots of b-functions.

Let us consider an embedded resolution $\pi : X' \to X$ of the singularities of $Y := f^{-1}(0)$. We set $f' = f \circ \pi$, and we denote by $b_{f'}$ the b function along the exceptional divisor. This is the g.c.d. of the local b function which in the case of a normal crossing have the property in the theorem $\{b_f = 0\} \subset \mathbb{Q}_{<0}$ as we can see in by the second example of section 1.2. After this observation the second claim in theorem 2.3 is a direct consequence of the following :

Theorem 2.9 *There exists an integer $N \geq 0$ such that $b_f(s)$ is a divisor of the product $b_{f'}(s)b_{f'}(s+1)\ldots b_{f'}(s+N)$.*

For that purpose we are going to consider the direct image of $\mathcal{M}_f$ by π. All what we have to know about direct image s contained in the following statements: Let $F : X \to Y$ be a projective morphism, and $\mathcal{M} = \mathcal{D}_X\mathcal{M}_0$ a $\mathcal{D}_X$-module generated by a coherent module $\mathcal{M}_0$ then :

- **Theorem 2.10** *a)* $\int^i \mathcal{M}$ *is a coherent* $\mathcal{D}_X$*-module for all* i.

  b) $Char \int^i \mathcal{M} \subset \varpi\rho^{-1} Char\mathcal{M}$ *were* ϖ, ρ *are the maps in the diagram:*

$$T^*X \xleftarrow{\ \rho\ } X \times_Y T^*Y \xrightarrow{\ \varpi\ } T^*Y$$

  For the proof of this result we refer to [17]. See also [26] for a proof of a) without microlocalisation, using only Grauert coherence theorem for $\mathcal{O}$-modules.

- **Proposition 2.11** *In the above situation if* V *is isotropic, then* $\varpi\rho^{-1}(V)$ *is also isotropic.*

The proof of theorem 2.9 now goes together with the proof of the following one :

Theorem 2.12 *The* $\mathcal{D}_X$ *module* $\mathcal{M}_f$ *is coherent and its characteristic variety of* $\mathcal{M}_f$ *is the closure* W_f *in* T^*X *of the following set*

$$\left\{ \left(x, s, \frac{df}{f} \right) \mid f(x) \neq 0, s \in \mathbb{C}. \right\}$$

Let us notice than in the previous section, the existence theorem coherence was provisionally only proved in the quasi homogeneous case.

Theorem 2.12 is true for f' by a direct checking. The equations of W_g for $g = x_1^{a_1} \dots x_n^{a_n}$ are $\frac{x_1\xi_1}{a_1} = \dots = \frac{x_n\xi_n}{a_n}$. We set $\mathcal{M}' = \int \mathcal{M}_{f'}$. By theorem 2.10, $\mathcal{N}'$ is coherent. It is isomorphic to $\mathcal{N}_f$, or to $\pi_*(\mathcal{N}_{f'})$ out of Y with characterictic over U variety being precisely $\{(x, s, \frac{df}{f}) \mid f(x) \neq 0, s \in \mathbb{C}.\}$, hence it contains W_f.

We shall admit the following result which follows from algebraic and geometric arguments (integral dependency of f on $J(f)$, normalization etc...)

Lemma 2.13 *The subvariety* W_0*, defined in* W_f *by the equations* $f(x)\xi_1 = \dots = f(x)\xi_n = 0$ *is lagrangean. Is is the union of the zero section and a subset of* $p^{-1}f^{-1}(0) \in T^*X$.

We can describe rather accurately $\mathcal{M}'_f$

Lemma 2.14 $\mathcal{M}'_f$ *is a subholonomic module. Its characteristic variety is the union of* W_f*, irreducible of dimension* $n+1$ *and of some lagrangean variety* Λ.

indeed by the second statement in 2.10, we have

$$char(\mathcal{M}'_f) \subset \varpi\rho^{-1} W_f \times_X (X \setminus Y) \bigcup \varpi\rho^{-1}(W_f \times_X Y) = W_f \bigcup \varpi\rho^{-1}(W_f \times_X Y),$$

and the second term is isotropic by proposition 2.11, hence its components not included in W_f are Lagrangean. Now we have only to consider the direct image in degree zero, the only one with support X.Let us look at the map :

$$\mathbb{C}_X \longrightarrow R^0 f_*(\mathcal{D}_{X \longleftarrow X'} \otimes^L_{\mathcal{D}_{X'}} \mathcal{M}_{f'})$$

There is a canonical section $\mathbf{1_{X \longleftarrow X'}} \in \mathcal{D}_{\mathbf{X \longleftarrow X'}}$ which correspond to the map $f^* \in Hom(f^{-1}\Omega_X, \Omega_{X'}$ and generates it as a bimodule. There fore we can write the image of 1 as $u = \mathbf{1_{X \longleftarrow X'}} \otimes \mathbf{f'^s}$. We set $\mathcal{M}''_f = \mathcal{D}_X[s] \cdot u$

Lemma 2.15 $\mathcal{M}''_f$ *is a coherent* $\mathcal{D}_X$*-module with a* $\mathcal{D}[s,t]$*module structure and we have a diagram :*

$$\mathcal{M}'_f \supset \mathcal{M}''_f \longrightarrow \mathcal{N}_f \tag{8}$$

in which the map on the right is well defined by $P(s)u \longrightarrow P(s)f^s$ *and surjective.*

The first statement comes from the fact that $\mathcal{M}''_f$ is an increasing union of finite type modules $\mathcal{D}_X[s]_{\leq N}$ in a coherent module $\mathcal{M}'_f$. The well definedness is checked at the generic point since $\mathcal{M}_f$ has no section with support $\subsetneq X$. The other statements are obvious.

As a consequence of the above diagram $\mathcal{M}'$ transmit to $\mathcal{M}_f$ the property of subholonomicity :

$$char\mathcal{N}_f = W_f \cup \text{ Some lagrangean } \Lambda_1 \tag{9}$$

We will apply to different modules the following two tatements

Lemma 2.16 *1) Let $\mathcal{P}$ be a $\mathcal{D}_X$- module which is also a $\mathcal{D}[s,t]$- module and such that $\mathcal{P}/t\mathcal{P}$ is holonomic. Then there is a non zero ppolynomial b such that $b\mathcal{P}/t\mathcal{P} = 0$, and we denote by $b(s, \mathcal{P}$ the largest unitary polynomial of this type.*

2) Let $\mathcal{L}$ be a coherent holonomic $\mathcal{D}_X$-module with a structure of $\mathcal{D}[s,t]$- module then $t^N\mathcal{L} = 0$ for some integer N.

The first statement is already proved and put here for fixing a notation.

Summary of the proof for 2) the decrasing sequence $t^j\mathcal{L}$ is stationnary as a sequence of modules of minimal dimension (argument not given like that in [17]). Therefore $\mathcal{L}' := \bigcap_{j\in\mathbb{N}} t^j\mathcal{L} = t^N\mathcal{L}$, and the surjective map $t : \mathcal{L}' \to \mathcal{L}'$ must be an isomorphism, by a multiplicity argument. Finally $\mathcal{L}' = 0$ comes from the equality $b(s,\mathcal{L}')t = tb(s-1,\mathcal{L}')$, hence $b(s-1,\mathcal{L}')\mathcal{L}'$ which forces $bs, \mathcal{L}$ to be constant.

Final part of the proof. For the relation $char\mathcal{M}_f = W_f$, we notice first that $Ext^j(\mathcal{M}_f, \mathcal{D}) = 0$ exept for $j = n-1, n$ because of relation (9). The relation $Ext^n\mathcal{M}_f, \mathcal{D}) = 0$ is also true because it is alwys holonomic, as a consequence of 2.16. Now $\mathcal{M}_f$ must have a pure dimensional characteristic variety which forces the result.

Now let us look again at the diagram (8): first because of the map we have $b_f(s) = b(s,\mathcal{M}_f) \mid b(s,\mathcal{M}''_f)$, and we also have

$$b(s,\mathcal{M}'_f) \mid b_{f'}(s) = b(s,\mathcal{M}_{f'})$$

Indeed there is by definition a map $g : \mathcal{M}_{f'})too\mathcal{M}_{f'})$ such that $b_{f'}(s) = t\circ g$ in $\mathcal{M}_{f'})$. Applying the functor $\int$ to g we get $b_{f'}(s) = t \circ \int g$ in $\mathcal{M}'_f$ which gives the divisibility we want. In order to conclude the proof it remains to show a relation of divisibility :

$$b(s,\mathcal{M}''_f) \mid b(s,\mathcal{M}'_f)b(s+1,\mathcal{M}'_f)\ldots b(s+N,\mathcal{M}'_f)$$

This last relation is obtained by applying to $\mathcal{M}'_f \supset \mathcal{M}''_f$ the following corollary of lemma 2.16:

Corollary 2.17 *Let $\mathcal{P} \supset \mathcal{P}'$ be two $\mathcal{D}[s,t]$-modules which are also coherent $\mathcal{D}_X$-modules and such that $\frac{\mathcal{P}}{t\mathcal{P}'}$ is holonomic. Then we have a divisibility relation*

$$b(s,\mathcal{P}') \mid b(s,\mathcal{P})b(s+1,\mathcal{P})\ldots b(s+N,\mathcal{P}').$$

Indeed by lemma 2.16 there is N such that $\mathcal{P} \supset \mathcal{P}' \supset \mathcal{P}$, and by iterated b functions we have

$$b(s,\mathcal{P})\mathcal{P} \subset t\mathcal{P},\ b(s,\mathcal{P})b(s+1,\mathcal{P})\mathcal{P} \subset t^2\mathcal{P}, \quad b(s,\mathcal{P})\ldots b(s+N,\mathcal{P})\cdot\mathcal{P} \subset t^{N+1}\mathcal{P} \subset \mathcal{P}'$$

as wished.

2.5 An application : holonomy of the modules of local cohomology.

The holonomy of $\mathcal{O}[\frac{1}{f}]$ is a rather easy consequence of the standard functional equation. See [13]. The starting point is the fact that $\mathcal{O}[\frac{1}{f}]$ is clearly finitely generated over $\mathcal{D}_X$ since we have :

$$\mathcal{O}\left[\frac{1}{f}\right] = \mathcal{D}_X \frac{1}{f^{k_0}}$$

if $b(-k) \neq 0$ when $k \geq k_o + 1$

In view of generalisation to local cohomology we need here to use the generalized functional equation for a more general result :

Theorem 2.18 *Let $\mathcal{M}$ be a holonomic module. Then its localisation $\mathcal{M}[\frac{1}{f}]$ is also holonomic.*

It is sufficient by exactness of localisation to assume that $\mathcal{M} = \mathcal{D}_X \cdot m$ for some $m \in \mathcal{M}$ We shall consider :

$$\mathcal{N}_\lambda := \frac{\mathcal{D}_X m f^s}{(s-\lambda)\mathcal{D}_X m f^s}$$

$\mathcal{D}_X m f^s$ admits a good $\mathcal{D}[s]$ coherent hence is coherent as a $\mathcal{D}[s]$-module. The quotient $\mathcal{N}_\lambda$ being annihilated by $s\lambda$ admit as well a $\mathcal{D}_X$-module finite presentation.

Out of $f^{-1}(0)$ this module is a tensor product of two holonomic modules

$$\mathcal{M} \otimes \frac{\mathcal{O}_X[\frac{1}{f}, s]f^s}{(s-\lambda)\mathcal{O}_X[\frac{1}{f}, s]f^s}$$

whose characteristic variety $char\mathcal{M}$ and $T^*_X X$ are tranversal. And by a known result (see [**?**]) this imply that $\mathcal{N}_\lambda$ is holonomic.

By similar argument $\mathcal{D}_X m f^s$ is subholonomic and by the same argument corollary 2.8 as in section the quotient $\frac{\mathcal{D}_X m f^s}{\mathcal{D}_X m f^{s+k}}$ is holonomic for k large enough. Its homorphic image by

$$\frac{\mathcal{D}_X m f^s}{\mathcal{D}_X m f^{s+k}} \longrightarrow \frac{\mathcal{D}_X m f^s}{\mathcal{D}_X m f^{\lambda+k}}$$

and finally $\mathcal{N}_\Lambda$ is also holonomic. We check with the local Bernstein equation :

$$P(s)m \otimes f^{s+1} = b(s)m \otimes f^s$$

that if $b(-k) \neq 0$ for any $k > r$, we have

$$\mathcal{D}_X(m/f^r) = \mathcal{D}_X m[1/f] = M[1/f]$$

This relation guarantees finiteness and then coherence over $\mathcal{D}$. Coherence comes from the surjective map $\mathcal{N}_r \longrightarrow \mathcal{D}_X(m/f^r)$.

Theorem 2.19 *For any reduced analytic subset $Y \subset$ the complex of local cohomology $R\Gamma_{[Y]}$ and $RM(*Y)$, have holonomic cohomology sheaf, if $\mathcal{M}$ is holonomic.*

Without entering into the detailed definitions I will recall two fact that show that this result can be deduced by very formal arguments from the case of an hypersurface, where we should recall that if $Y = f^{-1}(0)$, $R\Gamma_{[Y]}\mathcal{M}$ is just the complex:

$$\mathcal{M} \longrightarrow \mathcal{M}[1/f]$$

Now we treat any $R\Gamma_{[Y]}$ by induction on the number of equation. If $Y = Y_1 \cap Y_2$, with Y_1 a hypersurfacewe just use the fact that

$$R\Gamma_{[Y_1 \cap Y_2]}\mathcal{M} = R\Gamma_{[Y_1]}\left(R\Gamma_{[Y_2]}\mathcal{M}\right)$$

Example. In the case of a complete intersection Y defined by p equation $f_1 = \cdots = f_p = 0$, the sheaf $\mathcal{O}$ has only one non zero local cohomology

$$R^p(\mathcal{O}_X) = \frac{\mathcal{O}\left[\frac{1}{f_1 \ldots f_p}\right]}{\sum_{i=1}^n \mathcal{O}\left[\frac{1}{f_1 \ldots \check{f}_i \ldots f_p}\right]}.$$

3 Relationship with monodromy and vanishing cycles.

3.1 The case of an isolated singularity.

We sketch a proof of the result of Malgrange in [24]:

Theorem 3.1 *Let $f \in \mathcal{O}_{X,p}$ be an holomorphic germ with an isolated singularity. Then the reduced Bernstein polynomial $\tilde{b}_f = \frac{b(s)}{s+1}$ is the minimal polynomial of $-\partial_t t$ acting on the quotient $\frac{\tilde{H}''_f}{t\tilde{H}''_f}$ were $\tilde{H}''_f := \sum_{i \geq 0}(t\partial_t)^i H''_f$ is the saturated Brieskorn Lattice.*

Recall first the locally trivial fibration theorem due to Milnor : $f : B_\epsilon \cap f^{-1}(D_\eta) \to D_\eta$, with $0 < \eta \ll \epsilon \ll 1$. There is for each k, a flat connection for the fibre bundle based on D_η on $\bigcup_{t \in D_\eta} H^k(F_f, \mathbb{C})$, $F_f := f^{-1}(\eta)$, the Milnor fibre. By integrating this connection we recover the topological monodromy T_f on each $H^k(F_f, \mathbb{C})$. The theorem of the monodromy tells us that this action of T_f is quasi unipotent in particular that the eigenvalues of this action are roots of unity.

In the isolated singularity case the situation simplifies since the fiber $f^{-1}(\eta) = F_f$ has reduced cohomology only in degree $n-1$.[4]

Brieskorn shows in [9] that the above connection is the restriction to $D_\eta \setminus \{0\}$ of a meromorphic connection with regular singularities on a fibre bundle whose fibre at the origin is the above H''_f. In fact we have a series of inclusion which he denotes

$$H \hookrightarrow H' \hookrightarrow H''$$

We have here $H' = \frac{\Omega^{n-1}}{df \wedge \Omega^{n-2} + \Omega^{n-2}}$, where we denote for short $\Omega^i = \Omega^i_{X,x}$, and the second inclusion is given by $df\wedge$, so that the quotient $\frac{H''}{H'} = \frac{\Omega^n}{df \wedge \Omega^{n-1}}$, isomorphic to the jacobian quotient $\frac{\mathcal{O}}{f_1, \cdots, f_n}$ is a finite dimensional vector space of dimension μ. Similarly $\dim_{\mathbb{C}}(H/H') = \mu$.

Let us recall how is written the connection. If $d\eta = df \wedge \alpha \in \Omega^{n-1}$ which means in fact that the class $[\eta] \in H$, then we have $\nabla([\eta]) = [\alpha]$ written $\frac{d\eta}{df}$.

Similarly on H'' the connection can be written for $\omega \in \Omega^n : \nabla([\omega]) = d\left(\frac{\omega}{df}\right)$. Indeed if $[\omega]$ is in fact in H' let us write $\omega = df \wedge \eta$, and by tranposing via df the expression of the connection on H':

- If $d\eta = df \wedge \alpha$, $\nabla\omega = df \wedge \nabla\eta = df \wedge \alpha = d\eta$, and this expression extend in a unique way to H'.

[4] In the general case the monodromy theorem should be stated on the complex of local vanishing cycles of f.

- We know that there is a power f^k of f in the jacobian ideal. Therefore for an arbitrary $\omega \in \Omega^n$, we can again write $f^k\omega = df \wedge \eta$ hence the unique meromorphic expression :

$$\nabla(t^k\omega) = d\eta = t^k\nabla(\omega) + kt^{k-1}\omega, \quad \nabla(\omega) = \frac{1}{t^k}(d\eta - kt\omega)$$

Les us now come back to the $\mathcal{D}_X$ modules related to the the b function :

$$\mathcal{M}_f = \mathcal{D}[s]f^s = \mathcal{D}[-\partial_t t]\delta \subset \mathcal{N}_f = \mathcal{D}_{\underline{x},t}\partial_t^k\delta(t - f(x)) = \bigoplus_{k\in\mathbb{N}} \mathcal{O}_X\partial_t^k\delta(t - f(x))$$

We consider for any $\mathcal{D}_X$-module $\mathcal{M}$ is de Rham complex complex of form with coefficient in $\mathcal{M}$ with its natural differential :

$$0 \longrightarrow \mathcal{M} \longrightarrow \omega^1 \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow \dots \longrightarrow \Omega^n \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow 0$$

$$d(\omega \otimes m) = d\omega \otimes m + \sum dx_i \wedge \omega \otimes \frac{\partial}{\partial x_i} m$$

in the case of $\mathcal{M}_f$ or $\mathcal{N}_f$ the cohomology group have the stucture of $\mathbb{C}\{t\}\langle\partial_t\rangle$-modules and of $\mathcal{D}_t$ modules respectively. In [24] the following result is proved by direct calculations :

Lemma 3.2 *$H^1(\mathcal{N}_f) \simeq \mathbb{C}\{t\}$, and $H^n(\mathcal{N}_f) \simeq H'' \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$and the other cohomologies are zero.*

We concentrate on H^n and specifically on the way H''_f appears, and skip the rest of the proof. We filtrate $\mathcal{N}_f$ by

$$F^p = \bigoplus_{0\leq\leq k\leq p} \mathcal{O}_X\partial_t^k\delta(t - f(x))$$

and denote by $G^p := F^p(H^n\mathcal{N})$ the induced filtration. Let us notice that

$$\Omega^p \otimes_{\mathcal{O}} \mathcal{M}_f = \Omega^p \otimes_{\mathcal{O}} \left(\bigoplus_{k\in\mathbb{N}} \mathcal{O}_X\partial_t^k\delta(t - f(x))\right) \simeq \Omega^p[\partial_t]$$

so a typical element of $\Omega^p \otimes_{\mathcal{O}} \mathcal{N}_f$ is written $\alpha : \alpha_0 + \alpha_1\partial_t + \cdots + \alpha_r\partial_t^r$ with the differential

$$d\alpha = \sum_{i=0}^{r} d\alpha_j\partial_t^j + df \wedge \alpha_j\partial_t^{j+1}$$

We calculate G^0 which is the quotient of $\Omega^n = F^0(\Omega^n[\partial_t])$ by the elements $\omega \in \Omega^n$ which can be written as differential from $\Omega^{n-1}[\partial_t]$ i.e.

$$\alpha = d(\alpha_0 + \alpha_1\partial_t + \cdots + \alpha_r\partial_t^r)$$

We leave to the reader (or refer to [24]) the details which show that we can lower r to zero and finally find $\alpha_0 \in df \wedge d\Omega^{n-2}$ (Hint we use here the fact that we have an isolated singularity so that $(\Omega^\bullet, df)$ is a dual Koszul complex, with cohomology zero in degrees $\neq n$).

By similar methods we show an isomorphism $\partial_k : G^0 \to G^k$ from which the result follows. Since we can give a meanig to $\partial_t^{-1} : H''_f \to H'_f \subset H''_f$, the final result can also take the form

$$H^n(\mathcal{N}_f) \simeq H'' \otimes_{\mathbb{C}[\partial_t^{-1}]} \mathbb{C}[\partial_t^{-1}, \partial_t]$$

Let us summarize the end of the proof of Malgrange's result :

- The short exact sequence :

$$0 \longrightarrow \mathcal{M}_f \longrightarrow \mathcal{N}_f \longrightarrow \mathcal{N}_f/\mathcal{M}_f \longrightarrow 0$$

  yields an injection $H^n(\mathcal{M}_f) \hookrightarrow H^n(\mathcal{N}_f) = H''_f[\partial_t]$

- The image of $H^n(\mathcal{M})$ is nothing but the satured lattice $\tilde{G} = \sum_{k\geq 0}(\partial_t t)k)G^0$. The fact that this is a lattice is due to the **regularity** of the Gauss Manin connection. Conclusion $H^n(\mathcal{M}_f) = \tilde{H''_f}$ and the $\mathcal{D}_t$ structures coincide. The main result to be proven is now : $\tilde{b_f}$ is the minimal polynomial of the action of c on $\widetilde{H''_f}/t\widetilde{H''_f}$ A priori we know only that this polynomial divides $\tilde{b_f}$

- In fact we saw that $\tilde{b}_f$ is the minimal polynomial of the action on the module

$$\mathcal{L} = \frac{\mathcal{M}_f}{t\mathcal{M}_f + \mathcal{D}_X[t\partial_t]J(f)f^s}$$

  Since this module has its support at the origin a result on the structure of these modules show that $\tilde{b_f}$ is as well the minimal polynomial of the action of s on $H^n(\mathcal{L})$.

- To finish the proof it remain to show that the

$$\widetilde{H''_f}/t\widetilde{H''_f} \longrightarrow H^n(\mathcal{L}),$$

  this is a matter of an explicit calculation with the above map written as follows :

$$\frac{\Omega^n[s]}{\Omega^n[s]\mathcal{J}_f + \Omega^n[s]f} \longrightarrow \frac{\Omega^n[s]}{\Omega^n[s]\mathcal{J}_f + \Omega^n[s]f + \Omega^n[s]J(f)}$$

3.2 Bernstein polynomial and vanishing cycles.

Let us recall first the definition of the complex of vanishing cycles by Deligne. We consider the universal covering $\tilde{\mathbb{C}}^* \xrightarrow{j} \mathbb{C}^*$ and its pull back $\tilde{X}^* \xrightarrow{\bar{j}} X$ by $X \xrightarrow{f} \mathbb{C}$ then this complex is :

$$R\psi(\mathbb{C}_{\tilde{X}^*}) := \bar{i}^{-1}R\bar{j}_*\mathbb{C}_{\tilde{X}^*}$$

The cohomology of the fibre of this complex at a point $x \in f^{-1}(0)$ is the cohomology of the Milnor fibre at x, $H^*(F_{f,x}, \mathbb{C})$, and the monodromy correspond to the action on $R\psi$ induced by the automorphism $\log t \to \log t + 2i\pi$ of the unniversal covering. However contrary to the isolated case there are in general non a unique reduce cohomology group and also the fibre at x of the sheaf $R\psi$ take into account the Milnor fibres at nearby points $y \neq x$ in $sing(f^{-1}(0))$

In particular $\frac{\mathcal{M}_f}{t\mathcal{M}_f}$ still exists with the same meaning an our b_f is still the minimal polynomial of the action of $s = -\partial_t t$, but $H^n(\frac{\mathcal{M}_f}{t\mathcal{M}_f})$ does not describe $R\psi$. Here is how se can recuperate it.

We consider the 0^{th} term of the V filtration of $\mathcal{D}_{X\times\mathbb{C}}$ which is a coherent sheaf of ring:

$$V^0(\mathcal{D}_{X\times\mathbb{C}}) = \{P(x, t, \partial_s, t\partial_t)\}$$

and $\mathcal{M}_f$ is obviously a coherent $V^0(\mathcal{D}_{X\times\mathbb{C}})$- module with support on the graph of f.

Definition 3.3 *A lattice $M' \subset \mathcal{M}_f[\frac{1}{t}] = \mathcal{N}_f[\frac{1}{t}]$ is a coherent $V^0(\mathcal{D}_{X\times\mathbb{C}})$ module such M' that $M'[\frac{1}{t}] = \mathcal{M}_f[\frac{1}{t}]$. It is equivalent to ask that there are integers $k, \ell \in \mathbb{Z}$ such that*

$$t^k M \subset M' \subset t^\ell M$$

Lemma 3.4 *There is a unique lattice $\mathcal{M}'$ such that the eigen values λ of the action of $t\partial_t$ on M'/tM' satisfy $-1 < \Re\lambda \leq 0$*

The eigenvalues of the action $t\partial_t = -s-1$ on $\frac{\mathcal{M}_f}{t\mathcal{M}_f}$ are > -1 by theorem 2.3. Let $b(\theta)(= b_f(-\theta-1))$ be the minimal polynomial of this action. Let us choose a root $\alpha > 0$ and set $b(\theta) = (\theta-\alpha)^{m_\alpha} b_1(\theta)$. We consider the new lattice containing $\mathcal{M}_f$:

$$\widetilde{\mathcal{M}} := t^{-1}\mathcal{M}_f + (t\partial_t - \alpha)^{m_\alpha}\mathcal{M}_f.$$

A straightforward calculation using $b(t\partial_t)\mathcal{M}_f \subset t\mathcal{M}_f$ shows that the action of $t\partial_t$ on $\widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}}$ is annihilated by the polynomial $(\theta-\alpha)^{m_\alpha} b_1(\theta)$. By iterating this process we obtain a lattice $M' \supset M$ on which the minimal polynomial of the action of $t\partial_t$ has all its roots in $]-1, 0]$.

The main result in [24] is now the following

Theorem 3.5 *The complex with automorphism $(DR(M'/tM'), \exp(-2i\pi t\partial_t))$ is isomorphic with $(R\psi, T)$.*

In [18] Kashiwara obtains similar results for the vanishing cycle sheaf $R\psi\mathcal{F}^\bullet$ of a constructible sheaf which is the de Rham complex or the complex of solutions of a holonomic $\mathcal{D}_X$-module

3.3 Microlocal b-function

Let R_f be the set of root of $b_f(-s)/(-s+1)$, and $\alpha_f = \min R_f$. It can be shown that α_f is the largest $\alpha \in \mathbb{R}_{>0}$ such that $\int \frac{1}{|f|^{2\alpha}}$ is convergent.

In the case of isolated singularities, using the relationship between the spectrum and b and the symmetry of the spectrum, on can show that $R_f \subset [\alpha_f, n-\alpha_f]$, and also that the multiplicity of a root α satisfies $m_{alpha} \leq n - \alpha_f - \alpha + 1$. The main result in [30] is the following :

Theorem 3.6 *Let f an holomorphic germ with a possibly non isolated singularity. Then the relations $R_f \subset [\alpha_f, n-\alpha_f]$ and $m_\alpha \leq n - \alpha_f - \alpha + 1$ still hold true.*

In [30] Morihiko Saito introduces for this purpose the notion of microlocal b-function. We have seen after lemma 2.6 that $\mathcal{N}_f = \mathcal{D}_{x,t}\delta(t-f) \simeq \mathcal{O}\left[\frac{1}{t-f}\right]/\mathcal{O}$ is a free module over $\mathcal{O}_X[\partial_t]$ written as a direct sum :

$$\mathcal{N}_f = \bigoplus_{k\geq 0} \mathcal{O}_X \partial_t^k \delta$$

Following [30] we consider the action of the ring $\mathcal{R} = \mathcal{O}_X[t, \partial_t] \subset \mathcal{D}_{x,t}$, given by

$$tg(x)\partial_t^k\delta = fg(x)\partial_t^k\delta - kfg(x)\partial_t^{k-1}\delta, \tag{10}$$

$$\frac{\partial}{\partial x_i} g(x)\partial_t^k\delta = \frac{\partial g}{\partial x_i}\partial_t^k\delta - g\frac{\partial f}{\partial x_i}\partial_t^{k+1}\delta \tag{11}$$

and this yields the differential on the de Rham complex $\Omega^p \otimes_{\mathcal{O}} \mathcal{M}_f$ which act on α : $\alpha_0 + \alpha_1\partial_t + \cdots + \alpha_r\partial_t^r$ as :

$$d\alpha = \sum_{i=0}^{r} d\alpha_j \partial_t^j + df \wedge \alpha_j \partial_t^{j+1}$$

There is an action of the ring $\widetilde{\mathcal{R}} = \mathcal{O}_X[t, \partial_t, \partial_t^{-1}]$, on the free module :

$$\widetilde{\mathcal{N}}_f = \mathcal{O}_X[\partial_t, \partial_t^{-1}]\delta(t-f) = \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_X \partial_t^k \delta$$

given by the same formulas. Recall that b_f is the minimal polynomial for the action of $s = -\partial_t t$ on $\mathcal{N}_f / t\mathcal{N}_f$ and similarly Saito defines the microcal function $\widetilde{b}_f$ as the minimal polynomial for $\widetilde{\mathcal{N}}_f / t\widetilde{\mathcal{N}}_f$

$$\widetilde{b}_f(-\partial_t t) \in \mathcal{D}_X[\partial_t^{-1}, t\partial_t]\partial_t^{-1}\delta$$

Proposition 3.7 *The microlocal b-function coincides with* $\frac{b_f}{s+1}$.

The proof is rather elementary. Let illustrate it by proving that $\widetilde{b}_f$ divides $b_f/(s+1)$. The equation for b_f can be written by using $s+1 = -t\partial_t$:

$$(s+1)\frac{b_f(-\partial_t t)}{s+1}\delta = P[t\partial_t]t\delta = (s+1)P[t\partial_t]\partial_t^{-1}\delta$$

The multiplication by $(s+1)$ is invertible on $\widetilde{\mathcal{N}}_f$ since it is clear by construction for ∂_t. For t we notice using formula (10), that by grading by the power in ∂_t it reduces to the multiplication by f on $\mathcal{O}_X$.

About the proof of theorem 3.6
I will only give the a few ingredients refer to the paper the details.
On $\mathcal{N}_f$, M. Saito defines three filtrations

$$\text{Filtration induced by } V^{\bullet}\widetilde{\mathcal{R}} : G^p\widetilde{\mathcal{N}}_f = V^p\widetilde{\mathcal{R}}\delta = \partial_t^{-p}\mathcal{D}_X[t, t\partial_t]\delta$$

$$\text{Filtration by the order in } \partial_t : F^p\widetilde{\mathcal{N}}_f = \bigoplus_{k\le p} \partial_t^k\delta$$

A good V filtration

We refer to the course of Sabbah for the existence of a V- filtration indexed by a dircrete subset $A+\mathbb{Z}$ of $\mathbb{Q}$. We index it differently by requiring the nilpotency of $\partial_t t - \alpha$ instead of $t\partial_t - \alpha$ on the graded term of degree α.

An important relation due to the negativity of the roots of b is

$$F^0\widetilde{\mathcal{N}}_f \subset V^{>0}\widetilde{\mathcal{N}}_f$$

The V-filtration on $\widetilde{\mathcal{N}}_f$ has a specific property $\partial_t^j : V^\alpha(\widetilde{\mathcal{N}}_f) \longrightarrow V^{\alpha-j}(\widetilde{\mathcal{N}}_f)$ is an isomorphism. I skip the technical part of the proof which requires a use of a duality in the category of filted $\mathcal{D}$-modules. Let us just mention to finish that by defnition $\widetilde{b}_f$ is the minimal polynomial for the action of s on $gr_G^0(\widetilde{\mathcal{N}}_f)$. If $\alpha > n - \alpha_f$ we can prove that

$$G^0(gr_V^\alpha\widetilde{\mathcal{N}}_f) = G^1(gr_V^\alpha\widetilde{\mathcal{N}}_f)$$

so that $0 = Gr_G^0(gr_V^\alpha\widetilde{\mathcal{N}}_f) = gr_V^\alpha(Gr_G^0\widetilde{\mathcal{N}}_f)$ which is incompatible with $\widetilde{b}_f(-\alpha) = 0$

4 Semi-invariants on prehomogeneous spaces.

We refer for the detailed definitions and the main properties of prehomogeneous spaces to the book of T. Kimura [19] .

Recall that a *prehomogeneous space* is just an algebraic action $G \xrightarrow{\rho} GL(V)$ of an algebraic group G on a K-vector space V which admits a Zariski open orbit U. A *semi invariant* is a rational function $f \in K(V)$ such that there exists a one dimensional representation or character $\chi : G \to K^\star = GL(1)$ such that

$$\forall x \in V, \forall g \in G, \quad f(\rho(g)x) = \chi(g)f(x)$$

On a prehomogeneous space the character χ determines the semi invariant f, in particular a semi invariant is associated to the character 1 if and only if it is constant. The equations of the one codimensional components S_i of the complement $V \setminus U$ of the open orbit, are irreducible homogeneous polynomials and are semi invariants. In fact all the irreducible semi invariants are of this type. See [19, Theorem 2.9.] for details.

Let us now focus on the case of a reductive complex algebraic group. Such a group is the Zariski closure of a compact subgroup H. We may assume, after an appropriate change to coordinates (x), called unitary coordinates, that H included in the unitary group $U(n)$. In that situation one shows that the dual action

$$G \xrightarrow{\rho^\star = {}^t\rho^{-1}} GL(V)$$

is a prehomogeneus vector space. One also shows by a straightforward calculation in the dual coordinates (y) that if $h \in H$, then $\rho^\star(h) = \overline{\rho(h)}$ the complex conjugate and that if $f(x)$ is a semi invariant polynomial associated with a character χ, $f^\star(y) = \overline{f(\overline{y})}$ is a semi invariant associated with χ^{-1}.

Proposition 4.1 *In the above situation with homogeneous polynomial semi invariants $f, f^\star$ of degree d, there exists a non zero polynomial $b(s)$ of degree d such that*

$$f^\star(D_x)f(x)^{s+1} = b(s)f(x)^s \quad D_x = {}^t\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Proof The chain rule $\frac{\partial u}{\partial y_i}(g \cdot x) = \sum_j \left(g^{-1}\right)_{j,i} \frac{\partial}{\partial x_j}(u(g \cdot x))$ yields the usual formula for base change on vectors in accordance with the base change $y = g.x$ on coordinates :

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = {}^t g^{-1} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \tag{12}$$

Let us condense formula (12) into

$$D_y = {}^t g^{-1}\, D_x = \rho^\star(g) D_x \quad \text{or componentwise} \quad : \quad \frac{\partial}{\partial y_i} = ({}^t g^{-1}\, D_x)_i$$

Kimura writes $D_{\rho(g)x}$ for the more explicit $\rho^\star(g)D_x$ which is just a column of differential operators which are linear combinations of the $\frac{\partial}{\partial x_j}$ and $f^\star(D_{\rho(g)x})$, is just $f^\star(\rho^\star(g)D_x)$. In a similar way we associate to any polynomial differential operator its image by g^{-1}

$$P(D_y) = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{N}} C_I (\frac{\partial}{\partial y_1})^{i_1} \cdots (\frac{\partial}{\partial y_n})^{i_n}$$

$$P(D_{\rho(g)x}) = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{N}} c_I (\rho^\star(g)D_x)_1^{i_1}\cdots(\rho^\star(g)D_x)_n^{i_n}$$

obtained by applying the substitution (12) into $f^\star(D_y)$.

Now the chain rule formula reads :

$$\frac{\partial u}{\partial y_i}\circ\rho(g) = \sum_j \left(g^{-1}\right)_{j,i}\frac{\partial}{\partial x_j}(u\circ\rho(g)) = \left(\rho^\star(g)D_x\right)_i(u\circ\rho(g))$$

This reformulation allows us by a straighforward induction on the degree of P to write :

$$P(D_y)(u)\circ\rho(g) = P(D_{\rho(g)x})(u\circ\rho(g))$$

therefore in particular :

$$f^\star(D_y)(u)\circ\rho(g) = f^\star(\rho^\star(g)D_x)(u\circ\rho(g)) = \frac{1}{\chi(g)}f^\star(D_x)(u\circ\rho(g)) \tag{13}$$

If $u(x)$ is a semi invariant satisfying $u(\rho(g)x) = \chi_u(g)u(x)$ we get from the formula (13) and using the k-linearity of $f^\star(D_x)$:

$$f^\star(D_y)(u)(\rho(g)(x)) = \frac{1}{\chi(g)}f^\star(D_x)(\chi_u(g)u(x)) = \frac{\chi_u(g)}{\chi(g)}f^\star(D_x)(u)(x)$$

In particular applied to $u = f(y)^{s+1}$, we obtain

$$f^\star(D_y)(f(y)^{s+1})\circ\rho(g) = \frac{\chi(g)^{s+1}}{\chi(g)}f^\star(D_x)(f(x)^{s+1})$$

so that

$$\Big(\frac{1}{f(y)^s}f^\star(D_y)(f(y)^{s+1})\Big)_{|y=\rho(g)x} = \frac{1}{f(x)^s}f^\star(D_x)(f(x)^{s+1})$$

This proves that $x \to \frac{f^\star(D_x)(f(x)^{s+1})}{f(x)^s}$ is an absolute invariant hence a constant $b(s)$ depending on s. The fact that $b(s)$ is a polynomial is clear and we refer to [19] for the statement on its degree. □

We refer the interested reader to [34] and its bibliography for more informations on these b-functions as well as for the method of calculation of b_f in the case of an irreducible singular locus S by microlocal calculus and holonomy diagram. Let us give an accout of recent work about reducible examples involving linear free divisor, in [14], [15]. and also in [20], [21]. In the last two references, we find series of examples generalizing Cayley's identity.

***b* functions of reductive linear free divisors.**

The germ of an hypersurface $D = \{f^{-1}(0)\} \subset (X, x)$ is called free if the sheaf of ambient logarithmic vector fields tangent to D:

$$\mathrm{Der}(-\log D) = \{v \in \Theta_{X,x} \mid v(f) \in \mathcal{O}.f\}$$

which is a priori only reflexive of rank n, is free. In this case by K. Saito's criterion we can take the coefficient of a basis as a set of equations.

A linear vector field in $\mathbb{C}^n$ is a vector field associated to a square matrix $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ in the following way :

$$v = \sum a_{ij}x_i\partial_j = (x_1,\ldots,x_n)A(\partial_1,\ldots,\partial_n)^T (= \underline{x}A\overline{\partial},\ \text{for short})$$

A reduced hypersurface D in an n-dimensional complex vector space V is called a *linear free divisor* if the module $\mathrm{Der}(-\log D)$ has a basis of global degree 0 vector fields

$$\delta_k = A_k x = x^t A_k^t \partial_x \in \Gamma(V, \mathrm{Der}(-\log D))_0, \quad A_k \in \mathbb{C}^{n\times n}, \quad k = 1, \dots, n.$$

Then, by Saito's criterion

$$f = \det(\delta_1, \dots, \delta_n)$$

is a homogeneous defining equation for D of degree n. We show in [14] that such a divisor is the complement of the open orbit for a prehomogeneous vector space under the action a the linear algebraic group which is :

$$G_D := \{A \in Gl_n(\mathbb{C}) \mid A(D) = D\} = \{A \in Gl_n(\mathbb{C}) \mid f \circ A \in \mathbb{C} \cdot \ f\}$$

or rather its identity component G_D. We check also that its Lie algebra is identified with $\mathrm{Der}(-\log D)$, by $A \to \underline{x} A^T \overline{\partial}$. Now we have the following result

Theorem 4.2 *Assume that D is a reductive free divisor. The root of $b_f(s)$ are symmetric around -1, i.e. by $\alpha \leftrightarrow -\alpha - 2$.*

Summary of the proof. The dual representation is also linear free and its semi invariant satisfies $b_f = b_{f^*}$. On the other hand , by an elementary calculation based on the Fourier transforms of the operators involved in the functional equation one prove that $b_f(s) = b_{f^*}(-s-2)$. This gives the desired symmetry.

Note : In the case of an irreducible semi invariant, this result is known as fundamental Sato's theorem. In fact the proof of Sato can be adapted with some work to the reducible reduced case. However the proof in [14] is more elementary, and also is valid also in the case of non reduced divisors, in the same situation.

<u>Remarks</u>

- In [28] Narvaez prove by different method the same symmetry result for another class of divisors which is distinct from ours.

- A large amount of examples arise from the theory of representation spaces of quivers. Casually a quivers is a set Q_0 of vertives p_i and of of arrows Q_1, each arrow $\alpha \in Q_1$ having a source and a target. If we put in place of each vertex a vector space k^{d_i}, associated with a dimension vector $\mathbf{d}$, and an element of the representation space $Rep(Q, \mathbf{d})$ is just a collection of linear maps (A_α) between the k^{d_i} corresponding to the arrow α.

  There is an obvious action of the product of linear group $\prod GL(d_i)$ on this set of representation. A good dimension vector in that situation is a $\mathbf{d}$, for which we get in this way a reductive lfd $Rep(Q, \mathbf{d})$. And there are many such situations.

  A famous theorem of Gabriel state that the quivers which have only a finite number of irreducible representation type are exactly the Dynkin quivers, A_n, D_n, E_6, E_7, E_8 the roots of the Lie Algebra are exactly the above dimension vector and we obtain as many different divisor for each as there are non equivalent orientaton of the arrows.

  In [35] C. Sevenheck calculate a number of examples,, by relating these Bernstein polynomial to some spectral polynomial of good V-filtration associated to Gauss Manin systems as in [15]. In [20] A Lörincz gives a systematic way to calculate the b functions of the semi invariants of many quivers including all the Dynkin ones. His main tool is the so called Castling transformation which allows him to compute them recursively.

5 Arrangements of hyperplanes.

En construction

In [31] M. Saito shows the if $Q = \prod_{i=1}^{d}$ is a central indecomposable and essential arrangement we have

Theorem 5.1 *$MaxR_Q < 2 - \frac{2}{d}$ and the multiplicity of the root 1 is $m_1 = n$*

This prove again a theorem of Leykin quoted in [36]

Corollary 5.2 *The only integer root of b_Q is -1.*

The proof given in [36] is however much more elementary and also is an application of the study of modules $\mathcal{D} \cdot f^{\alpha}$ for complex values α.

In [**?**] Walther shows that Bernstein polynomial is not in general (contrary to what might suggest the generic case) a combinatorial invariant.

Let us also mention two results of Ph. Maisonobe on the Bernstein ideals of a free arrangement and of a generic one, see [22] and [23]:

Theorem 5.3 *Let V be a vector space and $\mathcal{A}$ a free arrangement of hyperplanes.Then the Bernstein ideal of the p-uple of the linear equations of this arrangement $H_1, \ldots, H_p$ is principal, generated by the polynomial:*

$$b_{\mathcal{A}}(s_1, \ldots, s_p) = \prod_{X \in L'\mathcal{A}} \prod_{i=0}^{2(card(J(X)-r(X)} \left(\sum_{i \in J(X))s_i + r(X) + j} \right)$$

where $L'(\mathcal{A})$ is the subset of irreducible elements in the lattice of intersections, and for $X \in L'(\mathcal{A})$ J(X) is the set of hyperplanes passing through X.

Theorem 5.4 *Let V be a vector space of dimension n and $\mathcal{A}$ a generic arrangement of $n+1$ hyperplanes. Then its Bernstein ideal is principal, generated by the polynomial:*

$$b_{\mathcal{A}}(s_1, \ldots, s_{n+1}) = \prod_{i=1}^{n+1}(s_i + 1) \prod_{k=0}^{n}(s_1 + s_2 + \cdots + s_{n+1} + n + k)$$

For a generic arrangement of p hyperplanes, with $p > n+1$ he has only a partial result.

6 Appendix : holonomic modules are spcialisable.

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