Exercises : group actions on S^1

Exercise 1. (warm-up) Let $f \in \mathrm{Homeo}^+(S^1)$, and suppose f has a periodic orbit of order $p \geq 1$. Prove that all periodic orbits of f have order p.

Exercise 2. i. (easy) Let $f \in \mathrm{Homeo}^+(S^1)$ have $\mathrm{rot}(f) = \dfrac{2}{3}$. Prove that

$$f \text{ is conjugate to a rotation} \Leftrightarrow f^3 = \mathrm{Id}\,.$$

Then, construct $g \in \mathrm{Homeo}^+(S^1)$ with $\mathrm{rot}(g) = \dfrac{2}{3}$ that is not conjugate to a rotation.
ii. (harder) Construct $f \in \mathrm{Homeo}^+(S^1)$ with $\mathrm{rot}(f) \notin \mathbb{Q}$, not conjugate to a rotation.

Exercise 3. Let $\varphi\colon \mathbb{Z} \to \mathbb{R}$. Suppose φ is a *quasi-morphism*, meaning there exists $C > 0$ such that for all $n, m \in \mathbb{Z}$, $|\varphi(m+n) - \varphi(m) - \varphi(n)| \leq C$.
Show that there exists $\tau \in \mathbb{R}$, and $D > 0$, such that for all $n \in \mathbb{Z}$, we have $|\varphi(n) - n \cdot \tau| \leq D$.

Exercise 4. (Properties of $\widetilde{\mathrm{rot}}$). Let $f \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

1. Show that $\widetilde{\mathrm{rot}}(f) := \lim\limits_{n \to +\infty} \dfrac{f^n(x)}{n}$ exists, and does not depend on x.
   (Possible hint : use exercise 3)
2. (easy) Show the following properties hold.
   - $\widetilde{\mathrm{rot}}(f^k) = k \cdot \widetilde{\mathrm{rot}}(f)$
   - $\widetilde{\mathrm{rot}}(gfg^{-1}) = \mathrm{rot}(g)$ for all $g \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$
   - $\widetilde{\mathrm{rot}}(f) = 0$ iff f has a fixed point. Generalize to the case where $\widetilde{\mathrm{rot}}(f) = \dfrac{p}{q} \in \mathbb{Q}$.
3. Find a real number $D > 0$ such that for all $f, g \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, we have

   $$|\widetilde{\mathrm{rot}}(fg) - \widetilde{\mathrm{rot}}(f) - \widetilde{\mathrm{rot}}(g)| \leq D.$$

   (harder) What is the best possible D ?
4. (hard) Prove that the map $\widetilde{\mathrm{rot}}\colon \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is continuous.

Remark : items 3. and 4. show that $\widetilde{\mathrm{rot}}$ is a continuous quasi-morphism. Finding the best constant D is (essentially !) the proof of the Milnor–Wood inequality...

Exercise 5. Prove that the definition of Euler number $\mathcal{E}(\rho)$ given in the lecture is independent of the choice of pants decomposition of the surface.

Exercise 6. Let $\pi_1(\Sigma_g) = \langle a_1, b_1, \dots a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$ be the standard presentation of a surface group, $g \geq 1$. For each generator s, let $\tilde{\rho}(s)$ denote a lift of $\rho(s)$ to $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$. Show the product of commutators

$$[\tilde{\rho}(a_1), \tilde{\rho}(b_1)] \dots [\tilde{\rho}(a_g), \tilde{\rho}(b_g)] \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$$

is independent of the choice of lifts, and equal to the map $x \mapsto x + \mathcal{E}(\rho)$ (Note : this gives another answer to the previous exercise, but you should try to do that one directly)

Exercise 7. Show that SO(2) is the unique maximal compact subgroup of $\text{Homeo}^+(S^1)$. (Hint : consider an invariant measure)

Exercise 8. Let R_α be an infinite order rotation, and let $f \in \text{Homeo}^+(S^1)$ be anything not equal to a rotation. Show the group generated by R_α and f contains a nonabelian free group.

Exercise 9. (Hölder's theorem, variation) Suppose that Γ acts freely on S^1 (meaning that each point has trivial stabilizer). Show that Γ is abelian, and semi-conugate to a subgroup of $SO(2)$.

Exercise 10. Show that there exists a unique (up to conjugacy) nontrivial action of $\text{PSL}(2, \mathbb{R})$ on S^1. **Challenge :** do not assume the homomorphism $\text{PSL}(2, \mathbb{R}) \to \text{Homeo}^+(S^1)$ is continuous (so in particular you can't use the previous exercise).
Alternatively : show there is a unique (up to semi-conjugacy) nontrivial action of the $(2, 3, 7)$-triangle group on S^1.

Exercise 11. (Challenge) Show that the rotation number of every element of the Thompson's group $T \subset \text{Homeo}^+(S^1)$ is rational. Which rational numbers are acheived ?

Exercise 12. (Open Question) Suppose Γ is a finitely generated group, and $\rho : \Gamma \to \text{Homeo}^+(S^1)$ an action so that the set of rotation numbers of elements in the image of ρ is *finite*. Is ρ necessarily semi-conjugate to a finite extension of a discrete subgroup of $\text{PSL}(2, \mathbb{R})$?
