Mapping class groups and curve complexes

Exercises

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1. Prove that a Dehn twist has infinite order in $MCG(S)$.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Use A to define a pseudo-Anosov map on a punctured torus.

3. Compute the dimension d of the curve complex $\mathcal{C}(S_{g,n})$ where $S_{g,n}$ is the surface of genus g with n punctures.

4. Find a free abelian group in $MCG(S_{g,n})$ with rank $d+1$. Can you find one that is not just Dehn multitwists?

5. Let $i(a,b)$ be the number of intersection points of two simple closed curves in minimal position in S. Given a bound $i(a,b) \leq k$, find a bound of the form $d_{\mathcal{C}(S)}(a,b) \leq k'$.

   Is there a bound in the opposite direction?

6. Exhibit pairs $a, b \in \mathcal{C}^{(0)}(S)$ with $d(a,b) = 1$, 2, 3, and 4. (Let S be $S_{2,0}$ or $S_{1,2}$, for example)

7. Show that the inclusion $\mathcal{C}(W) \to \mathcal{AC}(W)$ is a quasi-isometry, where $\mathcal{AC}(W)$ is the complex whose vertices are curves and properly embedded essential arcs, and simplices again correspond to disjoint sets.

8. If f is a pseudo-Anosov and $\lambda_\pm$ its stable and unstable laminations, prove that there exists $K = K(f)$ such that $d_W(\lambda_+, \lambda_-) \leq K$ for all $W \subsetneq S$.

9. If g is a geodesic in a δ-hyperbolic space X and $\pi : X \to g$ is a nearest-point projection, prove that there exists c such that, if $B_r(x)$ denotes a ball of radius r and $r < d(x,g)$ then $\operatorname{diam}(\pi(B_r(x))) \leq c$. Draw a convincing picture for $\mathbb{H}^2$.