Hyperbolic Structures on Surfaces

Exercises

10 July 2019

Exercise 1 (The hyperboloid model)**.** Consider the hyperboloid model $\mathbb{H}$ of the hyperbolic space, embedded into the Minkowski space $\mathbb{R}^{n,1}$.

(a) Show that the restriction of the Minkowski scalar product $\langle \cdot, \cdot \rangle_M$ induces a Riemannian metric $g^{\mathbb{H}}$ on $\mathbb{H}$.

(b) Determine the group of isometries $\mathbf{Isom}(\mathbb{H}, g^{\mathbb{H}})$.

(c) Let $v \in \mathbb{H}$ be a point, and $u \in T_v\mathbb{H}$ be a tangent vector. Determine the geodesic through v in the direction of u.

Exercise 2 (Möbius transformations)**.** Consider the Riemannian sphere $\widehat{\mathbb{C}}$.

(a) Show that the cross ratio function $cr : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is invariant under Möbius transformations.

(b) Show that every Möbius transformation $f \in \mathbf{Möb}(\widehat{\mathbb{C}})$ is conformal.

(c) Write down four simple conformal automorphisms.

(d) Show that $\mathbf{Möb}(\widehat{\mathbb{C}})$ acts transitively on $\widehat{\mathbb{C}}$, on ordered tuples of distinct points, and on ordered triples of distinct points.
Bonus: Show that the action on triples is free.

(e) Show that $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ lie on a circle or a line (i.e., a circle with infinite radius) if and only if $cr(z_1, z_2, z_3, z_4) \in \mathbb{R} \cup \{\infty\}$.

(f) Show that Möbius transformations map generalized circles to generalized circles.

(g) Prove that the Hilbert metric on the Klein model gives the hyperbolic metric.

Exercise 3 (The upper half plane model)**.** Consider the upper half plane model $(\mathcal{H}, g^{\mathbb{H}})$ of the hyperbolic 2-space.

(a) Determine the group of isometries $\mathbf{Isom}(\mathcal{H}, g^{\mathcal{H}})$.

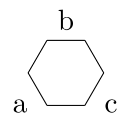
(b) Determine the distance in $\mathcal{H}$ between the points ia and ib for $a, b > 0$.

(c) Show that balls in $\mathcal{H}$ are also balls with respect to the Euclidean metric on $\mathbb{C}$.

(d) What can you say about the centers of the hyperbolic ball and its center with respect to the Euclidean metric?

(e) Prove that the sum of inner angles of a hyperbolic (geodesic) triangle is less than $180°$.

Exercise 4 (Hyperbolic polygons). (a) Let $a, b, c > 0$. Show that there is an (up to isometry) unique hyperbolic right angled hexagon, with the following side lengths:



(b) Show that there is an (up to isometry) unique ideal hyperbolic triangle.

(c) Show that there is an (up to isometry) unique one-parameter family of ideal hyperbolic quadrilaterals.

(d) Let $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. Express the cross ratios $cr(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ in terms of $cr(z_1, z_2, z_3, z_4)$ for any permutation $\sigma \in \mathfrak{S}_4$.