Hyperbolicity & Generalisations

IHES, July 2019

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E - Compute the stabiliser of 0, and of ∞ in $SL_2\mathbb{R} \curvearrowright \mathbb{H}^2 \cup (\mathbb{R}\cup\{\infty\})$.

E - Show that $SL_2\mathbb{R}$ acts transitively on the set of (unordered) triples of $\mathbb{R}\cup\{\infty\}$.

M - Show that the ideal triangle $(-1, \infty, 1)$ in the upper half plane model of $\mathbb{H}^2$, is δ-thin for some δ.

M - Show that $\mathbb{H}^2$ is Gromov-hyperbolic.

2 Show that $\mathbb{H}^2$ is not quasi-isometric to $PSL_2\mathbb{Z}$.

E-M

3 Show that if S_1 and S_2 are two finite generating sets of a group G, then $Cay_{S_1} G$ and $Cay_{S_2} G$ are quasi-isometric.

M

Write a proof of Švarc-Milnor lemma.

4 Let G be a finitely generated group.

E Show that if H is a finite index subgroup of G then H is quasi-isometric to G

M Deduce that all finitely generated, non abelian free groups are quasi-isometric to each other.

H? On the contrary, show that $\mathbb{Z}^2$ is not quasi-isometric to $\mathbb{Z}^3$.

(5) Show that a hyperbolic group G acts properly discontinuously
M and cocompactly on $(\partial G)^3 \setminus \Delta$

where $\Delta = \{(\xi_1, \xi_2, \xi_3) / \exists i \neq j \text{ with } \xi_i = \xi_j\}$.

(6) E - Show that $\mathbb{Z}^2$ is not hyperbolic

M - Show that if G is a hyperbolic group, then no subgroup is isomorphic to $\mathbb{Z}^2$.

(7) Show that every isometry of a tree either fix a point, or fix exactly two points in the boundary.

(8) M - Show that if G is hyperbolic, it is finitely presented.

?H - Show that if G is hyperbolic relative to P, and P is finitely presented, then G also.

(9) Show that if G is hyperbolic relative to P, and if $g \notin P$
E (but $g \in G$) then $P \cap (gPg^{-1})$ is finite.

M Assume G is hyperbolic relative to P, a finite index subgroup of G. What can then be said?

(9) Let G be a group hyperbolic relative to P.

M Show that P is a Lipschitz quasi retract of G : $\exists\ P \xrightarrow{i} G \xrightarrow{r} P$ such that $r \circ i = Id_P$ and r Lipschitz.

H? Deduce that if G is finitely presented then P is finitely presented.

(11) E Show that $PSL_2\mathbb{Z}$ is hyperbolic relatively to $\mathrm{Stab}\{\infty\}$.
(for its action on the upper half plane)

E Show that $\mathrm{Stab}_{PSL_2\mathbb{Z}}(\infty) \simeq \mathbb{Z}$.

M Show directly that most Dehn fillings of $PSL_2\mathbb{Z}$ are hyperbolic (compare to triangle groups).
Identify a set $\mathcal{F}$ of forbidden elements as in the statement.

(12) H Let G be hyperbolic relative to P, and $\widehat{\mathrm{Cay}}\, G$ be a cone-off Cayley graph (over left cosets of P).
Show that the action of G on $\widehat{\mathrm{Cay}}\, G$ is acylindrical.

(13) M Assume that "All hyperbolic groups are residually finite" (WHICH IS NOT PROVEN, and perhaps hard to believe...)
Prove that, under this assumption, all groups that are hyperbolic relative to residually finite groups would be residually finite.

(14) E Can $\mathbb{Z}^2$ act acylindrically on a hyperbolic space?

(15) M Recall that $MCG(\Sigma_g)$ is generated by Dehn twists about curves in a system of curves.
Show that $MCG(\Sigma_g)$ is not hyperbolic relative to a proper subgroup

(16) M Assume that G contains an hyperbolically embedded subgroup H which is free of rank 2.
Show that G is SQ-universal : any f.g group is a subgroup of a quotient of G.