IHES Summer School 2019 **July 8** **Yves Benoist**

Arithmeticity of Discrete subgroups A. **Discrete Subgroups**

We discuss here various facts on Zariski-dense and on discrete subgroups of $\mathrm{SL}(d,\mathbb{R})$. The exercises are independent of one another. Choose two or three exercises that match your personal background : (E)=easy, (M)=medium, (H)=hard, (O)=open.

Exercise 1. (E) Bruhat open cell Write $g \in G = \mathrm{SL}(2d,\mathbb{R})$ as a block matrix $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$. Let $P := \{g \in G \mid C_g = 0\}$ and $Q := \{g \in G \mid B_g = 0\}$.
(i) Prove that the product set QP is $QP = \{g \in G \mid \det A_g \neq 0\}$.
(ii) Prove that, for all g in G, one has $\det A_g = \det D_{g^{-1}}$.

Exercise 2. (E) Unipotent and semisimple elements Let $x \in G := \mathrm{SL}(d,\mathbb{R})$.
(i) Prove that x is unipotent (i.e. $(x-1)^d = 0$) if and only if there exists g in G such that $\lim\limits_{n\to\infty} g^n x g^{-n} = e$.
(ii) Prove that x is semisimple (i.e. diagonalizable over $\mathbb{C}$) if and only if the conjugacy class $\{gxg^{-1} \mid g \in G\}$ is closed in G.

Exercise 3. (E) Dimension 2 Let Γ be a non-solvable subgroup of $\mathrm{SL}(2,\mathbb{R})$. Prove that the group Γ is Zariski dense in $\mathrm{SL}(2,\mathbb{R})$.

Exercise 4. (M) Normal subgroups Prove that a Zariski dense subgroup Γ of $G = \mathrm{SL}(d,\mathbb{R})$ cannot contain an infinite abelian normal subgroup N.

Exercise 5. (M) Discrete or dense Let Γ be a Zariski dense subgroup of $\mathrm{SL}(d,\mathbb{R})$. Prove that the group Γ is either discrete or dense in G.

Exercise 6. (M) Discreteness of the normalizer Let Γ be a discrete Zariski dense subgroup of $G := \mathrm{SL}(d,\mathbb{R})$. Prove that the normalizer N of Γ in G is also a discrete Zariski-dense subgroup of G.

Exercise 7. (H) Zassenhaus neighborhoods Prove that there exists a neighborhood U of e in $G := \mathrm{GL}(d,\mathbb{R})$ such that, for every discrete subgroup Γ of G, the intersection $\Gamma \cap U$ is included in a connected nilpotent subgroup of G.
Indication: Prove first the existence of an open neighborhood Ω of 0 in the Lie algebra $\mathfrak{g} = \mathcal{M}(d,\mathbb{R})$ and of $c > 0$ such that for all X, Y in Ω, $\|\log(e^X e^Y e^{-X} e^{-Y})\| \leq c\|X\|\|Y\|$.

We discuss now various useful facts on lattices in Lie groups.

Exercise 8. (E) Commensurability Let Γ be a lattice in a Lie group G. Prove that a subgroup $\Gamma' \subset \Gamma$ is a lattice in G if and only if it has finite index in Γ.

Exercise 9. (M) Intersection with two subgroups Let G be a Lie group, Γ a discrete subgroup of G, and H_1, H_2 two closed subgroups of G such that $\Gamma \cap H_1$ is cocompact in H_1 and $\Gamma \cap H_2$ is cocompact in H_2. Prove that $\Gamma \cap H_1 \cap H_2$ is cocompact in $H_1 \cap H_2$.

Exercise 10. (M) Nilpotent groups Let G be a connected nilpotent Lie group and Γ be a lattice in G. Prove that the derived subgroup $[\Gamma, \Gamma]$ is a lattice in $[G, G]$. Indication: Assume first that G is a unipotent real algebraic group and check simultaneously by induction the equivalence for a discrete subgroup Γ of G:
Γ is cocompact in $G \iff \Gamma$ is a lattice in $G \iff \Gamma$ is Zariski dense in G.

Exercise 11. (M) Closed orbit Let G be a Lie group, $\Gamma \subset G$ a discrete subgroup, $X = G/\Gamma$ and $x_0 = \Gamma/\Gamma$ the basis point of X. Let $H \subset G$ a closed subgroup such that $\Gamma \cap H$ is a lattice in H. Prove that the orbit Hx_0 is closed in X. Indication: When $h_n x_0 \to \infty$ in Hx_0, find γ_n in $\Gamma \cap H$ such that $h_n \gamma_n h_n^{-1} \to e$.

Exercise 12. (H) Non irreducible lattices Let G_1, G_2 be two non-compact simple Lie groups and Γ be a lattice in $G := G_1 \times G_2$ such that $\Gamma \cap G_1$ is infinite. Prove that the subgroup $(\Gamma \cap G_1)(\Gamma \cap G_2)$ has finite index in Γ. Indication: First check that the projection N_1 of Γ on G_1 is discrete.

Exercise 13. (H) Lattices in $\mathbb{R}^2$ Let $\Lambda \subset \mathbb{R}^2$ be a lattice such that the set $\{xy \mid (x, y) \in \Lambda\}$ is closed and discrete in $\mathbb{R}$. Prove that there exist an integer $c \geq 1$ and $\lambda, \mu > 0$ such that $\Lambda \subset \{(\lambda m + \lambda n\sqrt{c}, \mu m - \mu n\sqrt{c}) \mid m, n \in \mathbb{Z}\}$. Indication: Apply Minkowski's theorem to the lattices $\Lambda_t := \{(tx, y/t) \mid (x, y) \in \Lambda\}$.

Exercise 14. (H) Lattices of matrices Let Λ be a lattice in the vector space $\mathcal{M}(d, \mathbb{R})$. Prove that the set $S_\Lambda := \{\det X \mid X \in \Lambda\}$ is either closed and discrete in $\mathbb{R}$ or it is dense in $\mathbb{R}$. Indication: Apply Ratner's theorem to the action of $H := \mathrm{SL}(d, \mathbb{R}) \times \mathrm{SL}(d, \mathbb{R})$ on the space of lattices in $\mathcal{M}(d, \mathbb{R})$. Check that H is a maximal subgroup in $\mathrm{SL}(d^2, \mathbb{R})$, and that, for all closed H-orbit $H\Lambda$, one has $S_\Lambda \subset t\mathbb{Z}$ for a $t > 0$.

IHES Summer School 2019 **July 9** **Yves Benoist**
Arithmeticity of Discrete subgroups C. **Arithmetic examples**

We discuss here important examples of arithmetic groups.

Choose two or three exercises to suit your taste in this sheet or in the previous one.

Exercise 1. (E) Linear groups Let $G = \mathrm{SL}(d,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$ and let $\Gamma := \{(g, g^\sigma) \mid g \in \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}])\}$, where σ is the Galois conjugation: for a, b matrices with rational coefficients, $(a + b\sqrt{2})^\sigma = a - b\sqrt{2}$.
(i) Prove that Γ is a discrete subgroup of G.
(ii) Prove that Γ is Zariski dense in G.
(iii) Check using Borel–Harish-Chandra's theorem that Γ is a lattice in G.

Exercise 2. (M) Orthogonal groups Let $G := \{g \in \mathrm{SL}(d,\mathbb{R}) \mid gJ_o\,{}^tg = J_o\}$ where J_o is the matrix $J_o = \mathrm{diag}(1, .., 1, -\sqrt{2})$ and let $\Gamma := G \cap \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}])$.
(i) Prove that Γ is a discrete subgroup of G.
(ii) Check using Borel–Harish-Chandra's theorem that Γ is a lattice in G.
(iii) Check that $\Gamma \smallsetminus \{e\}$ does not contain unipotent element.
(iv) Deduce that Γ is cocompact in G.

Exercise 3. (M) Unitary groups Let $G = \mathrm{SL}(2d,\mathbb{R})$ and Γ be the unitary group $\Gamma := \{g \in \mathrm{SL}(2d, \mathbb{Z}[\sqrt{2}]) \mid gJ\,{}^tg^\sigma = J\}$, where σ is the Galois conjugation and J is the antidiagonal matrix : $(J_{ij}) = (\delta_{i+j,2d+1})$.
(i) Prove that Γ is a discrete subgroup of G.
(ii) Let $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G \mid x \in \mathcal{M}(d,\mathbb{R})\}$. Prove that $\Gamma \cap U$ is a lattice in U.
(iii) Prove that Γ is Zariski dense in G.
(iv) Check using Borel–Harish-Chandra's theorem that Γ is a lattice in G.

Exercise 4. (H) Division algebra Let $D := \{d = \begin{pmatrix} m+n\sqrt{2} & 5p-5q\sqrt{2} \\ p+q\sqrt{2} & m-n\sqrt{2} \end{pmatrix} \mid m,n,p,q \in \mathbb{Z}\}$.
(i) Prove that D is a lattice in $\mathcal{M}(2,\mathbb{R})$ and a subring with no zero divisors.
Let $G := \mathrm{SL}(4,\mathbb{R})$ and $\Gamma := \{g = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \in G \mid d_1, d_2, d_3, d_4 \in D\}$.
(ii) Prove that Γ is a discrete subgroup of G.
(iii) Let $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G \mid x \in \mathcal{M}(2,\mathbb{R})\}$. Prove that $\Gamma \cap U$ is a lattice in U.
(iv) Prove that Γ is Zariski dense in G.
(v) Check using Borel–Harish-Chandra's theorem that Γ is a lattice in G.

Arithmeticity of Discrete subgroups D. **Intersecting Γ**

We discuss here discrete subgroups intersecting cocompactly a given subgroup.

Exercise 5. (E) Ping Pong Let $G = \mathrm{SL}(3,\mathbb{R})$. Prove that there exists a Zariski dense discrete subgroup Γ of G of infinite covolume. Indication: Choose Γ to be the free group generated by hyperbolic elements $\gamma_1, \gamma_2 \in G$ that play ping pong in $\mathbb{P}(\mathbb{R}^3)$.

Exercise 6. (M) Rank-one group Let $G = \mathrm{SL}(2,\mathbb{R})$ and $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\}$.
(i) Prove that there exists a discrete subgroup Γ of G of infinite covolume such that $\Gamma \cap U$ is cocompact in U.
(ii) Same question with $G = \mathrm{SL}(2,\mathbb{C})$ and $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{C}\}$.
Indication: Choose Γ_1 to be a lattice in U, choose Γ_2 to be the group generated by an hyperbolic element $\gamma_2 \in G$ that plays ping pong with Γ_1 and choose Γ to be the free product of Γ_1 and Γ_2.

Exercise 7. (H) Split torus Let $G = \mathrm{SL}(3,\mathbb{R})$ and $A = \{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in G\}$. Prove that there exists a Zariski dense discrete subgroup Γ of G of infinite covolume such that $\Gamma \cap A$ is cocompact in A.
Indication: Choose Γ_1 to be the group generated by the two matrices $\mathrm{diag}(\lambda, \lambda, \lambda^{-2})$ and $\mathrm{diag}(\lambda, \lambda^{-2}, \lambda)$ with $\lambda \gg 1$, choose Γ_2 to be the group generated by an hyperbolic element $\gamma_2 \in G$ that plays ping pong with Γ_1 and choose Γ to be the free product of Γ_1 and Γ_2.

Exercise 8. (O) Non split torus Let $G = \mathrm{SL}(3,\mathbb{R})$ and $B = \{\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \in G\}$.
Prove that there exists a Zariski dense discrete subgroup Γ of G of infinite covolume such that $\Gamma \cap B$ is cocompact in B and $\Gamma \cap A = \{e\}$.

Exercise 9. (O) Simple subgroup Let $G = \mathrm{SL}(3,\mathbb{R}) \supset H = \mathrm{SL}(2,\mathbb{R})$ and let Γ be a discrete Zariski dense subgroup of G such that $\Gamma \cap H$ is a lattice in H. Prove that Γ is a lattice in G.

Exercise 10. (O) Nori's conjecture Let G be a simple Lie group, $H \subset G$ be a simple subgroup with real rank ≥ 2, and Γ a discrete Zariski dense subgroup of G such that $\Gamma \cap H$ is a lattice in H. Prove that Γ is a lattice in G.