ℓ-adic cohomolgy of subdiagram unfoldings of Laurent polynomials and deformations of ℓ-adic sheaves

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We say ∇ has a pole of *Poincaré rank* m along the divisor $0\times X$ if m is the smallest integer such that

$$\nabla(\mathcal{F}) \subset \frac{1}{t^m}\Big(\sum_i \mathcal{O}_{\mathbb{C}\times X}dx_i + \mathcal{O}_{\mathbb{C}\times X}\frac{dt}{t}\Big)\otimes\mathcal{F},$$

where t is the coordinate for $\mathbb{C}$, and (x_i) the coordinate chart for X.

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for some matrices of homomorphic functions $\Omega_i(t,x)$ and $\Omega(t,x)$.

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for some matrices of homomorphic functions $\Omega_i(t,x)$ and $\Omega(t,x)$. From $dA + A \wedge A = 0$, we get

$$\sum_i \left(\frac{\partial \Omega_i}{\partial t} - \frac{1}{t}\left(\frac{\partial \Omega}{\partial x_i} - [\Omega, \Omega_i]\right)\right) dt dx_i$$

$$+ \quad \sum_{i<j} \left([\Omega_i, \Omega_j] + \frac{\partial \Omega_j}{\partial x_i} - \frac{\partial \Omega_i}{\partial x_j}\right) dx_i dx_j = 0.$$

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It follows that

$$\Big(\frac{\partial \Omega}{\partial x_i} - [\Omega, \Omega_i]\Big)|_{0 \times X} = 0,$$

$$d(\sum_i \Omega_i(0,x)dx_i) + (\sum_i \Omega_i(0,x)dx_i) \wedge (\sum_i \Omega_i(0,x)dx_i) = 0.$$

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$$\triangledown = \nabla|_{0\times X}, \qquad \mathrm{Res}(\nabla) = \nabla_{t\frac{\partial}{\partial t}}|_{0\times X}$$
$$\triangledown\triangledown = 0, \qquad \triangledown(R_\infty) = 0.$$

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The connection matrix is of the form

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Let

$$\Phi = (t\nabla)|_{0\times X} = \sum \Omega_i(0,x)dx_i : \mathcal{F}|_{0\times X} \to \mathcal{F}|_{0\times X} \otimes \Omega^1_X.$$

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$$R_0 = (t^2\nabla_{\frac{\partial}{\partial t}})|_{0\times X} = \Omega(0,x) \in \mathrm{End}(\mathcal{F}|_{0\times X}).$$

We have

$$[R_0, \Phi] = 0.$$

Let $\mathcal{E}$ be a trivial vector bundle on X, $\pi : \mathbb{P}^1 \times X \to X$ the projection. Suppose we have an integrable connection ∇ on $\pi^*\mathcal{E}$ with a logarithmic pole along $\infty \times X$, and a pole of Poicaré rank 1 along $0 \times X$, and holomorphic elsewhere.

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The connection $(\nabla, \pi^*\mathcal{E})$ can also be constructed from the tuple $(X, \mathcal{E}, \triangledown, R_0, R_\infty, \Phi)$.

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The connection $(\nabla, \pi^*\mathcal{E})$ can also be constructed from the tuple $(X, \mathcal{E}, \triangledown, R_0, R_\infty, \Phi)$. This gives rise the to the so-called Frobenius type structure.

A *Frobenius type structure* is a tuple $(X, \mathcal{E}, \nabla, R_0, R_\infty, \Phi)$ such that:

X: germ of complex manifold

$\mathcal{E}$: free $\mathcal{O}_X$ module of finite rank.

$R_0, R_\infty \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$.

$\Phi : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$: Higgs field.

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Let $\pi : \mathbb{P}^1 \times X \to X$ be the projection, t the coordinate chart at 0, $\tau = \frac{1}{t}$. We require that the connection

$$\begin{aligned} \nabla &= \pi^*\triangledown + \frac{\pi^*\Phi}{t} + (\frac{R_0}{t} - R_\infty)\frac{dt}{t} \\ &= \pi^*\triangledown + \tau\pi^*\Phi + (-\tau R_0 + R_\infty)\frac{d\tau}{\tau} \end{aligned}$$

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$$\triangledown\triangledown = 0, \quad \triangledown(R_\infty) = 0, \quad \Phi \wedge \Phi = 0, \quad [R_0, \Phi] = 0,$$
$$\triangledown(\Phi) = 0, \quad \triangledown(R_0) + \Phi = [R, R_\infty].$$

Any meromorphic integrable connection ∇ on a trivial vector bundle $\pi^*\mathcal{E}$ over $\mathbb{P}^1 \times X$ with a logarithmic pole along $\infty \times X$, a pole of Poincaré rank 1 along $0 \times X$, and no other poles is of the above form, and hence gives rise to a Frobenius type structure.

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Birkhoff problem: Let D be a disc, $(\mathcal{F}, \nabla)$ a trivial holomorphic bundle $\mathcal{F}$ on D equipped with an integrable connection with a pole of Poincaré rank 1 along 0.

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Birkhoff problem: Let D be a disc, $(\mathcal{F}, \nabla)$ a trivial holomorphic bundle $\mathcal{F}$ on D equipped with an integrable connection with a pole of Poincaré rank 1 along 0. Find a pair $(\widetilde{\mathcal{F}}, \widetilde{\nabla})$ such that $\widetilde{\mathcal{F}}$ is a trivial bundle on $\mathbb{P}^1$, $\widetilde{\nabla}$ is an integrable meromorphic connection with logarithmic pole at ∞ and no poles outside $\{0, \infty\}$, and $(\widetilde{\mathcal{F}}, \widetilde{\nabla})|_D \cong (\mathcal{F}, \nabla)$.

Theorem (Birkhoff problem for a family)

Let D be a disc, (X, x_0) a germ of complex manifold, $(\mathcal{F}, \nabla)$ a trivial holomorphic bundle on $D \times (X, x_0)$ equipped with an integrable meromorphic connection of Poincaré rank 1 along $0 \times X$.

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Algebraic version of the Birkhoff problem. Let (G, ∇) be a $\mathbb{C}[\tau, \tau^{-1}]$-module equipped with a connection having poles only at $0, \infty$ with regular singularity at ∞, *Algebraic version of the Birkhoff problem*. Let (G, ∇) be a $\mathbb{C}[\tau, \tau^{-1}]$-module equipped with a connection having poles only at $0, \infty$ with regular singularity at ∞, let G_0 be a free $\mathbb{C}[\tau]$-submodule of G with Poincaré rank 1 such that $G_0 \otimes_{\mathbb{C}[\tau]} \mathbb{C}[\tau, \tau^{-1}] = G$.

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on which the monodromy T acts.

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$$G_k H_\infty = (\mathcal{V}_\infty \bigcap \tau^{-k} G_0 + \tau^{-1} \mathcal{V}_\infty) / \tau^{-1} \mathcal{V}_\infty$$

on H_∞. $(G_0/\tau^k = \tau G_0/\tau^{k+1} \subset G_0/\tau^{k+1} \subset G.)$

Theorem (M. Saito's criterion)

Let N be the nilpotent part of the monodromy T. Suppose there exists a mixed Hodge structure on the nearby cycle of G at ∞ so that the Hodge filtration is $G.H_\infty$, and the weight filtration is the monodromy filtration $M.H_\infty$ of N. Then we can solve the Birkhoff problem.

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If $\mathcal{F}$ is a punctually pure lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{P}^1_{\mathbb{F}_q} - \{\infty, ...\}$, then the monodromy filtration on $\mathcal{F}_{\bar{\eta}_\infty}$ (considered as a representation of $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$) is the weight filtration, where η_∞ is the generic point of the henselization of $\mathbb{P}^1_{\mathbb{F}_q}$ at ∞.

ℓ-adic version of the "Birkhoff problem". Let η_0 be the generic point of the henselization of $\mathbb{P}^1_{\mathbb{F}_q}$ at 0 , and let $\rho : \mathrm{Gal}(\overline{\eta}_0/\eta_0) \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ be a $\overline{\mathbb{Q}}_\ell$-representation.

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$$\mathbf{w}_j = (w_{1j}, \dots, w_{nj}) \in \mathbb{Z}^n \ (j = 1, \dots, N)$$

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We suppose f is convenient and non-degenerate, that is, 0 lies in the interior of Δ, and for any face σ of Δ not containing 0, the equations

$$\frac{\partial f_\sigma}{\partial t_1} = \cdots = \frac{\partial f_\sigma}{\partial t_n} = 0$$

defines an empty subscheme in $\mathbb{T}^n = \mathrm{Spec}\,\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $f_\sigma = \sum_{\mathbf{w}_j \in \sigma} a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}$.

Let $g_1(t_1, \ldots, t_n), \ldots, g_m(t_1, \ldots, t_n)$ be a family of Laurent polynomials. Consider the deformation

$$F_x(t) = f(t_1, \ldots, t_n) + x_1 g_1(t_1, \ldots, t_n) + \cdots + x_m g_m(t_1, \ldots, t_n)$$

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of f. We say F is a *subdiagram deformation* of f if all exponents of monomials with nonzero coefficients in $g_1, \ldots, g_m$ lie in the interior of Δ. We say F is the *universal unfolding* of f if the images of $g_1, \ldots, g_m$ in the Jacobian quotient ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] / \left(\frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_n} \right)$ form a basis.

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$$(\Omega^{\cdot}_{\mathbb{T}^n \times \mathbb{A}^m / \mathbb{A}^m}[t, t^{-1}], e^{tF} \circ d \circ e^{-tF}).$$

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$$\begin{aligned} G &= \frac{\Omega^{n}_{\mathbb{T}^n\times\mathbb{A}^m/\mathbb{A}^m}[t,t^{-1}]}{\Big(d - tdF\wedge\Big)\Omega^{n-1}_{\mathbb{T}^n\times\mathbb{A}^m/\mathbb{A}^m}[t,t^{-1}]},\\ G^{(o)} &= \Omega^{n}_{\mathbb{T}^n}[t,t^{-1}]/\Big(d - tdf\wedge\Big)\Omega^{n-1}_{\mathbb{T}^n}[t,t^{-1}]. \end{aligned}$$

G is a free $C[x_1,\ldots,x_m][t,t^{-1}]$-module (trivial vector bundle over $(\mathbb{P}^1 - \{0,\infty\})\times\mathbb{A}^m$).

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$$\nabla_{\frac{\partial}{\partial x_j}} = e^{tF}\circ\frac{\partial}{\partial x_j}\circ e^{-tF},$$

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Using Saito's criterion, Douai and Sabbah prove that the Birkhoff problem is solvable for the pairs (G, G_0) (family version) and $(G^{(o)}, G_0^{(o)})$.

Next we consider the universal unfolding

$$F_x(t) = f(t_1, \ldots, t_n) + x_1 g_1(t_1, \ldots, t_n) + \cdots + x_m g_m(t_1, \ldots, t_n).$$

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The definition of the pair (G, G_0) relies on some transcendental procedure. Roughly speaking, G is the Fourier transform of the Gauss-Manin system for $F_x(t)$. There exists a neighborhood X of 0 in $\mathbb{C}^m$ such that G is a trivial holomorphic vector bundle on $(\mathbb{P}^1 - \{0, \infty\}) \times X$ equipped with a meromorphic connection ∇ with a regular singularity along $\infty \times X$, the Brieskorn lattice G_0 is a trivial holomorphic vector bundle on $(\mathbb{P}^1 - \{\infty\}) \times X$ such that $G_0|_{(\mathbb{P}^1 - \{0,\infty\}) \times X} = G$ and the connection ∇ has Poincaré rank 1 on G_0 along $0 \times X$.

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To get a Frobenius manifold structure on the universal unfolding parameter space, one need to find a primitive form to transplant the Frobenius type structure to the tangent sheaf of the universal unfolding parameter space. One also need a metric.

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Another approach is to start with the solution of the Birkhoff problem for a subdiagram deformation satisfying certain conditions. Then use a theorem of Hertling and Manin to show this solution has a universal deformation, which gives the Frobenius manifold structure.

In summary, we start with the Brieskorn lattice for f, which is obtained as the Fourier transform of the Gauss-Manin system associated to $f : \mathbb{T}^n \to \mathbb{A}^1$.

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Arithmetically, over characteristic p field. We start with the Deligne-Fourier transform $FT(Rf_!\overline{\mathbb{Q}}_\ell)$. It should satisfy conditions for applying Saito's criterion (being a lisse pure sheaf on $\mathbb{P}^1 - \{0, \infty\}$). Actually we can prove similar results for $FT(RF_!\overline{\mathbb{Q}}_\ell)$ for any subdiagram deformation F of f. We then study the deformation of $FT(Rf_!\overline{\mathbb{Q}}_\ell)$ with prescribed local monodromy at 0 and ∞.

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An analogue of the hypergeometric integral

$$f(y_1, \ldots, y_N) = \int_\Gamma e^{\sum_{j=1}^N y_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$

is

$$\mathrm{Hyp}(y_1, \ldots, y_N) = \sum_{t_1, \ldots, t_n \in \mathbb{F}_q^*} \psi\Big(\sum_{j=1}^N y_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}\Big).$$

This is the hypergeometric function over a finite field introduced by Gelfand-Graev.

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$$H : \mathbb{T}^n \times \mathbb{A}^N \to \mathbb{A}^1, \quad (t_1, \ldots, t_n, y_1, \ldots, y_N) \mapsto \sum_{j=1}^{N} y_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}.$$

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We define the *ℓ-adic GKZ hypergeometric sheaf* to be the object in $D_c^b(\mathbb{A}^N, \overline{\mathbb{Q}}_\ell)$ defined by

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By the Grothendieck trace formula, we have

$$\mathrm{Tr}\Big(\mathrm{Frob}_x, \mathrm{Hyp}_{\bar{x}}\Big) = \sum_{t_1, \ldots, t_n \in \mathbb{F}_q^*} \psi\Big(\sum_{j=1}^{N} y_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}\Big).$$

Theorem

(i) Hyp *is a mixed perverse sheaf on* $\mathbb{A}^N$ *of weights* $\leq n + N$ *and of rank* $(-1)^N n!\mathrm{vol}(\Delta)$ *(and we have a combinatorial formula for the rank of its weight* w *subquotient.)*

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(iii) Suppose 0 *lies in the interior of* Δ*. Then* $\mathcal{H}^{-N}(\mathrm{Hyp})|_V$ *is lisse, pure of weight* n*, and of rank* $n!\mathrm{vol}(\Delta)$*.*

$f = \sum_{j=1}^{N} a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}} \in \mathbb{F}_q[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$: a convenient nondegenerate Laurent polynomial.

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Enlarge the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$ by addings those exponents of monomials with nonzero coefficients in some g_i $(i = 1, \ldots, m)$.

This does not change the convex hull of the set $\{0, \mathbf{w}_1, \ldots, \mathbf{w}_N\}$ in $\mathbb{R}^n$.

$f = \sum_{j=1}^{N} a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}} \in \mathbb{F}_q[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$: a convenient nondegenerate Laurent polynomial.

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Enlarge the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$ by addings those exponents of monomials with nonzero coefficients in some g_i $(i = 1, \ldots, m)$. This does not change the convex hull of the set $\{0, \mathbf{w}_1, \ldots, \mathbf{w}_N\}$ in $\mathbb{R}^n$.

We may then write

$$\begin{aligned} g_i(t_1, \ldots, t_n) &= \sum_{j=1}^{N} b_{ij} t_1^{w_{1j}} \cdots t_n^{w_{nj}}, \\ t' F_x(t) &= \sum_{j=1}^{N} t'(a_j + \sum_{i=1}^{n} b_{ij} x_i) t_1^{w_{1j}} \cdots t_n^{w_{nj}}. \end{aligned}$$

$X = \mathbb{A}^m$.

$F : \mathbb{T}^n_X \to \mathbb{A}^1_X, \quad (t, x) \mapsto (F_x(t), x)$.

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When restricted to $\mathbb{P}^1_X - (\{0, \infty\} \times X)$, $\mathscr{H}^i\Big(FT(R\Phi_!\overline{\mathbb{Q}}_\ell)\Big) = 0$ *for* $i \neq n - 1$, *and* $\mathscr{H}^{n-1}\Big(FT(R\Phi_!\overline{\mathbb{Q}}_\ell)\Big)$ *is a pure lisse sheaf of weight* n.

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When restricted to $\mathbb{P}^1_X - (\{0,\infty\} \times X)$, $\mathscr{H}^i\left(FT(R\Phi_!\overline{\mathbb{Q}}_\ell)\right) = 0$ *for* $i \neq n-1$, *and* $\mathscr{H}^{n-1}\left(FT(R\Phi_!\overline{\mathbb{Q}}_\ell)\right)$ *is a pure lisse sheaf of weight* n. *For each fixed* x, *the restriction of* $\mathscr{H}^{n-1}\left(FT(R\Phi_!\overline{\mathbb{Q}}_\ell)\right)$ *to* $(\mathbb{P}^1 - \{0,\infty\}) \times \{x\}$ *is tamely ramified at* 0, *and has slopes* ≤ 1 *at* ∞.

For any rational point (τ, x) of $\mathbb{A}_X$, we have

$$\begin{aligned} & \mathrm{Tr}\Big(\mathrm{Frob}_{(\tau,x)}, (FT(R\Phi_! \overline{\mathbb{Q}}_\ell))_{\overline{(\tau,x)}}\Big) \\ = \; & -\sum_{t \in \mathbb{F}_q} \psi(\tau F(t,x)) \end{aligned}$$

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F: one of a following type of fields: a finite extension of $\mathbb{F}_\ell$, a finite extension of $\mathbb{Q}_\ell$, $\overline{\mathbb{F}}_\ell$, $\overline{\mathbb{Q}}_\ell$.
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We study deformations of $\mathcal{F}$, or equivalently of ρ_0.

Theorem

Let R be a complete noetherian local F-algebra with residue field F, let $\rho_s : \mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{GL}(R^r)$ be a representation such that there exist $P_{0,s} \in \mathrm{GL}(F^r)$ with the property $P_{0,s}^{-1}\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_{0,s} \equiv \rho_s \mod \mathfrak{m}_R$.

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Take $f(t_1, \dots, t_{n-1}) = t_1 + \cdots + t_{n-1} + \frac{1}{t_1 \dots t_{n-1}}$. It is proved by Barannikov that the Frobenius manifold structure on the universal unfolding of f is isomorphic to the one attached to the quantum cohomology of $\mathbb{P}_{\mathbb{C}}^{n-1}$.

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Let $\mathcal{F} = \mathcal{H}^{n-2}\Big(FT(Rf_! \overline{\mathbb{Q}}_\ell)\Big)$. Let ζ be a primitive n-th root of unity in k. We have

$$\mathcal{F}|_{\eta_\infty} \cong \bigoplus_{i=0}^{n-1} \mathcal{L}_\psi(n\zeta^i t)$$

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Consider the representation $\chi_i : \mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty) \to \mathbb{Q}_\ell[[s_i]]^*$ defined by the composite

$$\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty) \to \mathbb{Z}_\ell(1), \quad \sigma \mapsto \left(\frac{\sigma(\sqrt[\ell^n]{1/t})}{\sqrt[\ell^n]{1/t}}\right)$$

and the character $\mathbb{Z}_\ell(1) \to \mathbb{Q}_\ell[[s_i]]^*$ which maps a chosen generator of $\mathbb{Z}_\ell(1)$ to $1 + s_i$. It is the universal deformation of the trivial character $1 : \mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty) \to \mathbb{Q}_\ell^*$

By our theorem, there exists a representation

$$\rho : \pi_1(\mathbb{P}^1 - \{0, \infty\}) \to \mathrm{GL}\Big(n, \overline{\mathbb{Q}}_\ell[[s_1, \ldots, s_n]]/((1+s_1)\cdots(1+s_n)-1)\Big)$$

so that $\rho \mod (s_1, \ldots, s_n)$ is the representation corresponding to the sheaf $\mathcal{F}$, $\rho|_{\mathrm{Gal}(\bar{\eta}_0/\eta_0)} \cong U$,
$\rho|_{\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)} \cong \bigoplus_{i=0}^{n-1} \Big(\mathcal{L}_\psi(n\zeta^i x) \otimes \chi_i\Big)$ and $\det(\rho) = 1$.

Suppose we are given $\rho_0 : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(F^r)$ with $\det \rho_0 = 1$, and $P_{0,s} \in \mathrm{GL}(F^r)$ for each $s \in S$.

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$$\begin{aligned} R(A) = \{(\rho, (P_s)_{s \in S}) \quad | \quad & \rho : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A^r), P_s \in \mathrm{GL}(A^r), \\ & \rho \equiv \rho_0 \mod \mathfrak{m}_A, P_s \equiv P_{0,s}, \\ & \det(\rho) = 1 \mod \mathfrak{m}_A\}/\sim, \end{aligned}$$

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where two tuples $(\rho^{(i)}, (P_s^{(i)})_{s \in S})$ $(i = 1, 2)$ are equivalent if there exists $P \in \mathrm{GL}(A^r)$ such that $P \equiv I \mod \mathfrak{m}_A$ and

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For any $s \in S$, define

$$\begin{aligned} R_s(A) = \{\rho : \mathrm{Gal}(\bar{\eta}_s / \eta_s) \to \mathrm{GL}(A^r) \quad | \quad & \rho \equiv \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \mod \mathfrak{m}_A, \\ & \det(\rho) = 1\}. \end{aligned}$$

The functors $R, R_s : \mathcal{C} \to (\mathrm{Sets})$ satisfy Schlessinger's conditions (H1)-(H4), and hence are pro-representable. We have a canonical morphism of functors $R \to R_s$ given by

$$R(A) \to R_s(A), \quad (\rho, (P_s)) \to P_s^{-1} \rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P_s.$$

The fibers of this morphism are representations of $\pi_1(X - S, \bar{\eta})$ with prescribed local monodromy at s. These functors and morphism also appear in M. Kisin's work on modularity lifting problem (framed deformation of galois representations).

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Theorem

Suppose $\mathrm{End}(\mathcal{F}) = F$.
(i) The functors R_s $(s \in S)$ *are smooth.*
(ii) The morphism of functors $R \to \prod_{s \in S} R_s$ *is smooth.*
(iii) We have

$$\begin{aligned} \dim R(F[\epsilon]) &= -\chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F})) + |S| r^2 - 1, \\ \dim R_s(F[\epsilon]) &= -\chi(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{ad}^{(0)}(\rho_0)) + r^2 - 1 \end{aligned}$$

In the case $\mathcal{F} = \mathcal{H}^{n-2}\Big(FT(Rf_!\overline{\mathbb{Q}}_\ell)\Big)$ with $f = t_1 + \cdots + t_{n-1} + \frac{1}{t_1 \dots t_{n-1}}$, we have

$$\begin{aligned}
&\dim R(F[\epsilon]) = n(n-1) + 2n^2 - 1, \\
&\dim R_0(F[\epsilon]) = \dim R_\infty(F[\epsilon]) = n^2 - 1, \\
&\dim(\text{fiber of } R \to R_0 \times R_\infty) = n(n-1) + 1
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In the transcendental case, we deform the Brieskorn lattice associated to f by the Brieskorn lattice associated to the universal unfolding of f, and the deformation space is of dimension n, and we keep the logarithmic lattice at ∞ undeformed.

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In the transcendental case, we deform the Brieskorn lattice associated to f by the Brieskorn lattice associated to the universal unfolding of f, and the deformation space is of dimension n, and we keep the logarithmic lattice at ∞ undeformed. This suggests that we should choose a subspace of R_∞ of dimension n corresponding to the Brieskorn lattice of the universal unfolding of f, choose the trivial deformation subspace $\mathcal{C} \to (\mathrm{sets}),\ A \mapsto U$ of R_0, and try to find the element corresponding to the Frobenius type structure in the fiber of $R \to R_0 \times R_\infty$ above these two choices.