On the characteristic cycle of an étale sheaf

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25-26 septembre 2014, à l'IHES

Abstract

For an étale sheaf on a smooth variety over a perfect field of positive characteristic, the characteristic cycle is expected to be defined. In this series of lectures, we give a conditional definition and prove some of basic properties assuming the existence of a singular support satisfying certain local acyclicity conditions for families of morphisms to curves and to surfaces. For a sheaf on a surface, the ramification theory implies that the assumption is satisfied and we obtain unconditional results consequently.

Deligne describes, in unpublished notes [4], a theory of *characteristic cycles* of an étale sheaf assuming the existence of a closed subset of the cotangent bundle or a jet bundle satisfying a certain local acyclicity condition on morphisms to curves. He sketches or indicates proofs of the statements and formulates some conjectures. A crucial ingredient of his arguments lies in the continuity of the Swan conductor [9].

The main purpose of this lectures is to give proofs of his statements and conjectures, assuming the existence of a closed subset of the cotangent bundle, called a *singular support*, satisfying certain local acyclicity conditions for families of morphisms to curves and to surfaces. The statements include the Milnor formula (1.2) for an isolated characteristic point of a morphism to a curve and the Euler-Poincaré formula (3.4).

The ramification theory developed in [12] provides a closed subset of the cotangent bundle satisfying the required local acyclicity condition, on the complement of a closed subset of codimension $\geqq 2$. Using this, we obtain unconditional results for surfaces [13] including the Euler-Poincaré formula (3.4) without any assumption on ramification cf. [10], [3]. The two definitions of the characteristic cycle one by the Milnor formula and the other by the ramification theory are shown to be the same by reduction to the case of surfaces.

The proofs have two sides. A geometric side is the use of the universal family of hyperplane sections and its variant cf. [8]. The more abstract side including the continuity of the Swan conductor is based on the theory of vanishing cycles over a general base scheme developed in [7], [11]. In this course, the focus will be put on the geometric side.

After formulating the defining property of a singular support using local acyclicity of a family of morphisms to curves, we state the existence of the characteristic cycle characterized by the Milnor formula for the total dimension of the space of vanishing cycles at isolated characteristic points. We construct the characteristic cycle using the universal family of morphisms defined by pencils by taking an embedding to a projective space and prove the Milnor formula using the continuity of the Swan conductor. Finally, we state fundamental properties of the characteristic cycles including the Euler-Poincaré formula.

Contents

1 Singular support and the characteristic cycle

A singular support is defined to be a conic closed subset of the cotangent bundle such that a family of non-characteristic morphisms to curves with respect to it is local acyclic. The characteristic cycle is defined as a cycle supported on the singular support satisfying the Milnor formula for the total dimension of the vanishing cycles at an isolated characteristic points.

1.1 Singular support and the local acyclicity

Let E be a vector bundle on a scheme X. Recall that a closed subset S of E is said to be *conic* if it is stable under multiplication.

Definition 1.1. *Let X be a smooth scheme over a field k. Let $S = (S_i)_{i\in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X and let $T_i = S_i \cap T^*_X X \subset X$ be the intersections with the 0-section.*

1. *We say that a flat morphism $f\colon X \to C$ to a smooth curve C over k is* non-characteristic *with respect to S if the inverse image of S_i by the canonical map $df\colon X \times_C T^*C \to T^*X$ is a subset of the 0-section and if the restriction of f to T_i is* flat *for every $i \in I$.*

2. *Let*

$$(1.1)\qquad \begin{array}{ccc} W & \xrightarrow{f} & C \\ \downarrow & & \downarrow \\ X \times B & \xrightarrow{\mathrm{pr}_2} & B \end{array}$$

be a commutative diagram of flat morphisms of smooth schemes over k such that $W \to X \times B$ is étale and $C \to B$ is smooth of relative dimension 1. We say that $f\colon W \to C$ over B is non-characteristic *with respect to S if for every closed point b of B, the morphism $f_b\colon W_b = W \times_B b \to C_b = C \times_B b$ on the fibers is non-characteristic with respect to the pull-back of S by $T^*W_b \to T^*X$.*

Let k denote a perfect field of characteristic $p > 0$ and Λ a finite field of characteristic $\neq p$.

Definition 1.2. *Let X be a smooth scheme of dimension d over k and $\mathcal{K}$ be a constructible complex of Λ-modules on X.*

We say that a finite union $S = \bigcup_{i \in I} S_i$ of irreducible conic closed subsets of dimension d of the cotangent bundle T^*X is a singular support *of $\mathcal{K}$ if it satisfies the following condition in the case $d > 0$:*

(SS1) *For a commutative diagram* (1.1) *of flat morphisms of smooth schemes over k such that $W \to X \times B$ is étale and $C \to B$ is of relative dimension* 1*, the morphism $f \colon W \to C$ is* locally acyclic *relatively to the pull-back to W of $\mathcal{K}$ if the morphism $f \colon W \to C$ is* non-characteristic *with respect to S.*

If $d = 0$ and $\mathcal{K} \neq 0$, we require $S = T^*X$.

Let $SS\mathcal{K} \subset T^*X$ denote a singular support of $\mathcal{K}$.

Lemma 1.3. 1. *In the condition* (SS1)*, one can replace* locally acyclic *by* universally locally acyclic*.*

2. *If $f \colon Y \to X$ is a smooth morphism, the image $f^*SS\mathcal{K}$ of $SS\mathcal{K}$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$ is a singular support of $f^*\mathcal{K}$.*

3. *If a finite morphism $f \colon X \to Y$ is* unramified*, the image $f_*SS\mathcal{K}$ of $SS\mathcal{K}$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$ is a singular support of $f_*\mathcal{K}$.*

4. (local acyclicity of smooth morphism) *If $\mathcal{H}^q\mathcal{K}$ is locally constant for every q on the complement of a finite closed subset $Z \subset X$, the union of the* 0*-section $T^*_X X$ and the fiber $T^*_Z X$ is a singular support of $\mathcal{K}$.*

Ramification theory shows the existence of a singular support in codimension 1. Consequently, a singular support exists if $\dim X \leqq 2$ similarly as Lemma 1.3.4.

1.2 Characteristic cycle and the Milnor formula

Definition 1.4. *Let $f \colon X \to C$ be a morphism to a smooth curve C over k and u be a closed point of X. Let $\mathcal{K}$ be a constructible complex of Λ-modules on X and assume that a singular support $S = SS\mathcal{K}$ exists. We say that u is an* isolated characteristic point *of f relatively to $\mathcal{K}$ if there exists a neighborhood U of u such that the restriction of f to $U - \{u\}$ is non-characteristic with respect to the restriction of the singular support $S = SS\mathcal{K}$.*

Let u be an isolated characteristic point and ω be a basis of T^*C at $f(u)$. Then the section of T^*X defined on a neighborhood of u by $f^*\omega$ meets $S = SS\mathcal{K}$ properly on a neighborhood of the fiber $T^*_u X$. Consequently, for a linear combination $\sum_i a_i[S_i]$ of irreducible components of $S = \bigcup_i S_i$, the intersection number $(\sum_i a_i[S_i], f^*\omega)_{T^*X,u}$ is defined. Since S_i are assumed conic, it is independent of the choice of ω and will be denoted abusively as $(\sum_i a_i[S_i], df)_{T^*X,u}$.

Theorem 1.5.* *Assume that $S = SS\mathcal{K} = \bigcup_i S_i$ is a singular support of $\mathcal{K}$. Then, there exists a unique linear combination* $\mathrm{Char}\,\mathcal{K} = \sum_i a_i[S_i]$ *with $\mathbf{Z}[\frac{1}{p}]$-coefficients satisfying*

$$-\dim \mathrm{tot}\phi_u(\mathcal{K}, f) = (\mathrm{Char}\,\mathcal{K}, df)_{T^*X,u} \tag{1.2}$$

for every morphism $f \colon U \to C$ to a smooth curve C defined on an étale neighborhood U of a closed point u of X such that u is an isolated characteristic point of f.

The total dimension $\dim\mathrm{tot} = \dim + \mathrm{Sw}$ denotes the sum of the dimension with the Swan conductor. Here and in the following, * on a statement indicates that we assume the existence of a singular support satisfying the condition (SS1). The Milnor formula proved by Deligne in [1] is the case where $\mathcal{K} = \Lambda$ and Char $\mathcal{K} = (-1)^d[T^*_X X]$.

Examples 1. Let $U = X - D$ be the complement of a divisor $D = \bigcup_i D_i$ with simple normal crossing and $\mathcal{K} = j_!\mathcal{F}$ be the 0-extension of a locally constant constructible sheaf $\mathcal{F}$ on U tamely ramified along D. Then, we have

$$\mathrm{Char} j_!\mathcal{F} = (-1)^d \sum_I \mathrm{rank}\mathcal{F} \cdot [T^*_{D_I} X]$$

where $T^*_{D_I}X \subset T^*X$ denotes the conormal bundle of the intersection $D_I = \bigcup_{i\in I} D_i$. The formula (1.2) in this case is verified by E. Yang [14].

2. Assume $d = 1$ and let $Z \subset X$ be a finite closed subset as in Lemma 1.3.4. Then, we have

$$\mathrm{Char}\mathcal{K} = -\big(\mathrm{rank}\mathcal{K} \cdot [T^*X] + \sum_{x\in Z} a_x\mathcal{K} \cdot [T^*_x X]\big)$$

where $a_x\mathcal{K}$ denotes the Artin conductor $\dim \mathcal{K}_{\bar\eta} - \dim \mathcal{K}_x + \mathrm{Sw}_x\mathcal{K}$. The formula (1.2) in this case is a consequence of the induction formula for the Swan conductor.

2 Construction of characteristic cycle

We define the characteristic cycle first a priori depending on an embedding to a projective space by considering the universal family of pencils. We show the Milnor formula for morphisms defined by pencils using the continuity of Swan conductor. Together with the stability of the total dimension of vanishing cycles under deformation of morphisms to curves, it implies the independence of the choice of an embedding to a projective space and the Milnor formula in general.

2.1 Universal family of pencils

Assume that X is quasi-projective and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X,\mathcal{L})$ be a subspace of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee) = (E^\vee - \{0\})/\mathbf{G}_m = \mathrm{Proj}\ S^\bullet E$. We assume that the following condition on $E \subset \Gamma(X,\mathcal{L})$ is satisfied:

(E) On the base change to an algebraic closure $\bar k$ of k, for every pairs of distinct points $u \neq v$ of $X(\bar k)$, the composition

$$E \otimes \bar k \subset \Gamma(X,\mathcal{L}) \otimes \bar k \to \mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2\mathcal{L}_v \tag{2.1}$$

is a surjection.

The condition (E) is satisfied if we replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$ and E by $E^{(n)} = \mathrm{Im}(S^nE \to \Gamma(X,\mathcal{L}^{\otimes n}))$ for an integer $n \geqq 3$.

The dual projective space $\mathbf{P}^\vee = \mathbf{P}(E)$ parametrizes hyperplanes in $\mathbf{P}$. For a line $L \subset \mathbf{P}^\vee$, let $A_L \subset \mathbf{P}$ be the intersection of hyperplanes parametrized by L. The blow-up $X_L \to X$ at $X \cap A_L$ is an isomorphism on the complement $X^\circ_L = X - (X \cap A_L)$.

Let $p_L \colon X_L \to L$ denote the morphism defined by the pencil and $p_L^\circ \colon X_L^\circ \to L$ be the restriction.

First, we define the characteristic cycle $\mathrm{Char}_E \mathcal{K}$ a priori depending on E so that the Milnor formula for p_L holds for a generic pencil. Let $\mathbf{H} = \{(x, H) \in \mathbf{P} \times \mathbf{P}^\vee \mid x \in H\} \subset \mathbf{P} \times \mathbf{P}^\vee$ be the universal family of hyperplanes and

$$(X \times \mathbf{P}^\vee) \cap \mathbf{H} = X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$$

be the universal family of hyperplane sections. By the exact sequence $0 \to \Omega^1_{\mathbf{P}}(1) \to E \otimes \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(1) \to 0$, we identify $\mathbf{H} \subset \mathbf{P} \times \mathbf{P}^\vee$ with the projective space bundle $\mathbf{P}(T^*\mathbf{P})$ and $X \times_{\mathbf{P}} \mathbf{H}$ with its restriction $\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$.

We construct the universal family of morphisms defined by pencils. Let $\mathbf{G} = G(2, E)$ be the Grassmannian variety parametrizing lines in $\mathbf{P}^\vee$ and let $\mathbf{A} \subset \mathbf{P} \times \mathbf{G}$ be the universal linear subspace of codimension 2. The flag bundle $\mathbf{D} = \mathrm{Fl}(1, 2, E) \subset \mathbf{P}^\vee \times \mathbf{G}$ is the universal family of lines. We define $(X \times \mathbf{G})'$ by the left cartesian square of the diagram

$$\begin{array}{ccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & (X \times \mathbf{G})' & \longrightarrow & X \times \mathbf{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{D} & \longrightarrow & \mathbf{G}. \end{array} \tag{2.2}$$

The upper right arrow is the blow up at $(X \times \mathbf{G}) \cap \mathbf{A} = X \times_{\mathbf{P}} \mathbf{A}$ and is an isomorphism on the complement $(X \times \mathbf{G})^\circ = (X \times \mathbf{G}) - X \times_{\mathbf{P}} \mathbf{A}$. At the point of $\mathbf{G}$ corresponding to a line $L \subset \mathbf{P}^\vee$, the fiber of the right square defines $X \leftarrow X_L \to L$ and the restriction to the intersection with $(X \times \mathbf{G})^\circ$ defines $p_L^\circ \colon X_L^\circ \to L$.

Let $\widetilde{S} \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ denote the inverse image of $S = SS\mathcal{K} \subset T^*X$ by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ and let

$$\mathbf{P}(\widetilde{S}) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H} \tag{2.3}$$

be the projectivization. For an irreducible component of $S = \bigcup_i S_i$, let $T_i = S_i \cap T^*_X X \subset X$ be the intersection with the 0-section and let $\mathbf{P}^\vee_i = \mathbf{P}(E_i) \subset \mathbf{P}^\vee = \mathbf{P}(E)$ be the subspace defined by the kernel E_i of $E \subset \Gamma(X, \mathcal{L}) \to \Gamma(T_i, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{T_i})$. Define $\mathbf{R}(S) \subset X \times_{\mathbf{P}} \mathbf{H}$ to be the union

$$\mathbf{R}(S) = \bigcup_{i : \dim T_i > 0} (T_i \times \mathbf{P}^\vee_i) \subset X \times_{\mathbf{P}} \mathbf{H} \tag{2.4}$$

Define an open subset $(X \times \mathbf{G})^{\circ\circ} \subset (X \times \mathbf{G})^\circ$ to be the largest one such that the intersection Z with the inverse image of $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ is *quasi-finite* over $\mathbf{G}$ and that the intersection with the inverse image of $\mathbf{R}(S) \subset X \times_{\mathbf{P}} \mathbf{H}$ is empty.

Lemma 2.1.* 1. *For a point (u, L) of $(X \times \mathbf{G})^\circ$, the following conditions are equivalent:*

(1) *$u \in X_L^\circ$ is an isolated characteristic point of $p_L^\circ \colon X_L^\circ \to L$ with respect to $\mathcal{K}$.*

(2) *$(u, L) \in (X \times \mathbf{G})^{\circ\circ}$.*

2. *On the complement $X \times_{\mathbf{P}} \mathbf{H} - \mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S)$, the morphism $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is universally locally acyclic relatively to the pull-back of $\mathcal{K}$.*

Proof. 2. It follows from (SS1b) below. □

The condition (SS1) may be replaced by an equivalent condition on a family of smooth divisors. We say that an unramified morphism $Y \to X$ is *regular of codimension* r if it is a regular immersion of codimension r étale locally on Y.

Definition 2.2. *Let the notation be as in Definition* 1.1.

1. *We say that an unramified morphism* $i\colon Y\to X$ *regular of codimension* 1 *over* k *is* non-characteristic *with respect to* S *if the intersection of* $S_i\times_X Y$ *with the kernel of the surjection* $Y\times_X T^*X\to T^*Y$ *is a subset of the* 0*-section and if the unramified morphism* $T_i\times_X Y\to T_i$ *is also regular of codimension* 1 *for every* $i\in I$.

2. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Spec} k \end{array} \tag{2.5}$$

be a commutative diagram of flat morphisms of smooth schemes over k *such that* $Y\to X$ *is smooth and that the induced morphism* $i\colon Y\to X\times B$ *is unramified and is regular of codimension* 1. *We say that* $i\colon Y\to X\times B$ *over* B *is* non-characteristic *with respect to* S *if for every closed point* b *of* B*, the morphism* $i_b\colon Y_b=Y\times_B b\to X$ *is non-characteristic with respect to* S.

The condition in 1 that $T_i\times_X Y\to T_i$ is regular of codimension 1 implies that $S_i\times_X Y\to S_i$ is also regular of codimension 1. The condition (SS1) is equivalent to the following:

(SS1b) For a commutative diagram (2.5) as in Definition 2.2.2, the morphism $f\colon Y\to B$ is *locally acyclic* relatively to the pull-back to Y of $\mathcal{K}$ if the morphism $i\colon Y\to X\times B$ over B is *non-characteristic* with respect to S.

2.2 Continuity of Swan conductor

Definition 2.3. *Let* $f\colon Z\to S$ *be a quasi-finite morphism of schemes. We say that a function* $\varphi\colon Z\to \mathbf{Z}$ *is* flat *over* S *if for every geometric point* $x\to Z$ *and for every generalization* $s=f(x)\leftarrow t$ *of geometric points of* S*, we have*

$$\varphi(x)=\sum_{z\in Z_{(x)}\times_{S_{(s)}} t}\varphi(z) \tag{2.6}$$

where $Z_{(x)}$ *and* $S_{(s)}$ *denote strict localizations.*

If a constructible function φ is flat over S and if $Z\to S$ is *étale*, the function φ is locally constant.

The following is a partial generalization of the semi-continuity of Swan conductor by Deligne-Laumon [9].

Proposition 2.4. *Let*

$$\begin{array}{ccc} Z\subset X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

be a commutative diagram of morphisms locally of finite presentation of schemes. Assume that Z *is a closed subscheme of* X quasi-finite *over* S *and that* $g\colon Y\to S$ *is smooth of relative dimension* 1.

Let Λ be a finite field with characteristic invertible on S and let $\mathcal{K}$ be a constructible complex of Λ-modules on X. Assume that the restriction $f\colon X - Z \to Y$ is locally acyclic relatively to the restriction of $\mathcal{K}$ and that $p\colon X \to S$ is locally acyclic relatively to $\mathcal{K}$. Then, the function $\varphi\colon Z \to \mathbf{Z}$ defined by

$$\varphi(z) = \dim \operatorname{tot}\phi_z(\mathcal{K}|_{X_s}, f_s)$$

where $s = p(z)$ is constructible and flat *over S.*

Idea of Proof. We use the formalism of nearby cycles with general base scheme. Apply Deligne-Laumon [9] to $R\Psi_f\mathcal{K}$ using that the local acyclicity of $p\colon X \to S$ relatively to $\mathcal{K}$ implies the local acyclicity of $\overleftarrow{g}$ relatively to $R\Psi_f\mathcal{K}$.

Corollary 2.5.* *Let the notation be as in Lemma 2.1 and define a function $\varphi\colon Z \to \mathbf{Z}$ by*

$$\varphi(u, L) = -\dim \operatorname{tot}\phi_u(\mathcal{K}, p_L). \tag{2.7}$$

Then the function φ is constructible and flat over $\mathbf{G}$.

Proof. We apply Proposition 2.4 to

$$\begin{array}{ccc} Z \subset (X \times \mathbf{G})^{\circ\circ} & \longrightarrow & \mathbf{D} \\ & \searrow \quad \swarrow & \\ & \mathbf{G} & \end{array}$$

and the pull-back of $\mathcal{K}$. The local acyclicity of $(X \times \mathbf{G})^{\circ\circ} \to \mathbf{G}$ relatively to $\mathcal{K}$ is satisfied by the generic local acyclicity of Deligne [2]. Lemma 2.1.2 implies the local acyclicity of $(X \times \mathbf{G})^{\circ\circ} - Z \to \mathbf{D}$ relatively to $\mathcal{K}$. □

By Corollary 2.5, the function φ is constant a_i on a dense open subscheme $Z_i^\circ \subset Z$ of the inverse image of $\mathbf{P}(\widetilde{S}_i) \subset X \times_{\mathbf{P}} \mathbf{H}$ of each irreducible component of $S = \bigcup_i S_i$. Under the assumption (E), the restriction of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ to $\mathbf{P}(\widetilde{S}_i)$ is generically finite. Let $[\xi_i : \eta_i]_{\text{insep}}$ denote the inseparable degree of $\mathbf{P}(\widetilde{S}_i) \to p(\mathbf{P}(\widetilde{S}_i))$. We define the characteristic cycle by

$$\operatorname{Char}_E\mathcal{K} = \sum_i -\frac{a_i}{[\xi_i : \eta_i]_{\text{insep}}}[S_i]. \tag{2.8}$$

By the definition of $\operatorname{Char}_E\mathcal{K}$, for $p_L^\circ\colon X_L^\circ \to L$, the Milnor formula (1.2)

$$-\dim \operatorname{tot}\phi_u(\mathcal{K}, p_L) = (\operatorname{Char}_E\mathcal{K}, dp_L)_{T^*X,u} \tag{2.9}$$

holds at an isolated characteristic point u if $(u, L) \in Z$ is contained in $\coprod_i Z_i^\circ$.

Let $\mathbf{P}(\widetilde{\operatorname{Char}_E\mathcal{K}})$ denote the cycle $\sum_i -\dfrac{a_i}{[\xi_i : \eta_i]_{\text{insep}}}[\mathbf{P}(\widetilde{S}_i)]$. Then, the intersection number $(\operatorname{Char}_E\mathcal{K}, dp_L)_{T^*X,u}$ equals the intersection number $(\mathbf{P}(\widetilde{\operatorname{Char}_E\mathcal{K}}), X_L)_{X \times_{\mathbf{P}}\mathbf{H},u}$. We consider the cartesian diagram

$$\begin{array}{ccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & (X \times \mathbf{G})' & \longleftarrow & X_L \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{D} & \longleftarrow & L \\ & & \downarrow & & \downarrow \\ & & \mathbf{G} & \longleftarrow & [L]. \end{array}$$

The intersection number $(\mathrm{Char}_E\mathcal{K}, dp_L)_{T^*X,u} = (\mathbf{P}(\widetilde{\mathrm{Char}_E\mathcal{K}}), X_L)_{X\times\mathbf{PH},u}$ is further equal to degree at u of the fiber over $[L]$ of the pull-back of $\mathbf{P}(\widetilde{\mathrm{Char}_E\mathcal{K}})$ to $(X\times\mathbf{G})^{\circ\circ}$ by the top arrow, supported on a scheme Z quasi-finite over $\mathbf{G}$. Hence it is the value at (u, L) of another function on Z flat over $\mathbf{G}$.

Since two flat functions are equal if they are equal on a dense open, Corollary 2.5 implies that the Milnor formula (1.2) holds for $p_L^\circ\colon X_L^\circ \to L$ at every isolated characteristic point u.

2.3 Stability of the total dimension of the space of vanishing cycles

To complete the proof of Theorem 1.5, we show a stability of the total dimension of the space of vanishing cycles under small deformation of morphisms to curves. For two morphisms $f\colon X \to Y$ and $g\colon X \to Y$ of schemes and the closed subscheme $Z \subset X$ defined by an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say $g \equiv f \bmod \mathcal{I}_Z$ if the restrictions $f|_Z\colon Z \to Y$ and $g|_Z\colon Z \to Y$ are equal.

Proposition 2.6.* *Let $f\colon X \to C$ be a morphism to a smooth curve and u be an isolated characteristic point of f with respect to $\mathcal{K}$. Then, there exists an integer $N \geqq 1$ such that for every morphism $g\colon V \to C$ defined on an étale neighborhood V of u satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, the point u is also an isolated characteristic point of g and we have*

$$\dim \mathrm{tot}\phi_u(\mathcal{K}, g) = \dim \mathrm{tot}\phi_u(\mathcal{K}, f). \tag{2.10}$$

Proof. It is easy to see the existence of N such that $g \equiv f \bmod \mathfrak{m}_u^N$ implies that the point u is also an isolated characteristic point of g. Assume $g \equiv f \bmod \mathfrak{m}_u^N$, $V = X$ and $C = \mathbf{A}^1$. Define a homotopy $h\colon X\times\mathbf{A}^1 \to C\times\mathbf{A}^1$ connecting g to h by $h = (1-t)f + tg$. Then, it suffices to apply Proposition 2.4 to the diagram

$$\begin{array}{ccccc} \{u\}\times\mathbf{A}^1 \subset X\times\mathbf{A}^1 & & \xrightarrow{\ h\ } & & C\times\mathbf{A}^1\ . \\ & \searrow^{\mathrm{pr}_2} & & \swarrow_{\mathrm{pr}_2} & \\ & & \mathbf{A}^1 & & \end{array}$$

The morphism $\mathrm{pr}_2\colon X\times\mathbf{A}^1 \to \mathbf{A}^1$ is locally acyclic relatively to the pull-back of $\mathcal{K}$ the generic local acyclicity and $h\colon X\times\mathbf{A}^1 - Z \to C\times\mathbf{A}^1$ is locally acyclic relatively to the pull-back of $\mathcal{K}$ by (SS1). □

By approximation, one deduces easily from Proposition 2.6 that $\mathrm{Char}_E\mathcal{K}$ is independent of E and that the Milnor formula (1.2) for a general morphism is reduced to its special case (2.9) for a morphism defined by a pencil.

3 Properties of characteristic cycle

Lemma 3.1.* *For a finite unramified morphism $f\colon X \to Y$, we have*

$$\mathrm{Char} f_*\mathcal{K} = f_*\mathrm{Char}\mathcal{K}.$$

The right hand side denotes the image by the correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.

To prove further properties, we require that the singular support satisfies a local acyclicity condition for family of morphisms to surfaces.

Definition 3.2. *Let X be a smooth scheme over a field k and $S = (S_i)_{i\in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X.*

1. *We say that a smooth morphism $f\colon X \to P$ to a smooth surface P over k is* non-characteristic *with respect to S if the intersection of S_i with the image of $df\colon X\times_P T^*P \to T^*X$ is a subset of the 0-section and if the restriction to the intersection $T_i = S_i \cap T^*_X X \subset X$ with the 0-section of f is* flat *for every $i \in I$.*

2. *Let $P \to B$ be a smooth morphism of relative dimension 2 of smooth schemes over k and $W \to X \times B$ be an étale morphism over k. We say that a smooth morphism $f\colon W \to P$ over B is* non-characteristic *with respect to S if for every closed point b of B, the morphism $f_b\colon W_b = W \times_B b \to P_b = P \times_B b$ on the fibers is non-characteristic with respect to the pull-back of S by $T^*W_b \to T^*X$.*

Definition 3.3. *Let X be a smooth scheme of dimension d over k and $\mathcal{K}$ be a constructible complex of Λ-modules on X. We consider the following condition on the singular support $SS\mathcal{K} = \bigcup_{i\in I} S_i$:*

(SS2) *For a commutative diagram of smooth morphisms of smooth schemes over k*

$$\begin{array}{ccc} W & \xrightarrow{f} & P \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & B \end{array} \tag{3.1}$$

where $W \to X\times B$ is étale and $P \to B$ is of relative dimension 2, the morphism $f\colon W \to P$ is locally acyclic *relatively to the pull-back to W of $\mathcal{K}$ if the morphism $f\colon W \to P$ is* non-characteristic *with respect to S.*

If $\dim X \leqq 2$, (SS2) is satisfied trivially. For an integer $m > 2$, we define the condition (SSm) by replacing surface in the definition of (SS2) by smooth scheme of (relative) dimension m.

Similarly as (SS1), we have an equivalent condition:

(SS2b) For a commutative diagram (2.5) as in Definition 2.2.2 with $Y \to B$ smooth and with codimension 1 replaced by codimension 2, the morphism $f\colon Y \to B$ is *locally acyclic* relatively to the pull-back to Y of $\mathcal{K}$ if the morphism $f\colon Y\times X \to B$ is *non-characteristic* with respect to S.

We introduce a slightly stronger notion than the non-characteristicity for the immersion of a smooth divisor defined in Definition 2.2.1.

Definition 3.4. *Let $i\colon Y \to X$ be the closed immersion of a smooth divisor non-characteristic with respect to S. We say that $i\colon Y \to X$ is* strictly non-characteristic *with respect to S further if the divisor $T_i \cap Y \subset T_i$ is* reduced and irreducible.

Theorem 3.5.** *Assume that a singular support $SS\mathcal{K}$ of $\mathcal{K}$ satisfies also the condition* (SS2). *Let $i\colon Y \to X$ be a* strictly non-characteristic *regular immersion of a smooth divisor. Then $i^! SS\mathcal{K} \cup T^*_Y Y$ is a singular support of $i^*\mathcal{K}$ and we have*

$$\mathrm{Char}\, i^*\mathcal{K} = -i^! \mathrm{Char}\mathcal{K} \tag{3.2}$$

The right hand side of (3.2) is define by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$. Here and in the following, ** on a statement indicates that we assume the existence of a singular support satisfying the conditions (SS1) and (SS2).

Theorem 3.6. *Let $j\colon U \to X$ be the open immersion of the complement $U = X - D$ of a divisor D with simple normal crossings and $\mathcal{F}$ be a locally constant constructible sheaf on U such that the ramification along D is* strongly non-degenerate*. Then the characteristic cycle* Char $j_!\mathcal{F}$ *equals to that defined by ramification theory. In other words, the latter also satisfies the Milnor formula* (1.2).

Proof of Theorems. The proof goes Theorem 3.6 in dimension $\leqq 2 \Rightarrow$ Theorem 3.5 $\Rightarrow$ Theorem 3.6 in dimension > 2. We will only sketch the proof of the implication Theorem 3.6 in dimension $\leqq 2 \Rightarrow$ Theorem 3.5, if the time allows. The proof of Theorem 3.6 in dimension 2 uses a global argument, as in [4].

Corollary 3.7.** *Let $f\colon Y \to X$ be a smooth morphism. Then we have*

$$\text{Char} f^*\mathcal{K} = f^*\text{Char}\mathcal{K} \tag{3.3}$$

The right hand side denotes the image by the correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.

Theorem 3.8.** *Assume that X is projective and smooth and that $SS\mathcal{K}$ satisfies the conditions* (SSm) *for all $m \leqq d = \dim X$. Then we have*

$$\chi(X, \mathcal{K}) = (\text{Char } \mathcal{K}, T^*_X X)_{T^*X}. \tag{3.4}$$

Proof. We prove the equality (3.4) by induction on $\dim X$. If $\dim X = 0$, it is clear.

Lemma 3.9. *There exists a pencil L such that $p_L\colon X_L \to L$ has at most isolated characteristic points, that for a general hyperplane $H \in L$ the immersions $i\colon Y = X \cap H \to X$ and $i'\colon Z = X \cap A_L \to Y$ are strictly non-characteristic and that the isolated characteristic points of the morphism $p_L\colon X_L \to L$ are not contained in the inverse image of $X \cap A_L$.*

Let $X_L \to X$ be the blow-up at $X \cap A_L$. Then, we have

$$\chi(X, \mathcal{K}) = \chi(X_L, \mathcal{K}) - \chi(Z, \mathcal{K})$$

and

$$\chi(X_L, \mathcal{K}) = 2\chi(Y, \mathcal{K}) - \sum_u \dim \text{tot}\phi_u(\mathcal{K}, p_L)$$

by the Grothendieck-Ogg-Shafarevich formula [5]. By the hypothesis of induction, Theorem 3.5 and by the Milnor formula, we have

$$\chi(Y, \mathcal{K}) = (i^*\text{Char } \mathcal{K}, T^*_Y Y)_{T^*Y}, \quad \chi(Z, \mathcal{K}) = (i'^*\text{Char } \mathcal{K}, T^*_Z Z)_{T^*Z}$$

and

$$-\dim \text{tot}\phi_u(\mathcal{K}, p_L) = (\text{Char } \mathcal{K}, dp_L)_{T^*X,u}.$$

Substituting them and computing some Chern classes, we obtain (3.4). □

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