Symmetry groups of algebraic structures and their homology

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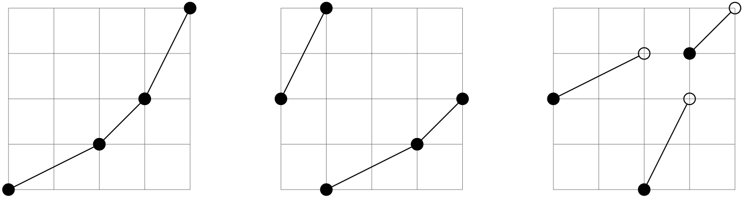
Topologie Algébrique et Applications

Montpellier — 26 Oct 2018

AN EXAMPLE

Thompson's groups



F $<$ T $<$ V

Theorem (Brown–Geoghegan 1984)

$$\mathsf{H}_\bullet \mathsf{F} = \mathbb{Z} \oplus \mathbb{Z}$$

Theorem (Ghys–Sergiescu 1987)

$$\mathsf{H}_\bullet \mathsf{T} = \mathsf{H}_\bullet^{\mathsf{S}^1}(\Lambda \mathsf{S}^3)$$

Proposition (Brown 1992, Kapoudjian 2002)

$$\mathsf{H}_\bullet \mathsf{V} \otimes \mathbb{Q} = 0, \quad \mathsf{H}_1 \mathsf{V} = \mathsf{H}_2 \mathsf{V} = \mathsf{H}_3 \mathsf{V} = 0$$

Theorem (S.–Wahl)

$$\mathsf{H}_\bullet \mathsf{V} = 0$$

Homological stability and stable homology I

A diagram

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \dots$$

of groups is **homologically stable** if the induced homomorphisms

$$\mathrm{H}_\bullet G_r \longrightarrow \mathrm{H}_\bullet G_{r+1}$$

are isomorphisms for r large (possibly depending on $\bullet$).

The **stable homology** is

$$\operatorname{colim}_r \mathrm{H}_\bullet G_r \cong \mathrm{H}_\bullet \operatorname{colim}_r G_r,$$

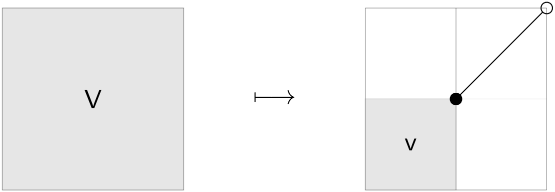
the homology of the group $G_\infty = \operatorname{colim}_r G_r$.

Thompson's group V and stabilization

There is a diagram

$$\mathsf{V}_0 \longrightarrow \mathsf{V}_1 \longrightarrow \mathsf{V}_2 \longrightarrow \mathsf{V}_3 \longrightarrow \cdots$$

of groups such that $\mathsf{V}_r \cong \mathsf{V}$ for all $r \geqslant 1$.



$\mathsf{H}_\bullet \mathsf{V} = 0 \because$ homological stability holds and the stable homology vanishes

ALGEBRAIC THEORIES

Algebraic theories (Lawvere)

An **algebraic theory** is an essentially surjective functor $\mathbf{T}$ that is defined on the category of finite sets and that preserves coproducts.

$$\mathbf{T}_r = \mathbf{T}(\{1, \ldots, r\}) = \textbf{free } \mathbf{T}\textbf{–algebra on } r \textbf{ generators}$$

The theory **Gr** of groups is the functor that sends a set to the free group generated by that set.

Similarly, there are algebraic theories **Ab** of abelian groups, modules over a given ring, rings, commutative rings, Lie rings, monoids, sets...

Homological stability and stable homology II

$\mathbf{T}$ = an algebraic theory

$\mathbf{T}_r$ = free $\mathbf{T}$–algebra on r generators

$$\mathrm{Aut}(\mathbf{T}_0) \longrightarrow \mathrm{Aut}(\mathbf{T}_1) \longrightarrow \mathrm{Aut}(\mathbf{T}_2) \longrightarrow \mathrm{Aut}(\mathbf{T}_3) \longrightarrow \cdots$$

Question

Are the groups $\mathrm{Aut}(\mathbf{T}_r)$ homology stable for $r \to \infty$?

Problem

Compute the stable homology $\mathrm{H}_\bullet \mathrm{Aut}(\mathbf{T}_\infty) \cong \mathrm{H}_\bullet \Omega^\infty_0 \mathrm{K}(\mathbf{T})$!

The theory of sets

$\{1, \ldots, r\}$ = (free) set on r generators

$\mathrm{Aut}(\{1, \ldots, r\}) = \mathrm{S}_r$ symmetric group

Theorem (Nakaoka, Barratt–Priddy–Quillen–Segal)

The groups S_r are homology stable ($r \to \infty$) and

$$\mathrm{H}_\bullet \mathrm{S}_\infty = \mathrm{H}_\bullet \Omega_0^\infty \mathbb{S}.$$

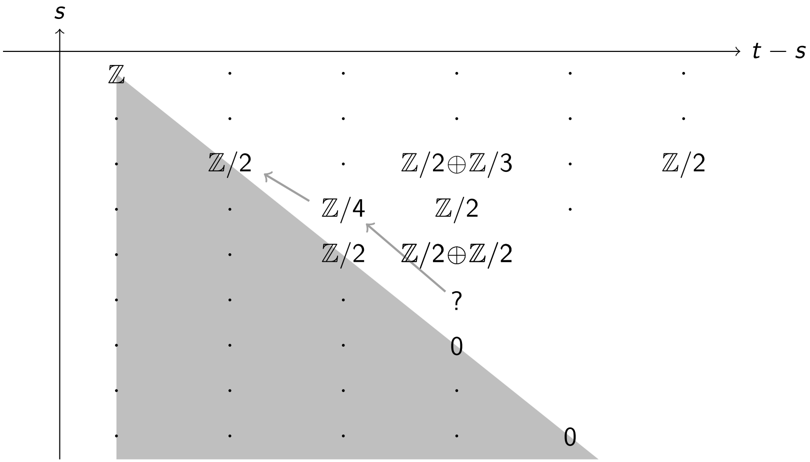
$$E^1_{s,t} = H_{t-s}(S_{-s}, S_{-s-1}) \Longrightarrow H_{t-s}(S_\infty)$$

s

$t-s$

$\mathbb{Z}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\mathbb{Z}/2$	$\cdot$	$\mathbb{Z}/2$	$\cdot$	$\mathbb{Z}/2$
$\cdot$	$\cdot$	$\cdot$	$\mathbb{Z}/3$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

$$E^1_{s,t} = H_{t-s}(GL_{-s}\,\mathbb{F}_2, GL_{-s-1}\,\mathbb{F}_2) \Longrightarrow H_{t-s}(GL_\infty\,\mathbb{F}_2)$$



The theory of abelian groups

$\mathbb{Z}^r$ free abelian group of rank r

$\mathrm{Aut}(\mathbb{Z}^r) = \mathrm{GL}_r(\mathbb{Z})$ general linear group

Theorem (Charney–Maazen–van der Kallen, Borel)

The groups $\mathrm{GL}_r(\mathbb{Z})$ are homology stable ($r \to \infty$) and

$$\mathrm{H}_\bullet(\mathrm{GL}_\infty(\mathbb{Z}); \mathbb{Q}) = \Lambda^\bullet(\, y_5, y_9, y_{13}, \ldots\,).$$

The theory of groups

F_r free group on r generators

$\mathrm{Aut}(\mathsf{F}_r)$ automorphism group of free group

Theorem (Hatcher–Vogtmann, Galatius)

The groups $\mathrm{Aut}(\mathsf{F}_r)$ are homology stable ($r \to \infty$) and

$$\mathsf{H}_\bullet \mathrm{Aut}(\mathsf{F}_\infty) = \mathsf{H}_\bullet \Omega_0^\infty \mathbb{S}.$$

CANTOR ALGEBRAS

The theory of Cantor (“$\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$”) algebras

$\mathsf{C}_{a,r}$ = free a–**ary Cantor algebra** (“$X^a \xrightarrow{\cong} X$”) on r generators

$\mathsf{V}_{a,r} = \mathrm{Aut}(\mathsf{C}_{a,r})$ = **Higman–Thompson group**

$\mathsf{V}_{a,r} \cong \mathsf{V}_{a,r+(a-1)}$ and $\mathsf{V}_{2,1} = \mathsf{V}$

Theorem (S.–Wahl)

The groups $\mathsf{V}_{a,r}$ are homology stable ($r \to \infty$) and

$$\mathsf{H}_\bullet \mathsf{V}_{a,r} = \mathsf{H}_\bullet \Omega_0^\infty \mathbb{S}/(a-1).$$

$\mathbb{S}/(a-1)$ = mod $a-1$ **Moore spectrum**

MAPPING CLASS GROUPS

The theory of nilpotent groups of class $c \geqslant 2$

$\mathsf{N}_c^r = \mathsf{F}_r/\Gamma_{c+1}\mathsf{F}_r$ = free nilpotent group of class c on r generators

$$\mathsf{N}_2^2 = \text{ Heisenberg group } = \left\{ \begin{bmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, \star \in \mathbb{Z} \right\}$$

$\mathrm{Aut}(\mathsf{N}_c^r)$ = automorphism groups of free nilpotent groups

Theorem (S.)

The groups $\mathrm{Aut}(\mathsf{N}_c^r)$ are homology stable ($r \to \infty$) and

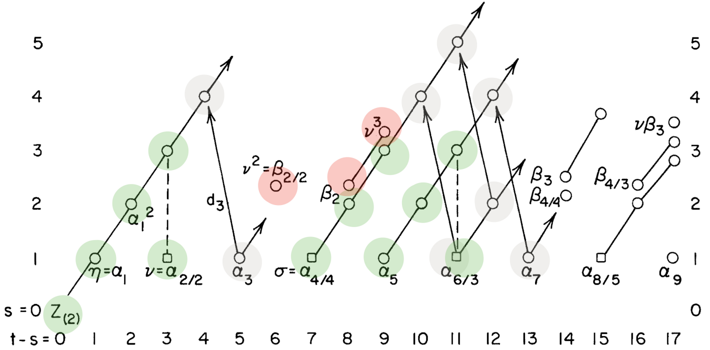
$$\mathsf{H}_\bullet(\mathrm{Aut}(\mathsf{N}_c^\infty);\mathbb{Q}) = \mathsf{H}_\bullet(\mathsf{GL}_\infty(\mathbb{Z});\mathbb{Q})$$

The algebraic K–theory perspective

$\mathsf{F}_r = \mathsf{N}^r_\infty \to \cdots \to \mathsf{N}^r_c \to \cdots \to \mathsf{N}^r_1 = \mathbb{Z}^r$

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & \\
 & & \mathsf{K}(\mathbf{Nil}_3) \arrow[d] & \\
 & & \mathsf{K}(\mathbf{Nil}_2) \arrow[d] & \\
\mathbb{S} \arrow[r, equal] & \mathsf{K}(\mathbf{Gr}) \arrow[r] \arrow[ur, bend left] \arrow[uur, bend left] \arrow[uuur, bend left] & \mathsf{K}(\mathbf{Ab}) \arrow[r, equal] & \mathsf{K}(\mathbb{Z})
\end{tikzcd}
$$

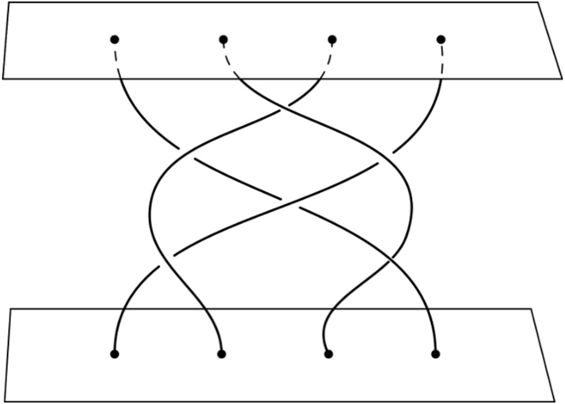
THE NOVIKOV SPECTRAL SEQUENCE FOR P=2



HOMOTOPY GROUP $\pi_{t-s}S^{\circ}$

Z	$\frac{Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{Z}{2}$	$\frac{Z}{16}$	$\frac{2Z}{2}$	$\frac{3Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{2Z}{2}$	$\frac{Z}{32} \oplus \frac{Z}{2}$	$\frac{2Z}{2}$	$\frac{4Z}{2}$

BRAIDS



Racks and quandles

A **rack** is a set R together with a binary operation

$$\triangleright \colon R \times R \longrightarrow R$$

such that all left-multiplications are automorphisms.

Each group G gives a quandle via conjugation $x \triangleright y = xyx^{-1}$.

A **quandle** is a rack Q such that $Q \xrightarrow{\triangle} Q \times Q \xrightarrow{\triangleright} Q$ is the identity.

Free quandles and their automorphisms

$$\mathsf{Q}_r = \{x \in \mathsf{F}_r \mid x \text{ conjugate to one of the } r \text{ generators}\}$$

$$\mathsf{Aut}(\mathsf{Q}_r) = \langle\, \mathsf{S}_r, \mathsf{B}_r \,\rangle \leqslant \mathsf{Aut}(\mathsf{F}_r)$$

Theorem (Vershinin 1998)

The stable homology $\mathsf{H}_\bullet(\mathsf{Aut}(\mathsf{Q}_\infty))$ has $\mathsf{H}_\bullet\Omega_0^\infty\mathbb{S}$ as a direct summand.

Proof

Every set is a quandle via $x \rhd y = y$. $\square$

M.S. Twisted homological stability for extensions and automorphism groups of free nilpotent groups. J. K-Theory 14 (2014) 185–201

M.S. The rational stable homology of mapping class groups of universal nil-manifolds. Ann. Inst. Fourier (to appear)

M.S. and N. Wahl. The homology of the Higman–Thompson groups. arXiv:1411.5035

Thank you!