Mapping class group representations in combinatorial quantization

Matthieu Faitg

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- There exists a Poisson structure $\{\cdot,\cdot\}$ on $C^\infty(\mathcal{A}_f/\mathcal{G})$.
- Fock and Rosly found a combinatorial description $\big((\mathcal{A}_f/\mathcal{G})^{\mathrm{FR}}, \{\cdot,\cdot\}_{\mathrm{FR}}\big)$ of $(\mathcal{A}_f/\mathcal{G}, \{\cdot,\cdot\})$ based on a discretization of S by a fat graph.
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  $\rightarrow$ notion of lattice gauge field theory based on a (Lie) group.
- Alekseev-Grosse-Schomerus and Buffenoir-Roche introduced a quantization of $\big((\mathcal{A}_f/\mathcal{G})^{\mathrm{FR}}, \{\cdot,\cdot\}_{\mathrm{FR}}\big)$, which is called combinatorial quantization.
  $\rightarrow$ notion of lattice gauge field theory based on a quantum group, or more generally on a Hopf algebra.

Preliminaries, notations

Let H be a finite-dimensional ribbon Hopf algebra, with universal R-matrix $R \in H \otimes H$ and ribbon element v.

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- Let $M = \sum_{i,j} E^i_j \otimes m_{ij} \in \mathrm{M}_m(\mathbb{C}) \otimes A$, $N = \sum_{i,j} E^i_j \otimes n_{ij} \in \mathrm{M}_n(\mathbb{C}) \otimes A$ where E^i_j are the elementary matrices. We define:

$$M_1 = \sum_{i,j} E^i_j \otimes \mathrm{I}_n \otimes M^i_j \in \mathrm{M}_m(\mathbb{C}) \otimes \mathrm{M}_n(\mathbb{C}) \otimes A$$

$$N_2 = \sum_{i,j} \mathrm{I}_m \otimes E^i_j \otimes N^i_j \in \mathrm{M}_m(\mathbb{C}) \otimes \mathrm{M}_n(\mathbb{C}) \otimes A.$$

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- In the sequel, we assume everywhere that H is a finite-dimensional factorizable ribbon Hopf algebra.

Example: $H = \overline{U}_q(\mathfrak{sl}(2))$

Let q be a primitive $2p$-th root of unity ($p > 2$).
$\overline{U}_q = \overline{U}_q(\mathfrak{sl}(2))$ is the $\mathbb{C}$-algebra generated by E, F, K modulo

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$
$$E^p = F^p = 0, \quad K^{2p} = 1.$$

$\overline{U}_q$ is a Hopf algebra:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K$$
$$\varepsilon(E) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = 1$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}$$

We have $\dim(\overline{U}_q) = 2p^3$. There is a R-matrix (in an extension of $\overline{U}_q$), a ribbon element $v \in \overline{U}_q$, and $\overline{U}_q$ is factorizable.
$\overline{U}_q$ is not semisimple.

The loop algebra $\mathcal{L}_{0,1}(H)$

Let $\mathrm{T}(H^*)$ be the tensor algebra of H^* and let $j : H^* \to \mathrm{T}(H^*)$ the canonical embedding. We denote $\overset{I}{M} = j(\overset{I}{T})$.

Definition

The loop algebra $\mathcal{L}_{0,1}(H)$ is the quotient of $\mathrm{T}(H^*)$ by the following *fusion relations*

$$\overset{I\otimes J}{M}_{12} = \overset{I}{M}_1 (\overset{IJ}{R'})_{12} \overset{J}{M}_2 (\overset{IJ}{R'^{-1}})_{12}$$

for all finite-dimensional H-modules I, J.

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Reflection equation

The following exchange relation holds:

$$\overset{IJ}{R}_{12} \overset{I}{M}_1 (\overset{IJ}{R'})_{12} \overset{J}{M}_2 = \overset{J}{M}_2 \overset{IJ}{R}_{12} \overset{I}{M}_1 (\overset{IJ}{R'})_{12}.$$

Properties of $\mathcal{L}_{0,1}(H)$

- The following right action of H

$$\forall\, h \in H, \quad \overset{I}{M} \cdot h = \sum_{(h)} \overset{I}{h'}\,\overset{I}{M}\,S(\overset{I}{h''})$$

endows $\mathcal{L}_{0,1}(H)$ with the structure of a H-module-algebra.

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- If we endow H with the right adjoint action, then

$$\begin{array}{rccl} \Psi_{0,1}: & \mathcal{L}_{0,1}(H) & \to & H \\ & \overset{I}{M} & \mapsto & (\overset{I}{T} \otimes \mathrm{Id})(RR') = (\overset{I}{a_i b_j}) b_i a_j \end{array}$$

is an isomorphism of H-module-algebras.

- In particular, $\mathcal{L}_{0,1}^{\mathrm{inv}}(H) \cong \mathcal{Z}(H)$.

Example: $\mathcal{L}_{0,1}(\overline{U}_q)$

Let $\mathcal{X}^+(2)$ be the fundamental representation of $\overline{U}_q$, and let

$$R := \overset{\mathcal{X}^+(2),\mathcal{X}^+(2)}{R} = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \hat{q} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad M := \overset{\mathcal{X}^+(2)}{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\mathcal{L}_{0,1}(\overline{U}_q)$ admits the following presentation

$$\left\langle a, b, c, d \;\middle|\; \begin{array}{ccc} & R_{12}M_1R_{21}M_2 = M_2R_{12}M_1R_{21} & \\ ad - q^2bc = 1, & b^p = c^p = 0, & d^{2p} = 1 \end{array} \right\rangle.$$

The monomials $b^ic^jd^k$ with $0 \leq i, j \leq p-1$, $0 \leq k \leq 2p-1$ form a basis.

The handle algebra $\mathcal{L}_{1,0}(H)$

Consider $\mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$ (free product of two copies of $\mathcal{L}_{0,1}(H)$), and let $j_1, j_2 : \mathcal{L}_{0,1}(H) \to \mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$ be the canonical embeddings. We denote $\overset{I}{A} = j_1(\overset{I}{M})$ and $\overset{I}{B} = j_2(\overset{I}{M})$.

Definition

The handle algebra $\mathcal{L}_{1,0}(H)$ is the quotient of $\mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$ by the following *exchange relations*:

$$\overset{IJ}{R}_{12}\overset{I}{B}_1(\overset{IJ}{R'})_{12}\overset{J}{A}_2 = \overset{J}{A}_2\overset{IJ}{R}_{12}\overset{I}{B}_1(\overset{IJ}{R^{-1}})_{12}$$

for all finite-dimensional H-modules I, J.

Properties of $\mathcal{L}_{1,0}(H)$

- The following right action of H

$$\forall\, h \in H, \quad \overset{I}{A} \cdot h = \sum_{(h)} \overset{I}{h'} \overset{I}{A} S(\overset{I}{h''}), \;\; \overset{I}{B} \cdot h = \sum_{(h)} \overset{I}{h'} \overset{I}{B} S(\overset{I}{h''})$$

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  endows $\mathcal{L}_{1,0}(H)$ with the structure of a H-module-algebra.
- There is an isomorphism of algebras:

$$\begin{array}{rccl} \Psi_{1,0}: & \mathcal{L}_{1,0}(H) & \to & \mathcal{H}(H^*) \\ & \overset{I}{A} & \mapsto & (a_i \overset{I}{b_j}) b_i a_j \\ & \overset{I}{B} & \mapsto & (\overset{I}{a_i}) b_i \overset{I}{T} (\overset{I}{b_j}) a_j. \end{array}$$

  where $\mathcal{H}(H^*)$ is the Heisenberg double of H^*.

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  where $\mathcal{H}(H^*)$ is the Heisenberg double of H^*.
  $\implies$ $\mathcal{L}_{1,0}(H)$ has a faithful representation on H^*.

In particular $\mathcal{L}_{1,0}(H)$ is (isomorphic to) a matrix algebra.

Definition of $\mathcal{L}_{g,n}(H)$

Let $\mathrm{mod}_r(H)$ be the category of finite-dimensional right H-modules and $\widetilde{\otimes}$ be the braided tensor product in $\mathrm{mod}_r(H)$.

Definition

$$\mathcal{L}_{g,n}(H) = \mathcal{L}_{1,0}(H)^{\widetilde{\otimes} g} \,\widetilde{\otimes}\, \mathcal{L}_{0,1}(H)^{\widetilde{\otimes} n} \in \mathrm{mod}_r(H).$$

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Explicitly, with generators and relations:

$$\begin{cases} \overset{I\otimes J}{U}(i)_{12} = \overset{I}{U}(i)_1\, R_{21}\, \overset{J}{U}(i)_2\, R_{21}^{-1} & \text{for } 1 \leq i \leq g+n, \\ R_{12}\, \overset{I}{U}(i)_1\, R_{12}^{-1}\, \overset{J}{V}(j)_2 = \overset{J}{V}(j)_2\, R_{12}\, \overset{I}{U}(i)_1\, R_{12}^{-1} & \text{for } 1 \leq i < j \leq g+n, \\ R_{12}\, \overset{I}{B}(i)_1\, R_{21}\, \overset{J}{A}(i)_2 = \overset{J}{A}(i)_2\, R_{12}\, \overset{I}{B}(i)_1\, R_{12}^{-1} & \text{for } 1 \leq i \leq g, \end{cases}$$

where $U(i), V(i)$ are $A(i)$ or $B(i)$ if $1 \leq i \leq g$ and are $M(i)$ if $g+1 \leq i \leq g+n$.

Alekseev isomorphism and representations of $\mathcal{L}_{g,n}(H)$

Theorem

There is an explicit isomorphism of algebras

$$\alpha_{g,n} : \mathcal{L}_{g,n}(H) \to \mathcal{L}_{1,0}(H)^{\otimes g} \otimes \mathcal{L}_{0,1}(H)^{\otimes n}.$$

Composing with $\Psi_{1,0}^{\otimes g} \otimes \Psi_{0,1}^{\otimes n}$, we get an isomorphism

$$\Psi_{g,n} : \mathcal{L}_{g,n}(H) \to \mathcal{H}(H^*)^{\otimes g} \otimes H^{\otimes n}.$$

It follows that every indecomposable representation of $\mathcal{L}_{g,n}(H)$ is of the form

$$(H^*)^{\otimes g} \otimes I_1 \otimes \ldots \otimes I_n$$

where $I_1, \ldots, I_n$ are indecomposable representations of H.

Characterization and representation of the invariants

Consider the matrices

$$\overset{I}{C} = \overset{I}{C}(1) \ldots \overset{I}{C}(g) \overset{I}{M}(g+1) \ldots \overset{I}{M}(g+n)$$

with $\overset{I}{C}(i) = \overset{I}{v}{}^2 \overset{I}{B}(i) \overset{I}{A}(i)^{-1} \overset{I}{B}(i)^{-1} \overset{I}{A}(i)$ for all finite-dim H-module I.

Theorem

$$x \in \mathcal{L}^{\mathrm{inv}}_{g,n}(H) \iff \forall\, I,\; x\overset{I}{C} = \overset{I}{C}x.$$

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Theorem

$$x \in \mathcal{L}_{g,n}^{\mathrm{inv}}(H) \iff \forall\, I,\; x\overset{I}{C} = \overset{I}{C}x.$$

If $(V, \triangleright)$ is a representation of $\mathcal{L}_{g,n}(H)$, define

$$\mathrm{Inv}(V) = \left\{ v \in V \;\middle|\; \forall\, I,\; \overset{I}{C} \triangleright v = 1_{\dim(I)} v \right\}.$$

By definition, $\mathrm{Inv}(V)$ is stable under $\mathcal{L}_{g,n}^{\mathrm{inv}}(H)$, and thus provides a representation.

Mapping class group of Σ_g

Let:

- Σ_g be the compact orientable surface of genus g,
- $D \subset \Sigma_g$ be an imbedded open disk,
- $C = \partial(\Sigma_g \setminus D)$.

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Consider the simple closed curves e, b_i, x_i $(1 \leq i \leq g)$ drawn on the blackboard. It is known that the Dehn twists about these curves generate $\mathrm{MCG}(\Sigma_g)$ and $\mathrm{MCG}(\Sigma_g \setminus D)$ (the Humphries generators).

Moreover, there exists a presentation of $\mathrm{MCG}(\Sigma_g)$ and $\mathrm{MCG}(\Sigma_g \setminus D)$ with generators $\tau_e, \tau_{b_i}, \tau_{x_i}$ (Wajnryb's presentation).

Action on the fundamental group

Put a basepoint on C and let a_i, b_i be the generators of $\pi_1(\Sigma_g \setminus D)$ drawn on the blackboard, such that

$$C = b_1 a_1^{-1} b_1^{-1} a_1 \ldots b_g a_g^{-1} b_g^{-1} a_g.$$

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$$\begin{aligned}
&\tau_e(a_1) = e^{-1} a_1 e, \ \ \tau_e(b_1) = e^{-1} b_1 e, \ \ \tau_e(b_2) = e^{-1} b_2, \\
&\tau_{b_i}(a_i) = b_i^{-1} a_i, \\
&\tau_{x_1}(b_1) = b_1 a_1, \\
&\tau_{x_j}(a_{j-1}) = x_j^{-1} a_{j-1} x_j, \ \ \tau_{x_j}(b_{j-1}) = b_{j-1} x_j, \ \ \tau_{x_j}(b_j) = x_j^{-1} b_j,
\end{aligned}$$

for all $1 \leq i \leq g$ and $2 \leq j \leq g$, and with $e = b_1 a_1^{-1} b_1^{-1} a_1 b_2 a_2^{-1} b_2^{-1}$, $x_j = a_{j-1} b_j a_j^{-1} b_j^{-1}$.

Lift of Dehn twists to $\mathcal{L}_{g,0}(H)$

We "lift" the Dehn twists by replacing curves of $\pi_1(\Sigma_g \setminus D)$ by matrices with coefficients in $\mathcal{L}_{g,0}(H)$.
For the Humphries generators, the non-trivial values are

$$\widetilde{\tau}_e(\overset{I}{A}(1)) = \overset{I}{E}^{-1}\overset{I}{A}(1)\overset{I}{E}, \ \widetilde{\tau}_e(\overset{I}{B}(1)) = \overset{I}{E}^{-1}\overset{I}{B}(1)\overset{I}{E}, \ \widetilde{\tau}_e(\overset{I}{B}(2)) = \overset{I}{v}\overset{I}{E}^{-1}\overset{I}{B}(2),$$

$$\widetilde{\tau}_{b_i}(\overset{I}{A}(i)) = \overset{I}{v}\overset{I}{B}(i)^{-1}\overset{I}{A}(i),$$

$$\widetilde{\tau}_{x_1}(\overset{I}{B}(1)) = \overset{I}{v}^{-1}\overset{I}{B}(1)\overset{I}{A}(1),$$

$$\widetilde{\tau}_{x_j}(\overset{I}{A}(j-1)) = \overset{I}{X}_j^{-1}\overset{I}{A}(j-1)\overset{I}{X}_j, \ \widetilde{\tau}_{x_j}(\overset{I}{B}(j-1)) = \overset{I}{v}^{-1}\overset{I}{B}(j-1)\overset{I}{X}_j,$$

$$\widetilde{\tau}_{x_j}(\overset{I}{B}(j)) = \overset{I}{v}\overset{I}{X}_j^{-1}\overset{I}{B}(j),$$

for all $1 \leq i \leq g$ and $2 \leq j \leq g$, and with
$\overset{I}{E} = \overset{I}{v}^4\overset{I}{B}(1)\overset{I}{A}(1)^{-1}\overset{I}{B}(1)^{-1}\overset{I}{A}(1)\overset{I}{B}(2)\overset{I}{A}(2)^{-1}\overset{I}{B}(2)^{-1}$,
$\overset{I}{X}_j = \overset{I}{v}^2\overset{I}{A}(j-1)\overset{I}{B}(j)\overset{I}{A}(j)^{-1}\overset{I}{B}(j)^{-1}$.

Elements associated to Dehn twist automorphisms

Since $\mathcal{L}_{g,0}(H)$ is a matrix algebra, the automorphisms lifting the Dehn twists are inner.

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Theorem

Let γ be a simple closed curve in $\Sigma_g \setminus D$. Then for all $x \in \mathcal{L}_{g,0}(H)$,

$$\widetilde{\tau}_\gamma(x) = v_{\widetilde{\gamma}}^{-1} \, x \, v_{\widetilde{\gamma}},$$

where $\widetilde{\gamma}$ is the lift of the curve γ in $\mathcal{L}_{g,n}(H)$ and $v_{\widetilde{\gamma}}$ is the ribbon element v over $\widetilde{\gamma}$.

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where $\widetilde{\gamma}$ is the lift of the curve γ in $\mathcal{L}_{g,n}(H)$ and $v_{\widetilde{\gamma}}$ is the ribbon element v over $\widetilde{\gamma}$.

It can be shown that a suitably normalized lift of a simple closed curve satisfies the defining relation of $\mathcal{L}_{0,1}(H)$, thus there exists a morphism of module-algebras

$$\begin{array}{rccc} j_{\widetilde{\gamma}} : & \mathcal{L}_{0,1}(H) & \to & \mathcal{L}_{g,0}(H) \\ & M & \mapsto & \widetilde{\gamma}. \end{array}$$

Since $v \in \mathcal{Z}(H) \cong \mathcal{L}_{0,1}^{\mathrm{inv}}(H)$, we define $v_{\widetilde{\gamma}} = j_{\widetilde{\gamma}}(v) \in \mathcal{L}_{g,0}^{\mathrm{inv}}(H)$.

Projective representation of $\mathrm{MCG}(\Sigma_g)$

Recall that we have a representation

- of $\mathcal{L}_{g,0}(H)$ on $V = (H^*)^{\otimes g}$, which we denote ρ
- and of $\mathcal{L}_{g,0}^{\mathrm{inv}}(H)$ on $\mathrm{Inv}(V)$, which we denote ρ_{inv}.

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Theorem

1) The assignment

$$\begin{array}{rcl} \mathrm{MCG}(\Sigma_g \setminus D) & \to & \mathrm{GL}(V) \\ \tau_\gamma & \mapsto & \rho(v_{\widetilde{\gamma}}^{-1}) \end{array}$$

is a projective representation.

2) The assignment

$$\begin{array}{rcl} \mathrm{MCG}(\Sigma_g) & \to & \mathrm{GL}(\mathrm{Inv}(V)) \\ \tau_\gamma & \mapsto & \rho_{\mathrm{inv}}(v_{\widetilde{\gamma}}^{-1}) \end{array}$$

is a projective representation.

Example: the torus Σ_1

- The mapping class groups are

$$\mathrm{MCG}(\Sigma_1 \setminus D) = B_3 = \left\langle \tau_a, \tau_b \,\middle|\, \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \right\rangle,$$
$$\mathrm{MCG}(\Sigma_1) = \mathrm{SL}_2(\mathbb{Z}) = \left\langle \tau_a, \tau_b \,\middle|\, \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b,\ (\tau_a\tau_b)^6 = 1 \right\rangle.$$

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$$\mathrm{MCG}(\Sigma_1) = \mathrm{SL}_2(\mathbb{Z}) = \left\langle \tau_a, \tau_b \,\middle|\, \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b,\ (\tau_a\tau_b)^6 = 1 \right\rangle .$$

- The representation space of $\mathcal{L}_{1,0}^{\mathrm{inv}}(H)$ (and of $\mathrm{MCG}(\Sigma_1)$) is

$$\mathrm{Inv}(H^*) = \mathrm{SLF}(H) = \{\varphi \in H^* \,|\, \forall\, x, y \in H,\ \varphi(xy) = \varphi(yx)\} \,.$$

Example: the torus Σ_1

- The mapping class groups are

$$\mathrm{MCG}(\Sigma_1 \setminus D) = B_3 = \langle \tau_a, \tau_b \,|\, \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \rangle\,,$$
$$\mathrm{MCG}(\Sigma_1) = \mathrm{SL}_2(\mathbb{Z}) = \left\langle \tau_a, \tau_b \,\middle|\, \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b,\ (\tau_a\tau_b)^6 = 1 \right\rangle.$$

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$$\mathrm{Inv}(H^*) = \mathrm{SLF}(H) = \{\varphi \in H^* \,|\, \forall\, x, y \in H,\ \varphi(xy) = \varphi(yx)\}\,.$$

- We can compute formulas for the action of v_A^{-1}, v_B^{-1} on H^*.

Example: the torus Σ_1

Theorem

- There is a representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{SLF}(H)$ given by:

$$\tau_a \mapsto \rho_{\mathrm{inv}}\left(v_A^{-1}\right), \quad \tau_b \mapsto \rho_{\mathrm{inv}}\left(v_B^{-1}\right).$$

- If $S(\varphi) = \varphi$ for all $\varphi \in \mathrm{SLF}(H)$, then this is in fact a projective representation of $\mathrm{PSL}_2(\mathbb{Z})$.
- This representation is equivalent to the Lyubashenko-Majid representation.

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More precisely,

$$v_A^{-1} v_B^{-1} v_A^{-1} = v_B^{-1} v_A^{-1} v_B^{-1} \text{ in } \mathcal{L}_{1,0}(H),$$

$$\rho_{\mathrm{inv}}\left(v_A^{-1} v_B^{-1}\right)^3 = \frac{\mu^l(v^{-1})}{\mu^l(v)} S,$$

where S is the antipode on H^* ($S(\varphi) = \varphi \circ S$) and μ^l is a left integral.

GTA basis of $\mathrm{SLF}(\overline{U}_q)$

$\overline{U}_q$ is not semisimple. Let

- $\mathcal{X}^{\epsilon}(s)$ be the simple $\overline{U}_q$-module of dimension p with highest weight εq^{s-1} ($\varepsilon \in \{\pm\}$, $1 \leq s \leq p$),
- $\mathcal{P}^{\varepsilon}(s)$ be the projective cover of $\mathcal{X}^{\varepsilon}(s)$ (usually called a PIM).

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We have $\dim(\mathrm{SLF}(\overline{U}_q)) = \dim(\mathcal{Z}(\overline{U}_q)) = 3p - 1$.

The GTA basis of $\mathrm{SLF}(\overline{U}_q)$ contains two types of elements:

- $\chi_s^\varepsilon = \mathrm{tr}(\overset{\mathcal{X}^\varepsilon(s)}{T})$ is the character of $\mathcal{X}^\varepsilon(s)$ ($1 \leq s \leq p$),
- $G_s = \mathrm{tr}(H_s^+) + \mathrm{tr}(H_{p-s}^-)$, where H_s^ε is a submatrix of $\overset{\mathcal{P}^\varepsilon(s)}{T}$ ($1 \leq s \leq p-1$).

Key property: the multiplication rules in this basis are remarkably simple.

Projective representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{SLF}(\overline{U}_q)$

The action of $\tau_a, \tau_b \in \mathrm{SL}_2(\mathbb{Z})$ on the GTA basis can be computed explicitly:

Theorem

$$\tau_a \chi_s^\epsilon = v_{\mathcal{X}^\epsilon(s)}^{-1} \chi_s^\epsilon, \qquad \tau_a G_{s'} = v_{\mathcal{X}^+(s')}^{-1} G_{s'} - v_{\mathcal{X}^+(s')}^{-1} \hat{q} \left(\frac{p-s'}{[s']} \chi_{s'}^+ - \frac{s'}{[s']} \chi_{p-s'}^- \right)$$

and

$$\tau_b \chi_s^\epsilon = \xi \epsilon (-\epsilon)^{p-1} s q^{-(s^2-1)} \left(\sum_{\ell=1}^{p-1} (-1)^s (-\epsilon)^{p-\ell} \left(q^{\ell s} + q^{-\ell s} \right) \left(\chi_\ell^+ + \chi_{p-\ell}^- \right) \right.$$
$$\left. + \chi_p^+ + (-\epsilon)^p (-1)^s \chi_p^- \right) + \xi \epsilon (-1)^s q^{-(s^2-1)} \sum_{j=1}^{p-1} (-\epsilon)^{j+1} [j][js] G_j,$$
$$\tau_b G_{s'} = \xi (-1)^{s'} q^{-(s'^2-1)} \frac{\hat{q}p}{[s']} \sum_{j=1}^{p-1} (-1)^{j+1} [j][js'] \left(2G_j - \hat{q} \frac{p-j}{[j]} \chi_j^+ + \hat{q} \frac{j}{[j]} \chi_{p-j}^- \right).$$

with $\epsilon \in \{\pm\}$, $0 \leq s \leq p$, $1 \leq s' \leq p-1$ and $\xi \in \mathbb{C} \setminus \{0\}$.

The multiplication rules are used to compute the action of τ_b.

Decomposition of the representation

Let $\mathcal{V} = \mathrm{vect}\big(\chi_s^+ + \chi_{p-s}^-, \chi_p^\pm\big)_{1 \leq s \leq p-1}$, of dimension $p+1$. Since:

- $\forall \psi \in \mathcal{V},\ \forall z \in \mathcal{Z}(\overline{U}_q),\ \psi(z?) \in \mathcal{V}$,
- $\mathcal{V}$ is an ideal $\mathrm{SLF}(\overline{U}_q)$,

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Theorem

There exists a projective representation $\mathcal{W}$ of $\mathrm{SL}_2(\mathbb{Z})$, of dimension $p-1$, such that

$$\mathrm{SLF}(\overline{U}_q) = \mathcal{V} \oplus \left(\mathbb{C}^2 \otimes \mathcal{W}\right).$$

$\mathcal{W}$ admits a basis $(w_s)_{1 \leq s \leq p-1}$ such that the action is given by

$$\tau_a\, w_s = v_{\mathcal{X}^+(s)}^{-1} w_s, \qquad \tau_b\, w_s = \xi(-1)^s q^{-(s^2-1)} \frac{\hat{q}p}{[s]} \sum_{j=1}^{p-1} (-1)^{j+1} [j][js] w_j.$$

- M. Faitg, *A note on symmetric linear forms and traces on the restricted quantum group* $\overline{U}_q(\mathfrak{sl}(2))$, arXiv:1801.07524.
  $\rightarrow$ Properties of the GTA basis.

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- Higher genus surfaces: work in progress.

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Thanks for listening!