Homology of the hypertree poset

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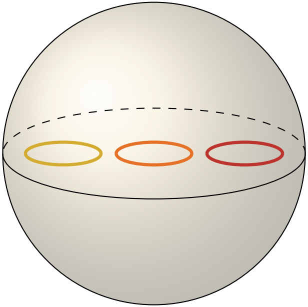
Motivation : $P\Sigma_n$

- F_n generated by $(x_i)_{i=1}^n$
- $P\Sigma_n$, pure symmetric automorphism group of the free group
  - group of automorphisms of F_n which send each x_i to a conjugate of itself,
  - group of motions of a collection of n coloured unknotted, unlinked circles in 3-space, where the components return to their original positions.



$A_{i,j}$: pull the ith circle through the jth circle

$$\leftrightarrow$$

$$\alpha_{i,j} : x_i \mapsto x_j x_i x_j^{-1}$$
$$x_k \mapsto x_k, k \neq i$$

[Goldsmith, 1981]

Historical background

1996 McCullough and Miller, *Symmetric automorphisms of free products*
→ Definition of a contractible complex on which $P\Sigma_n$ (and $OP\Sigma_n$) acts

2001 Brady, McCammond, Meier and Miller, *The pure symmetric automorphisms of a free group form a duality group*
→ Description of the fundamental domain of this complex as the geometric realisation of the Whitehead (hypertree) poset

2004 McCammond and Meier, *The hypertree poset and the l^2-Betti numbers of the motion group of the trivial link*
→ Compute of the Euler characteristic of the hypertree poset to explicitly describe the l^2-Betti numbers of $P\Sigma_n$, following a theorem of Davis, Januszkiewicz and Leary

Historical background

2006 Jensen, McCammond and Meier, *The integral cohomology of the group of loops*

2007 Jensen, McCammond and Meier, *The Euler characteristic of the Whitehead automorphism group of a free product*

2007 Chapoton, *Hyperarbres, arbres enracinés et partitions pointés*
$\rightarrow$ Conjectures the link between homology of the hypertree poset and pre-Lie operad

Summary

1. Posets and hypertrees
   - Homology of a poset
   - Hypertrees
   - Hypertree poset
2. Computation of the homology of the hypertree poset
   - Species
   - Counting strict chains using large chains
   - Pointed and hollow hypertrees
   - Relations between chains of hypertrees
   - Dimension of the homology
3. From the hypertree poset to rooted trees
   - PreLie species
   - Character for the action of the symmetric group on the homology of the poset

Posets and hypertrees

Homology of a poset

Let us consider a finite poset P.

Definition

A strict k-chain in a poset P on I is a $k+1$-tuple $(a_0, a_1, \ldots, a_k)$, where a_i are elements of P different from the minimum $\hat{0}$ or the maximum $\hat{1}$ (if there are some) and $a_i \prec a_{i+1}$. We denote by C_k^s the vector space of strict $k+1$-chains.

To every poset P, one can associate a simplicial complex (nerve of the poset seen as a category) whose k-faces are the k-chains of P.

Definition

The homology of a poset is the homology of its associated simplicial complex, named order complex.

We consider in this talk only pure poset:

Definition

A poset is pure if all its maximal chains (i.e. not strictly contained in another chain) have the same length.

Definition

A pure poset is Cohen-Macaulay if its (reduced) homology is concentrated in top degree.

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If a poset is Cohen-Macaulay and admits an action of a group G on its elements, the character for the action of G on the unique non trivial homology group of the poset is given by:

Proposition (Hopf trace formula)

$$\chi_{\tilde{H}_n} = \sum_{k=0}^{n} (-1)^{n-k} \chi_{C_k^s}, \textit{ with } \dim C_0^s = 1.$$

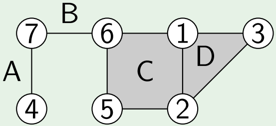
Hypergraphs and hypertrees

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- *V is a finite set (vertices)*
- *E is a collection of subsets of cardinality at least two of elements of V (edges).*

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

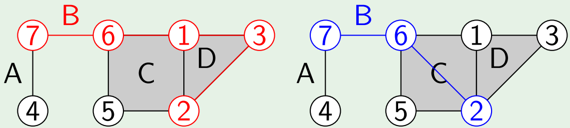
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from a vertex v_b to a vertex v_e in H is an alternating sequence of vertices and edges beginning by v_b and ending by v_e:

$$(v_b, \ldots, e_i, v_i, e_{i+1}, \ldots, v_e)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks



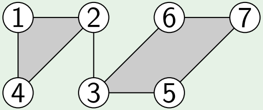
Hypertrees

Definition

A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- *there exists a walk from v to w in H with distinct edges e_i, (H is connected),*
- *and this walk is unique, (H has no cycles).*

Example of a hypertree



The hypertree poset

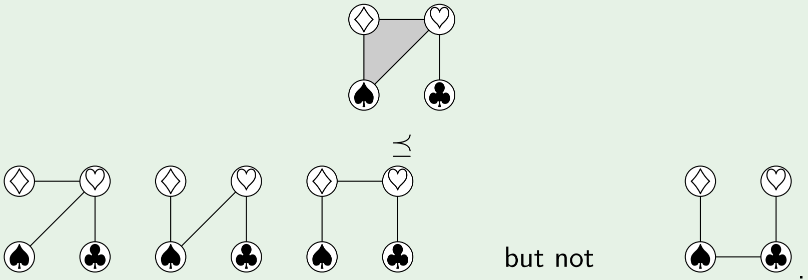
Definition

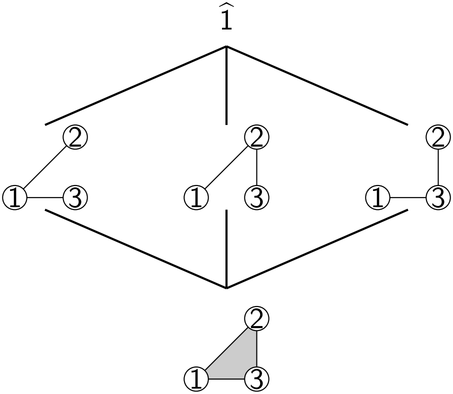
Let I be a finite set of cardinality n, S and T be two hypertrees on I.

$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices




$\hat{1}$
2
1 3
2
1 3
2
1 3
2
1 3

- Graded poset by the number of edges [McCullough and Miller 1996],
- There is a unique minimum $\hat{0}$,
- $\mathrm{HT(I)}$ = hypertree poset on I,
- $\mathrm{HT_n}$ = hypertree poset on n vertices.
- $\mathrm{HT_n}$ is Cohen-Macaulay [McCammond and Meier 2004]
- Euler characteristic : $(n-1)^{n-2}$ [McCammond and Meier 2004]

Goal:

- New computation of the homology dimension
- Computation of the action of the symmetric group on the homology (Conjecture of Chapoton 2007)

Computation of the homology of the hypertree poset

1 Posets and hypertrees

2 Computation of the homology of the hypertree poset
- Species
- Counting strict chains using large chains
- Pointed and hollow hypertrees
- Relations between chains of hypertrees
- Dimension of the homology

3 From the hypertree poset to rooted trees

What are species?

Definition

A *species* F *is a functor from the category of finite sets and bijections to itself. To a finite set* I*, the species* F *associates a finite set* $\mathsf{F}(I)$ *independent from the nature of* I*.*

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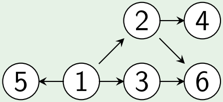
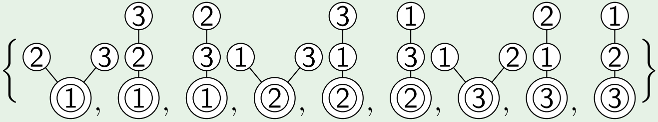
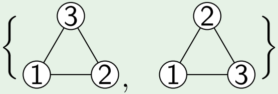
Counterexamples

The following sets are not obtained using species:

- $\{(1, \mathbf{3}, 2), (2, 1, \mathbf{3}), (2, \mathbf{3}, 1)(3, 1, \mathbf{2})\}$(set of permutations of $\{1, 2, 3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1, 2, 3, 4, 5, 6\}$)

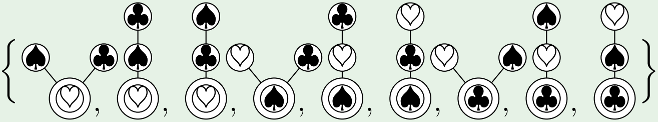
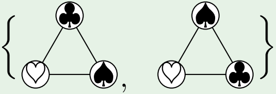


Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists Assoc on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (Species of non-empty sets Comm)
- $\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets Perm)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists Assoc on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets Comm)
- $\{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (Species of pointed sets Perm)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

These sets are the image by species of the set $\{\clubsuit,\heartsuit,\spadesuit\}$.

Operations on species and associated series

Proposition

Let F and G be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)

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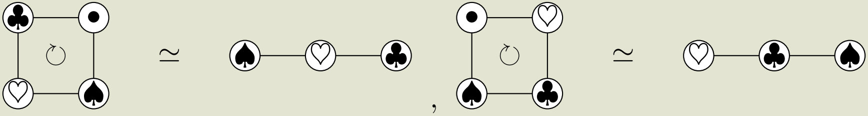
Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$



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- $(F \cdot G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, (product)

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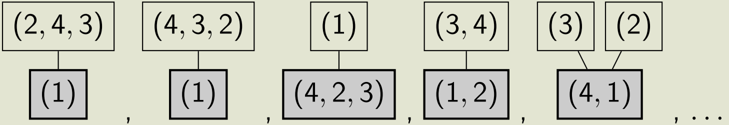
Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$



Definition

To a species F, we associate its generating series:

$$C_F(x) = \sum_{n\geq 0} \#F(\{1,\ldots,n\})\frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\text{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\text{Comm}} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\text{Perm}} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\text{PreLie}} = \sum_{n\geq 0} n^{n-1}\frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{Cycles} = -\ln(1-x)$.

Definition

The cycle index series *of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:*

$$Z_F(\mathfrak{p}) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- *with F^σ, is the set of F-structures fixed under the action of σ,*
- *and σ_i, the number of cycles of length i in the decomposition of σ into disjoint cycles.*

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1-p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\text{Comm}} = \exp(\sum_{k \geq 1} \frac{p_k}{k}) - 1$.

Operations on cycle index series

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$\begin{aligned} Z_{F+G} &= Z_F + Z_G, & Z_{F\cdot G} &= Z_F \times Z_G, \\ Z_{F\circ G} &= Z_F \circ Z_G, & Z_{F'} &= \frac{\partial Z_F}{\partial p_1}. \end{aligned}$$

Definition

The suspension Σ of a cycle index series $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \ldots).$$

Counting strict chains using large chains

Let P be a pure finite poset, with an action of a group G on its elements.

Definition

A large k-chain of P_n is a $k+1$-tuple $(a_0, a_1, \ldots, a_k)$, where a_i are (possibly maximal or minimal) elements of P_n and $a_i \preceq a_{i+1}$.

Denoting by χ_k^l (resp. χ_k^s) the character for the action of G on large (resp. strict) k-chains, we have:

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Proposition (BDO,16)

if $\hat{0}, \hat{1} \in P$,

$$\chi_k^l = \sum_{i \geq 0} \binom{k+2}{i+1} \chi_{i-1}^s,$$

if $\hat{0} \in P$, $\hat{1} \notin P$,
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$$\chi_k^l = \sum_{i \geq 0} \binom{k+1}{i} \chi_{i-1}^s,$$

if $\hat{0}, \hat{1} \notin P$

$$\chi_k^l = \sum_{i \geq 0} \binom{k}{i} \chi_i^s.$$

The Euler characteristic μ_χ for the action of G on the pure finite poset P is then given by:

Proposition (BDO,16)

if $\hat{0}, \hat{1} \in P$, $$\mu_\chi = \chi'_{-3}$$	*if* $\hat{0} \in P$, $\hat{1} \notin P$, *if* $\hat{1} \in P$, $\hat{0} \notin P$, $$\mu_\chi = -\chi'_{-2}$$	*if* $\hat{0}, \hat{1} \notin P$ $$\mu_\chi = \chi'_{-1} - 1.$$

Conclusion :

It is enough to compute the character for the action of the symmetric group on n-chains.

Back to hypertrees

Theorem (McCammond, Meier, 04)

The poset $\widehat{\mathrm{HT}_\mathrm{n}}$ is Cohen-Macaulay, i.e. its reduced homology is concentrated in higher degree.

Corollary

$$\chi_{\tilde{H}_{n-3}} = (-1)^n \chi^{I}_{-2}$$

The hypertrees will now be on n vertices.

Pointed hypertrees

Definition

Let H be a hypertree on I. H is:

- *rooted in a vertex s if the vertex s of H is distinguished,*

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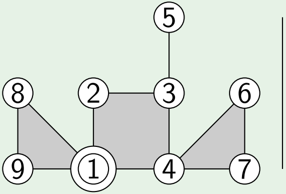
Example of pointed hypertrees



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Let H be a hypertree on I. H is:

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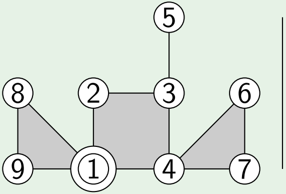
Example of pointed hypertrees



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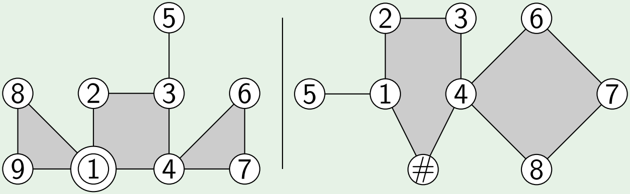

Relations between species of hypertrees

Theorem (BDO 13)

The species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{H}_k^p = X \cdot \mathcal{H}_k' \tag{1}$$

$$\mathcal{H}_k^p = X \cdot \mathrm{Comm} \circ \mathcal{H}_k^c + X, \tag{2}$$

$$\mathcal{H}_k^c = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c \circ \mathcal{H}_k^p, \tag{3}$$

$$\left(\mathcal{H}_{k-1}^p - x\right) \circ \mathcal{H}_k^p + \mathcal{H}_k = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p + \mathcal{H}_k^p, \tag{4}$$

Relations between species of hypertrees

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$$\left(\mathcal{H}_{k-1}^p - x\right) \circ \mathcal{H}_k^p + \mathcal{H}_k = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p + \mathcal{H}_k^p, \tag{4}$$

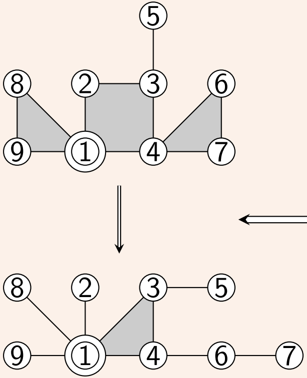
Proof.

1. Rooting a species F is the same as multiplying the singleton species X by the derivative of F,

Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

$$\mathcal{H}_k^p = X \cdot \mathrm{Comm} \circ \mathcal{H}_k^c + X,$$



Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{C}_k^p = x \cdot \exp\left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1\right), \tag{5}$$

$$(\mathcal{C}_{k-1}^p - x)(\mathcal{C}_k^p) + \mathcal{C}_k = (\mathcal{C}_{k-1} - x)(\mathcal{C}_k^p) + \mathcal{C}_k^p, \tag{6}$$

$$x \cdot \mathcal{C}_k' = \mathcal{C}_k^p, \tag{7}$$

Lemma

The generating series of $\mathcal{H}_0^p$ is given by:

$$\mathcal{C}_0^p = x \exp(e^{\mathcal{C}_0^p} - 1)$$

Hence, we have $\mathcal{C}^p_{-1} = xe^x$ and $\mathcal{C}_{-1} = e^x$.

Lemma

The generating series of $\mathcal{H}_0^p$ is given by:

$$\mathcal{C}_0^p = x \exp(e^{\mathcal{C}_0^p} - 1)$$

Hence, we have $\mathcal{C}_{-1}^p = xe^x$ and $\mathcal{C}_{-1} = e^x$. This implies with the previous theorem:

Theorem (McCammond-Meier, 04)

The dimension of the top homology group of the hypertree poset is $(n-1)^{n-2}$.

This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.

From the hypertree poset to rooted trees

1 Posets and hypertrees

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From the hypertree poset to rooted trees

1. This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices. The operad (species $\mathcal{S}$ with a natural transformation $\mathcal{S} \circ \mathcal{S} \to \mathcal{S}$) whose vector space are PreLie(n) is PreLie.
2. This operad is anticyclic (Chapoton, 07): There is an action of the symmetric group $\mathfrak{S}_n$ on PreLie($n-1$) which extends the one of $\mathfrak{S}_{n-1}$.
3. Moreover, there is an action of $\mathfrak{S}_n$ on the homology of the poset of hypertrees on n vertices.

Question

Is the action of $\mathfrak{S}_n$ on PreLie(n-1) the same as the action on the homology of the poset of hypertrees on n vertices?

Character for the action of the symmetric group on the homology of the poset

Using relations on species established previously, we obtain:

Proposition

The series Z_k, Z_k^p, Z_k^a and Z_k^{pa} satisfy the following relations:

$$Z_k + Z_{k-1}^p \circ Z_k^p = Z_k^p + Z_{k-1} \circ Z_k^p, \tag{8}$$

$$Z_k^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z_{k-1}^p \circ Z_k^p - Z_k^p}{Z_k^p} \right), \tag{9}$$

$$p_1 \frac{\partial Z_k}{\partial p_1} = Z_k^p. \tag{10}$$

Theorem (BDO 13, conjecture of Chapoton)

The cycle index series Z_{-1}, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series M associated with the anticyclic structure of PreLie *by:*

$$Z_{-1} = p_1 - \Sigma M = \mathrm{Comm} \circ \Sigma\, \mathrm{PreLie} + p_1 \left(\Sigma\, \mathrm{PreLie} + 1 \right). \tag{11}$$

The cycle index series Z^p_{-1} is given by:

$$Z^p_{-1} = p_1 \left(\Sigma\, \mathrm{PreLie} + 1 \right). \tag{12}$$

Other works and Open questions

- Same method applied to semi-pointed partition posets [BDO 16]
- Hypertree posets in type B

Pointed partition poset [Chapoton-Vallette 06, Vallette 07]

Set partitions whose parts are pointed,
ordered by refinment:

$$\{(p_1, F_1), \dots, (p_k, F_k)\} \leq \{(q_1, G_1), \dots, (q_l, G_l)\} \tag{13}$$

if $\forall j,\ G_j = \sqcup_{m=1}^{n_j} F_{i_m}$ and $q_j \in \{p_{i_1}, \dots, p_{i_{n_j}}\}$

Proposition (Chapoton-Vallette 06)

The character for the action of the symmetric group on the homology of the pointed partition poset is given by $\Sigma PreLie$.

Other works and Open questions

- Same method applied to semi-pointed partition posets [BDO 16]
- Hypertree posets in type B
- Study of the structure on chains in the hypertree poset ?

Why?

Get new tools to study properties of operads by looking at decorated hypertree poset !

$\rightarrow$ Vallette 07 : link Koszulness and Cohen-Macaulayness

$\rightarrow$ Bellier-Millès - BDO - Hoffbeck 18+ : link PBW and CL-shellability

Thank you for your attention !

[Oge13] Bérénice Oger Action of the symmetric groups on the homology of the hypertree posets. Journal of Algebraic Combinatorics, february 2013.