Local groups and related C^*-algebras

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Local group

Definition

Let $\mathcal{G}$ be a discrete set and $e \in \mathcal{G}$, W a subset of $\mathcal{G} \times \mathcal{G}$, and let $m : W \longrightarrow L$ and $i : L \longrightarrow L$ be the mappings, $m(a, b) = ab$ and $i(a) = a^{-1}$. Then the system $(\mathcal{G}, e, W, m, i)$ is called a local group (denoted by $\mathcal{G}$) if the following conditions are satisfied:

1) *if $(a, b), (b, c) \in W$ and $(ab, c) \in W$, then $a(bc) \in W$ and $(ab)c = a(bc)$;*
2) *for all $a \in L$ (a, e) and $(e, a) \in W$ and $ae = ea = a$;*
3) *for all $a \in L$ (a, a^{-1}) and $(a^{-1}, a) \in W$ и $aa^{-1} = a^{-1}a = e$.*

Standard constructions

Let $\mathcal{G}$ be a local group and $\mathfrak{A}$ be a C^*-algebra. The map $\pi : \mathcal{G} \longrightarrow \mathfrak{A}$ is a $*$-representation of $\mathcal{G}$ into $\mathfrak{A}$ if

- $\pi(e) = I$,
- $\pi(a^{-1}) = (\pi(a))^*$,
- $\pi(a)\pi(b)\pi(b^{-1}) = \begin{cases} \pi(ab)\pi(b^{-1}) & \text{if} \quad (a,b) \in W; \\ 0, & \text{if} \quad (a,b) \notin W. \end{cases}$
- $\pi(a^{-1})\pi(a)\pi(b) = \begin{cases} \pi(a^{-1})\pi(ab) & \text{if} \quad (a,b) \in W; \\ 0, & \text{if} \quad (a,b) \notin W. \end{cases}$

A collection $\mathcal{B} = \{B_g\}_{g \in \mathcal{G}}$ with $\cdot : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ and $* : \mathcal{B} \longrightarrow \mathcal{B}$ is a Fell bundle on $\mathcal{G}$ if for all $a, b \in \mathcal{G}$ and $A, B \in \mathcal{B}$:

1) $B_a B_b \subseteq B_{ab}$ для $(a, b) \in W$,
2) $\cdot : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ associative and bi-linear for $(a, b) \in W$ and $(ab, c) \in W$,
3) $\|AB\| \leq \|A\|\|B\|$,
4) $(B_a)^* \subseteq B_{a^{-1}}$,
5) $* : B_a \longrightarrow B_{a^{-1}}$ is conjugate-linear,
6) B_e is a C^*-algebra.
7) $A^*A \in B_e$.

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7) $A^*A \in B_e$.

Let Γ be a discrete group and $P \subset \Gamma$ be a subset. Define

$$\Gamma_P = \{a \in \Gamma : aP \cap P \neq \emptyset\}.$$

Proposition

Γ_P is a local group.

Example of a local group

Let G be a compact abelian group, $\mathfrak{A}$ be a C^*-algebra and

$$\tau : G \longrightarrow \mathrm{Aut}\,\mathfrak{A}$$

strongly continuous representation. Then

$$\mathfrak{A} \hookrightarrow C(G, \mathfrak{A}); A \mapsto \widehat{A};\ \widehat{A}(g) = \tau(g)(A).$$

Let Γ be an additive group of characters of G and

$$\widehat{A} \simeq \sum_{a \in \Gamma} A_a \chi^a,\ A_a = \int_G \widehat{A} \chi^{-a} d\mu. \quad \text{Let } \mathfrak{A}_a = \{A_a \in \mathfrak{A} : A \in \mathfrak{A}\}.$$

Definition

τ-spectrum of $\mathfrak{A}$ is defined to be the set $\mathrm{Sp}_\tau \mathfrak{A} = \{a \in \Gamma : \mathfrak{A}_a \neq \emptyset\}$.

Proposition

$\mathrm{Sp}_\tau \mathfrak{A}$ is a local group.

Algebra $C_r^*(P)$

Let Γ be a discrete group and $P \subset \Gamma$ be a subset. Let $\{U_a\}$ be the group of unitary shift operators on $l^2(\Gamma)$, $U_a e_b = e_{ab}$. Define

$$T_a = J^* U_a J, \quad J : l^2(P) \hookrightarrow l^2(\Gamma).$$

The family $\{T_a\}_{a \in \Gamma}$ generates the algebra $C_r^*(P)$.

Let $\mathrm{Mon}(P)$ be the semigroup generated by $\{T_a\}$. If $W \in \mathrm{Mon}(P)$ and $We_b = e_c$ then $\mathrm{ind} W = cb^{-1}$.

Proposition

The index of W is well defined and
if $W_1 W_2 \neq 0$ then $\mathrm{ind}(W_1 W_2) = \mathrm{ind}(W_1)\mathrm{ind}(W_2)$.

Proposition

The semigroup $\mathrm{Mon}(P)$ *is inverse and the set* $\{\mathrm{ind}(\mathrm{Mon}(P))\}$ *is a local group and coincides with* Γ_P*.*

Let $\mathrm{Mon}_a(P) = \{W \in \mathrm{Mon}(P) : \mathrm{ind}\, W = a\}$ and let B_a be an operator space generated by $\mathrm{Mon}_a(P)$.

Theorem

- *The collection* $\{B_a\}_{a\in\Gamma_P}$ *is a Fell bundle over local group* Γ_P*.*
- $\{B_a\}_{a\in\Gamma_P}$ *is a grading for* $C_r^*(P)$*,* $C_r^*(P) = \overline{\bigoplus_{a\in\Gamma_P} B_a}$*.*
- *There exists a conditional expectation* $\Phi : C_r^*(P) \longrightarrow B_e$*.*
- B_e *is a Cartan subalgebra in* $C_r^*(P)$*.*

Theorem

Let Γ be a discrete abelian group and $P \subset \Gamma$ be a subset such that $P \cup (-P)$ generates Γ. Then there exists a strongly continuous representation

$$\tau : G \longrightarrow \mathrm{Aut}\, C_r^*(P),$$

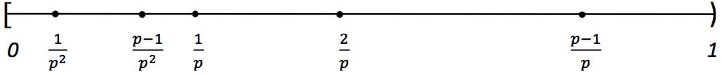
such that

$$\mathrm{Sp}_\tau C_r^*(P) = \Gamma_P, \;\; G \simeq \widehat{\Gamma}.$$

Example 1

Let $\Gamma = \mathbb{Q}_p = \{\frac{m}{p^k}\}_{k\in\mathbb{N},m\in\mathbb{Z}}$. Define

$$P_0 = [0,1)\cap\mathbb{Q}_p, \quad \text{then} \quad \Gamma_{P_0} = (-1,1)\cap\mathbb{Q}_p.$$



$$l^2(P_0), \quad \{e_{\frac{m}{p^k}}\}_{m<p^k}, \quad T_a e_{\frac{m}{p^k}} = \begin{cases} e_{a+\frac{m}{p^k}} \text{ if } a+\frac{m}{p^k}\in P_0; \\ 0, \text{ if } a+\frac{m}{p^k}\notin P_0. \end{cases}$$

Lemma

For all $k\in\mathbb{N}$ $l^2(P_0)$ can be decomposed as $\bigoplus\limits_{0\le a<\frac{1}{p^k}} H_a, \quad a\in P_0,$

and every H_a is invariant subspace for $T_{\frac{1}{p^k}}$ with dimension p^k.

Example 1

Let $\mathfrak{A}_k$ be the C^*-algebra generated by $T_{\frac{1}{p^k}}$.

Corollary

$$\mathfrak{A}_k \simeq M_{p^k}(\mathbb{C}).$$

Note that $T_{\frac{1}{p}} = T^p_{\frac{1}{p^2}}$, hence

$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_3 \ldots, \quad C^*_r(P_0) = \overline{\bigcup_k \mathfrak{A}_k}.$$

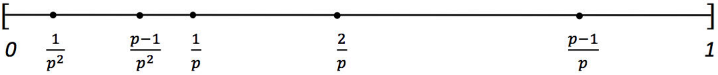
Theorem

$C^*_r(P_0)$ is an UHF-algebra.

Example 2

Again $\Gamma = \mathbb{Q}_p = \{\frac{m}{p^k}\}_{k\in\mathbb{N}, m\in\mathbb{Z}}$,

$$P_1 = [0,1] \cap \mathbb{Q}_p, \quad \text{then} \quad \Gamma_{P_1} = [-1,1] \cap \mathbb{Q}_p.$$



$$l^2(P_1) = l^2(P_0) \oplus \mathbb{C}\{e_1\}, \ \{e_{\frac{m}{p^k}}\}_{m \le p^k}, \ T_a e_{\frac{m}{p^k}} = \begin{cases} e_{a+\frac{m}{p^k}} \text{ if } a + \frac{m}{p^k} \in P_1; \\ 0, \text{ if } a + \frac{m}{p^k} \notin P_1. \end{cases}$$

Lemma

For all $k \in \mathbb{N}$ $l^2(P_1)$ can be decomposed as $H_0 \oplus (\bigoplus_{0 \le a < \frac{1}{p^k}} H_a)$, $a \in P_1$, and every H_a is invariant subspace for $T_{\frac{1}{p^k}}$ with dimension p^k, and H_0 is invariant subspace with dimension $p^k + 1$.

Example 2

Let $\mathfrak{A}_k$ be the C^*-algebra generated by $T_{\frac{1}{p^k}}$.

Corollary

$$\mathfrak{A}_k \simeq M_{p^k+1}(\mathbb{C}) \oplus M_{p^k}.$$

Again $T_{\frac{1}{p}} = T^p_{\frac{1}{p^2}}$, hence

$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_3 \dots, \quad C^*_r(P_1) = \overline{\bigcup_k \mathfrak{A}_k}.$$

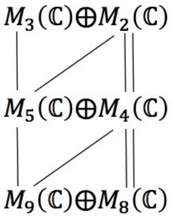
Theorem

$C^*_r(P_1)$ is an AF-algebra. $C^*_r(P_1)$ contains the algebra of compact operators K and the following short sequence is exact

$$0 \longrightarrow K \longrightarrow C^*_r(P_1) \longrightarrow C^*_r(P_0) \longrightarrow 0.$$

Example 2

In case $p = 2$ we have the following diagram



In both algebras the semigroup

$$\mathrm{Mon}(P) = E_0 \cup E_1,$$

where E_0 is semilattice of idempotents and E_1 is set of nilpotent operators.

Example 3

Let $P = \{2, 3, 5, 7, \dots\}$ be the set of all primes and $C_r^*(P)$ be the corresponding algebra. Define $X = \{n \in \mathbb{Z} : 2 + n \in P\}$

Proposition

- $\Gamma_P \neq \mathbb{Z}$.
- *Operators* $\{T_n\}_{n \in X}$ *are nilpotent.*
- *Operators* $\{T_n T_n^*\}_{n \in X}$ *are one dimensional projectors onto* e_{n+2}, *and* $T_n^* T_n$ *is one dimensional projector onto* e_2.
- $C_0(\Gamma_P) \oplus \mathbb{C}I \subset B_0$.

Proposition

Let the Polignac's conjecture is true. Then $\Gamma_P = 2\mathbb{Z} \cup (X \cup (-X))$ *and* $C_r^*(P)$ *is not simple.*

Thank you!