Homology of Ample Groupoids and the Matui Conjecture

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Introduction

This work was inspired by a sequence of recent papers by Matui ([M12], [M15], [M16]) on certain algebraic invariants associated to ample groupoids: the homology and the topological full group.

In [M12] Matui proved that if Γ is the groupoid associated to an action of $\mathbb{Z}$ on the Cantor set which is both minimal and topologically free or if Γ is the groupoid associated to an irreducible shift of finite type then:

$$(*) \qquad K_0(C_r^*(\Gamma)) \cong \bigoplus_{n=0}^{\infty} H_{2n}(\Gamma) \quad \text{and} \quad K_1(C_r^*(\Gamma)) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(\Gamma),$$

where $H_*(\Gamma)$ is the Crainic-Moerdijk homology of Γ (see [CM]).

An étale groupoid Γ is said to be *effective* (or essentially principal) if $(\operatorname{Iso}\Gamma)^0 = \Gamma^{(0)}$. We define an *ample* groupoid to be an étale groupoid with a basis of compact open sets.

In [M16] Matui conjectured that (*) holds if Γ is ample, minimal and effective such that $\Gamma^{(0)}$ is a compact (the HK conjecture).

Matui proved that if Γ_1, Γ_2 are ample groupooids such that for $i = 1, 2$, $\Gamma_i^{(0)}$ Cantor; Γ_i satisfies (*); and $C_r^*(\Gamma_i)$ is nuclear and in the UCT class, then $\Gamma_1 \times \Gamma_2$ also satisfies (*).

We observed that isomorphism (*) holds for many ample groupoids which do not satisfy Matui's hypotheses.

Let $\mathfrak{M}$ denote the class of ample groupoids for which (*) holds.
There are examples in $\mathfrak{M}$ which are not minimal, not effective and for which $\Gamma^{(0)}$ is not compact.
$\mathfrak{M}$ includes many discrete groups but not $\mathbb{Z}_n$ for $n > 1$.
We proved that $\mathfrak{M}$ is closed under equivalence of ample groupoids.
After completing our work, we learned that a counterexample to Matui's conjecture was found by Scarparo (see [S]).

The question we consider here is whether path groupoids of higher rank graphs or more generally higher rank Renault-Deaconu groupoids with zero dimensional unit space belong to $\mathfrak{M}$.

We know of no natural map $H_i(\Gamma) \to K_i(C_r^*(\Gamma))$ for $i > 0$.

We say X is zero dimensional if it has a basis of compact open sets.

Ample groupoid homology *d'après* Crainic & Moerdijk

Let $\psi : Y \to X$ be a local homeomorphism where X, Y are zero-dimensional spaces and let A be an abelian group. Define $\psi_* : C_c(Y, A) \to C_c(X, A)$ by

$$\psi_*(f)(x) = \sum_{x=\psi(y)} f(y).$$

Let Γ be an ample groupoid and for $n \geq 1$ set

$$\Gamma^{(n)} := \{(\gamma_1, \ldots, \gamma_n) : s(\gamma_i) = r(\gamma_{i+1})\}.$$

For $n > 1$ and $0 \leq i \leq n$ define $d_i = d_i^n : \Gamma^{(n)} \to \Gamma^{(n-1)}$ by

$$d_i^n(\gamma_1, \ldots, \gamma_n) := \begin{cases} (\gamma_2, \ldots, \gamma_n) & \text{if } i = 0, \\ (\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_n) & \text{if } 0 < i < n, \\ (\gamma_1, \ldots, \gamma_{n-1}) & \text{if } i = n. \end{cases}$$

Define $\partial_n : C_c(\Gamma^{(n)}, A) \to C_c(\Gamma^{(n-1)}, A)$ by $\partial_n := \sum (-1)^n (d_i^n)_*$ for $n > 1$ and $\partial_1 = s_* - r_*$ and for $n = 1$. One checks that $\partial_{n-1}\partial_n = 0$ and we denote the homology of the complex by $H_*(\Gamma, A)$ and write $H_*(\Gamma) := H_*(\Gamma, \mathbb{Z})$.

Properties of the homology

- Let Γ and Σ be ample groupoids and let $\psi : \Gamma \to \Sigma$ be an étale groupoid morphism. Then ψ_* induces a homomorphism

$$\psi_* : H_*(\Gamma) \to H_*(\Sigma).$$

- A groupoid equivalence $\Gamma \sim \Sigma$ where Γ and Σ are ample groupoids induces an isomorphism $H_*(\Gamma) \cong H_*(\Sigma)$.
- If an ample groupoid Γ can be expressed as increasing union of a sequence of clopen subgroupoids $\{\Gamma_n\}$ such that $\Gamma^{(0)} \subset \Gamma_n$ for each n, then

$$H_*(\Gamma) \cong \lim (i_n)_*(\Gamma_n)$$

where $i_n : \Gamma_n \to \Gamma$ denotes the inclusion map.
- If Γ is an AF groupoid then

$$H_n(\Gamma) \cong \begin{cases} K_0(C^*(\Gamma)) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Let Λ be a row-finite higher rank graph with no sources. Then there is a natural map $H_*(\Lambda) \to H_*(\mathcal{G}_\Lambda)$.

Higher rank Renault-Deaconu groupoids

Let X be a locally compact Hausdorff space and let $k > 0$. A map $\sigma : \mathbb{N}^k \times X \to X$ is called an action of $\mathbb{N}^k$ on X by local homeomorphisms if

1. For each $n \in \mathbb{N}^k$, the map $\sigma^n := \sigma(n, \cdot)$ is a local homeomorphism.
2. For $m, n \in \mathbb{N}^k$, $\sigma^{m+n} = \sigma^m \circ \sigma^n$ and $\sigma^0 = \mathrm{id}_X$.

The Renault-Deaconu groupoid associated to the action is given by

$$\Gamma(X, \sigma) := \{(x, m - n, y) : \sigma^m(x) = \sigma^n(y)\}.$$

Identifying X with $\Gamma(X, \sigma)^{(0)}$ via $x \mapsto (x, 0, x)$; we have

$$r(x, n, y) = x, \quad s(x, n, y) = y \quad \text{and}$$
$$(x, n, y)(y, m, z) = (x, m + n, z).$$

Suppose that if X is zero dimensional. Then $\Gamma = \Gamma(X, \sigma)$ is ample and the skew product $\Gamma \times_c \mathbb{Z}^k$ is an AF groupoid.

Skew product

The canonical cocycle $c : \mathcal{G}(X,\sigma) \to \mathbb{Z}^k$ is given by $c(x,n,y) := n$. We form the skew-product groupoid $\mathcal{G} \times_c \mathbb{Z}^k$ with structure maps:

$$r((x,n,y),p) = (x,p), \quad s((x,n,y),p) = (y,p+n) \quad \text{and}$$
$$((x,n,y),p)((y,m,z),p+n) = ((x,m+n,z),p).$$

There is an action $\widetilde{\sigma}$ of $\mathbb{N}^k$ on $\widetilde{X} = X \times \mathbb{Z}^k$ by local homeomorphisms given by $\widetilde{\sigma}^q(x,p) = (\sigma^q(x), p+q)$ and an isomorphism $\mathcal{G} \times_c \mathbb{Z}^k \cong \mathcal{G}(\widetilde{X},\widetilde{\sigma})$ given by

$$((x,m,y),p) \mapsto ((x,p),m,(y,p+m)).$$

There is also a natural action of $\mathbb{Z}^k$ on $\Gamma \times_c \mathbb{Z}^k$ such that $(\Gamma \times_c \mathbb{Z}^k) \rtimes \mathbb{Z}^k$ is equivalent to Γ; hence, we have

$$H_*(\Gamma) \cong H_*((\Gamma \times_c \mathbb{Z}^k) \rtimes \mathbb{Z}^k).$$

Moreover, $C^*(\Gamma)$ is strong Morita equivalent to $C^*((\Gamma \times_c \mathbb{Z}^k) \rtimes \mathbb{Z}^k)$ and

$$K_*(C^*(\Gamma)) \cong K_*(C^*((\Gamma \times_c \mathbb{Z}^k) \rtimes \mathbb{Z}^k)).$$

There is a natural action of $\mathbb{Z}^k$ on

$$K_0(C^*(\Gamma \times_c \mathbb{Z}^k)) \cong H_0(\Gamma \times_c \mathbb{Z}^k) \cong \lim(C_c(X,\mathbb{Z}), \sigma_*^n).$$

Theorem

$H_n(\Gamma) \cong H_n(\mathbb{Z}^k, H_0(\Gamma \times_c \mathbb{Z}^k))$ for all n. (So $H_n(\Gamma) = 0$ for $n > k$.)

An application of Kasparov's spectral sequence yields the following:

Theorem

There is a spectral sequence that converges to the K-theory of $C^*(\Gamma)$:

$$H_p(\mathbb{Z}^k, K_q(C^*(\Gamma \times_c \mathbb{Z}^k))) \implies K_{p+q}(C^*(\Gamma)).$$

Note $H_p(\mathbb{Z}^k, K_q(C^*(\Gamma \times_c \mathbb{Z}^k))) = 0$ if $p > k$ or q is odd.
The differentials are of the form $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q-r+1}$.

Corollary

If $k = 1, 2$, then Γ is in the class $\mathfrak{M}$. So if Λ is a k-graph for $k = 1, 2$, then $\mathcal{G}_\Lambda$ is in $\mathfrak{M}$.

References

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Merci!

Higher rank graphs

Let $k \in \mathbb{N} := \{0, 1, 2, \dots\}$.

Definition (see [KP00])

Let Λ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. Then (Λ, d) is a *k-graph* if it satisfies the factorization property:
For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$.

Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \mathrm{Obj}\,(\Lambda)$, the set of *vertices*.

An element $\lambda \in \Lambda^{e_i}$ is called an *edge*.

We assume throughout that Λ is *row-finite* and *source-free*; that is, for all $v \in \Lambda^0, n \in \mathbb{N}^k$, $v\Lambda^n := r^{-1}(v) \cap \Lambda^n$ is finite and nonempty.

The path groupoid

The infinite path space Λ^∞ is the set of k-graph morphisms $x : \Omega_k \to \Lambda$.
Shift map: for $q \in \mathbb{N}^k$ define the local homeomorphism $\sigma^q : \Lambda^\infty \to \Lambda^\infty$ by

$$\sigma^q(x)(m,n) = x(m+q, n+q) \qquad \text{for } (m,n) \in \Omega_k.$$

We define the path groupoid $\mathcal{G}_\Lambda \subset \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty$ by

$$\mathcal{G}_\Lambda := \{(x, m-n, y) : \sigma^m(x) = \sigma^n(y) \text{ for some } m,n \in \mathbb{N}^k\}.$$

The unit space is identified with Λ^∞ via the map $x \mapsto (x, 0, x)$.

Λ is *cofinal* if for every $v \in \Lambda^0$ and $x \in \Lambda^\infty$, there are $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n,n)$ and $r(\lambda) = v$. If Λ is cofinal, $\mathcal{G}_\Lambda$ is minimal.

For $v \in \Lambda^0$ the *local periodicity group at* v, denoted $P_\Lambda(v)$, is the set of all $m - n \in \mathbb{Z}^k$ such that $m, n \in \mathbb{N}^k$ and $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^\infty$.

Λ is *aperiodic* if $P_\Lambda(v) = 0$ for all $v \in \Lambda^0$. If Λ is aperiodic, $\mathcal{G}_\Lambda$ is *topologically principal*, that is, points with trivial isotropy are dense.