CCR flows associated to closed convex cones

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E_0-semigroups

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In Arveson's theory of E_0-semigroups, the cone is $[0, \infty)$. Here we consider the higher dimensional analogue.

The semigroup P is a closed convex in $\mathbb{R}^d$ such that $P - P = \mathbb{R}^d$ and $P \cap -P = \{0\}$.

Definition

By an E_0-semigroup on $B(\mathcal{H})$ over P, we mean a family $\alpha := \{\alpha_x\}_{x \in P}$ of unital normal $*$-endomorphisms of $B(\mathcal{H})$ such that

(1) *for $x, y \in P$, $\alpha_x \circ \alpha_y = \alpha_{x+y}$, and*

(2) *for $A \in B(\mathcal{H})$ and $T \in \mathcal{L}^1(\mathcal{H})$, the map $P \ni x \to Tr(T\alpha_x(A)) \in \mathbb{C}$ is continuous.*

We identify E_0-semigroups acting on different Hilbert spaces if they are unitarily equivalent.

cocycle conjugacy

The basic equivalence relation in the theory of E_0-semigroups is that of cocycle conjugacy.

Definition

Let $\alpha := \{\alpha_x\}_{x \in P}$ be an E_0-semigroup on $B(\mathcal{H})$. By an α-cocycle we mean a strongly continuous family of unitaries $\{U_x\}_{x \in P}$ such that $U_x \alpha_x(U_y) = U_{x+y}$. Let $U := \{U_x\}_{x \in P}$ be an α-cocycle. Define for $x \in P$,

$$\beta_x(.) = U_x \alpha_x(.) U_x^*.$$

Then $\beta := \{\beta_x\}_{x \in P}$ is an E_0-semigroup on $B(\mathcal{H})$. Such an E_0-semigroup is called a cocycle perturbation of α.

Cocycle conjugacy is an equivalence relation.

Examples

Let us recall the symmetric Fock space and the Weyl operators. For a Hilbert space $\mathcal{H}$, $\Gamma(\mathcal{H})$ denotes the symmetric Fock space. For $u \in \mathcal{H}$, let

$$e(u) := \sum_{n=0}^{\infty} \frac{u^{\otimes n}}{\sqrt{n!}}.$$

We have the following.

1. The set $\{e(u) : u \in \mathcal{H}\}$ is linearly independent and total in $\Gamma(\mathcal{H})$.
2. For $u, v \in \mathcal{H}$, $\langle e(u)|e(v)\rangle = e^{\langle u|v\rangle}$.

For $u \in \mathcal{H}$, there exists a unique unitary operator denoted $W(u)$ on $\Gamma(\mathcal{H})$ such that

$$W(u)e(v) := e^{-\frac{||u||^2}{2} - \langle u|v\rangle} e(u+v).$$

CCR relations

The operators $\{W(u) : u \in \mathcal{H}\}$ satisfy the following relations called the canonical commutation relations.

$$W(u)W(v) = e^{-iIm(\langle u|v\rangle)}W(u+v).$$

An important fact is that the von-Neumann algebra generated by $\{W(u) : u \in \mathcal{H}\}$ is σ-weak dense in $B(\Gamma(\mathcal{H}))$.

Proposition

Let $V \in B(\mathcal{H})$ be an isometry on $\mathcal{H}$. Then there exists a unique normal unital $*$-endomorphism, denoted α, of $B(\Gamma(\mathcal{H}))$ such that

$$\alpha(W(u)) = W(Vu).$$

CCR flows

Proposition

Let $V := \{V_x\}_{x \in P}$ be a strongly continuous isometric representation of P on $\mathcal{H}$. Then there exists a unique E_0-semigroup denoted $\alpha^V = \{\alpha_x\}_{x \in P}$ on $B(\Gamma(\mathcal{H}))$ such that

$$\alpha_x(W(u)) = W(V_x u).$$

The E_0-semigroup α^V is called the CCR flow associated to the isometric representation V.

The first example of an isometric representation of P that one can think of is the left regular representation of P on $L^2(P)$. This example can be generalised.

P-modules

Let $A \subset \mathbb{R}^d$ be a P-module. We say that A is a P-module if

- the subset A is closed,
- A is non-empty and proper, and
- for $x \in P$, $A + x \subset A$.

Let $k \in \{1, 2, \cdots, \} \cup \{\infty\}$ and $\mathcal{K}$ be a Hilbert space of dimension k. Consider the Hilbert space $L^2(A, \mathcal{K})$. The cone P acts as isometries on $L^2(A, \mathcal{K})$ via translations. The associated isometric representation will be denoted by $V^{(A,k)}$ and the associated CCR flow will be denoted by $\alpha^{(A,k)}$.

The main result

Theorem (Anbu Arjunan and S.S)

Let A_1, A_2 be P-modules and $k_1, k_2 \in \{1, 2, \cdots, \} \cup \{\infty\}$. Then the following are equivalent.

1. *The CCR flows $\alpha^{(A_1,k_1)}$ and $\alpha^{(A_2,k_2)}$ are conjugate.*
2. *The CCR flow $\alpha^{(A_1,k_1)}$ And $\alpha^{(A_2,k_2)}$ are cocycle conjugate.*
3. *There exists $z \in \mathbb{R}^d$ such that $A_1 = A_2 + z$ and $k_1 = k_2$.*

An invariant – Gauge group

Let $\alpha := \{\alpha_x\}_{x \in P}$ be an E_0-semigroup on $B(\mathcal{H})$. A strongly continuous family of unitaries $U := \{U_x\}_{x \in P}$ is called a gauge cocycle of α if

(1) for $x, y \in P$, $U_x \alpha_x(U_y) = U_{x+y}$, and

(2) for $x \in P$, $U_x \in \alpha_x(B(\mathcal{H}))'$.

That is if we perturb α by U, we get α. The set of gauge cocycles, denoted $G(\alpha)$, forms a group where the multiplication is given by

$$\{U_x\}.\{V_x\} = \{U_x V_x\}.$$

The gauge group is computable for the CCR flow associated to a P-module which is where groupoids play a key role.

Formula for the gauge group

Let A be a P-module and $V^A := \{V_x\}_{x \in P}$ be the isometric representation of P on $L^2(A)$. Define

$$E_x := V_x V_x^*$$
$$M := \{V_x, V_x^* : x \in P\}'$$
$$\mathcal{U}(M) := \text{unitary group of } M.$$

Let $\lambda \in \mathbb{R}^d$ and $u \in \mathcal{U}(M)$. Set

$$U_x^{(\lambda,u)} := e^{i\langle \lambda | x \rangle}\Gamma(uE_x^{\perp} + E_x).$$

Proposition

The map

$$\mathbb{R}^d \times \mathcal{U}(M) \ni (\lambda, u) \to U^{(\lambda,u)} \in G(\alpha)$$

is a topological isomorphism.

What is M ?. Let A be a P-module. Set

$$G_A := \{z \in \mathbb{R}^d : A + z = A\}.$$

The isotropy group G_A acts on $L^2(A)$ by translations.

Proposition

The von-Neumann algebra M is generated by G_A. In particular, M is abelian.

It is in the proof of the above proposition where groupoids play a key role.

A paper of Muhly and Renault

"C^*-algebras associated to multivariable Wiener-Hopf operators", Muhly and Renault, Transactions of the AMS, 1982.

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In the above paper, Mulhy and Renault studied the C^*-algebra generated by $\{\int f(x)V_x dx : f \in L^1(P)\}$, denoted $\mathcal{W}(P)$, where $V := \{V_x\}_{x \in P}$ is the left regular representation of P on $L^2(P)$. In particular, they obtained a groupoid realisation of $\mathcal{W}(P)$.

Universal groupoid

Computing the commutant of $\{V_x, V_x^* : x \in P\}$ is the same as computing the commutant of the C^*-algebra generated by $\{\int f(x)V_x dx : f \in L^1(P)\}$.
It is desirable to have a universal groupoid from which information about the C^*-algebra generated $\{\int f(x)V_x dx : f \in L^1(P)\}$ can be obtained for different isometric representations V.
A common feature that isometric representations associated to P-modules share is that they have **commuting range projections**.

contd...

Let $V := \{V_x\}_{x \in P}$ be a strongly continuous family of isometries with commuting range projections. For $z \in \mathbb{R}^d$, write $z = x - y$ with $x, y \in P$. Set

$$W_z := V_y^* V_x.$$

(1) For $z \in \mathbb{R}^d$, W_z is well-defined and is a partial isometry.

(2) The range projections $E_z = W_z W_z^*$ forms a commuting family of projections.

(3) $C^*\{\int f(z) W_z dz : f \in L^1(\mathbb{R}^d)\} = C^*\{\int f(x) V_x dx : f \in L^1(P)\}$.

$$\mathcal{C}(\mathbb{R}^d) := \{\text{closed subsets of } \mathbb{R}^d\}.$$

Endow $\mathcal{C}(\mathbb{R}^d)$ with the Fell topology. The group $\mathbb{R}^d$ acts on $\mathcal{C}(\mathbb{R}^d)$ by translations. Set

$$X_u := \{A \in \mathcal{C}(\mathbb{R}^d) : 0 \in A, -P + A \subset A\}.$$

Then X_u is compact in $\mathcal{C}(\mathbb{R}^d)$. Consider the transformation groupoid $\mathcal{C}(\mathbb{R}^d) \rtimes \mathbb{R}^d$ and restrict to X_u. Denote the resulting groupoid by $\mathcal{G}_u$. For $f \in C_c(\mathbb{R}^d)$, let

$$\widetilde{f}(A, x) = f(x).$$

Then $\widetilde{f} \in C_c(\mathcal{G}_u)$.

Proposition

(1) *The groupoid $\mathcal{G}_u$ has a Haar system.*

(2) *The C^*-algebra of $\mathcal{G}_u$ is generated by $\{\widetilde{f} : f \in C_c(\mathbb{R}^d)\}$.*

Let $V := \{V_x\}_{x \in P}$ be an isometric representation on $B(\mathcal{H})$ with commuting range projections.

Theorem (2014, S.S)

There exists a unique $*$-homomorphism $\pi : C^*(\mathcal{G}_u) \to B(\mathcal{H})$ with range $C^*\{\int f(z)W_z dz : f \in L^1(\mathbb{R}^d)\}$ such that

$$\pi(\widetilde{f}) = \int f(-z)W_z dz$$

for $f \in C_c(\mathbb{R}^d)$.

Let $V^A := \{V_x\}_{x \in P}$ be the isometric representation associated to the P-module A. Then $-A \in X_u$. The representation of $C^*(\mathcal{G}_u)$ corresponding to V^A is π_{-A} where π_{-A} is the induced representation of $\mathcal{G}_u$ at the point $-A$.

A Theorem of Connes asserts that if $\mathcal{G}$ is a groupoid and $u \in \mathcal{G}^{(0)}$ then the commutant of $\pi_u(C^*(\mathcal{G}))$ is generated by the isotropy group of u. This is the conceptual explanation of Proposition 4.

With the explciity knowledge of gauge group and with a few tricks, one can complete the proof of Theorem 1.

Merci beaucoup !